Mathematics 700 Homework
due Wednesday, October 6

The following are problems on finding the matrices of linear maps. There are examples in Chapter 10 of the text that are relevant to these problems.

1. Let \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be given by
   \[
   T(x, y) := (x - 2y, -4x + y, 7x - 11y).
   \]
   Then find the matrix of \( T \) with respect to the standard bases of each of \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \). (Recall that the standard basis of \( \mathbb{R}^n \) is the basis \( e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), e_3 = (0, 0, 1, \ldots, 0) \ldots \).)

2. With \( T \) as in the last problem find the matrix of \( T \) with respect to the bases \( V = \{ (1, 2), (3, 2) \} \) of \( \mathbb{R}^2 \) and \( W := \{ (1, 1, 1), (0, 2, 1), (0, 0, 3) \} \) of \( \mathbb{R}^3 \).

3. Letting \( \mathcal{P}_3 \) be the real polynomials of degree \( \leq 3 \) and using the standard basis \( V := \{ 1, x, x^2, x^3 \} \) of \( \mathcal{P}_3 \), find the matrices, the rank and the nullity of the following linear maps
   (a) \( (Tp)(x) = p(x - 2) \),
   (b) \( (Cp)(x) = (x + 1)^3 p \left( \frac{x - 1}{x + 1} \right) \),
   (c) \( Ap = p + p' + p'' + p''' \),
   (d) \( (Pp)(x) = e^{-x} \int_{-\infty}^{x} p(t) e^t \, dt \),
   (e) \( (Bp)(x) = \int_{0}^{x} p'(t) \, dt \),
   (f) \( (Vp)(x) = \frac{d}{dx} \int_{0}^{x} p(t) \, dt \).

4. Let \( M_{2 \times 2} \) be the vector space of \( 2 \times 2 \) matrices over the real numbers. Let
   \[
   A := \begin{bmatrix}
   a_{11} & a_{12} \\
   a_{21} & a_{22}
   \end{bmatrix}.
   \]
   Use for \( M_{2 \times 2} \) the the ordered basis basis \( V := \{ E_{11}, E_{12}, E_{21}, E_{22} \} \) where
   \[
   E_{11} = \begin{bmatrix}
   1 & 0 \\
   0 & 0
   \end{bmatrix},
   E_{12} = \begin{bmatrix}
   0 & 1 \\
   0 & 0
   \end{bmatrix},
   E_{21} = \begin{bmatrix}
   0 & 0 \\
   1 & 0
   \end{bmatrix},
   E_{22} = \begin{bmatrix}
   0 & 0 \\
   0 & 1
   \end{bmatrix}.
   \]
   Then find the matrices of the following linear maps from \( M_{2 \times 2} \) to itself.
   (a) \( LX = AX \),
   (b) \( RX = XA \),
   (c) \( CX = AX - XA \),

1
(d) \( TX = X^t \) (the transpose of \( X \)),
(e) \( SX = \frac{1}{2}(X + X^t) \) and find the rank and nullity of this map, and
(f) \( GX = \frac{1}{2}(X - X^t) \) and find the rank and nullity of this map.

5. The set of complex numbers \( \mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \} \) is a two dimensional vector space over the real numbers \( \mathbb{R} \). Using the basis \( B = \{ 1, i \} \) for this real vector space find the matrices of the following linear maps

(a) \( Jz = iz \),
(b) \( Cz = \overline{z} \) (where \( \overline{z} \) is the complex conjugate of \( z \)),
(c) \( Tz = (2 + 3i)z \),
(d) \( Mz = (a + bi)z \), and
(e) \( Rz = \frac{1}{2}(z + \overline{z}) \).