Mathematics 700 Homework
Due Monday, September 27

Quotient Spaces. If there is any one single idea the characterizes modern algebra it is the idea of a quotient structure. The following problems introduce the linear algebra version of this concept. You have may have seen other version as in the integers mod n, the quotient of a group by a normal subgroup, and the quotient of a ring by an ideal.

Let V be a vector space over the field $\mathbf{F}$ and W a subspace of V. Then define an equivalence relation $\sim_W$ by

$$v_1 \sim_W v_2 \quad \text{if and only if} \quad v_2 - v_1 \in W.$$

Problem 1. Show that this is an equivalence relation. (Recall a relation $\sim$ on a set V that $\sim_W$ is an ***equivalence relation*** iff the three conditions (1) $x \sim x$ for all $x \in V$ (i.e. it is ***reflective***) (2) $x \sim y$ implies $y \sim x$ for all $x, y \in V$ (i.e. $\sim$ is ***symmetric***) and (3) $x \sim y$ and $y \sim z$ implies $x \sim y$ (i.e. $\sim$ is ***transitive***).) □

Denote by $[v]_W$ the equivalence class of $v \in V$ under the equivalence relation $\sim_W$. That is

$$[v]_W := \{u \in V : u \sim_W v\}.$$

Problem 2. Show $[v]_W = v + W$ where $v + W = \{v + w : w \in W\}$. □

Let V/W be the set of all equivalence classes of $\sim_W$. That is

$$V/W := \{[v]_W : v \in V\} = \{v + W : v \in V\}.$$

The equivalence class $[v]_W = v + W$ is often called the ***coset of*** v ***in*** V/W.

Problem 3. Let $V = \mathbf{R}^2$ and let W be the subspace of points of V of points (x, y) with $y = -2x$. Then draw pictures of the coset of $(1, 1)$ in V/W and the coset of $(3, -2)$ in V/W. What is a geometric description of the coset of $v \in \mathbf{R}$ in V/W? □

Define a sum and scalar multiplication in V/W by

$$[v_1]_W + [v_2]_W := [v_1 + v_2]_W \quad c[v]_W := [cv]_W$$

where $v_1, v_2, v \in V$ and $c \in \mathbf{F}$.

Problem 4. Show this is well defined. The term ***well defined*** is used in mathematics to mean "is independent of the choices made in the definition". In this particular case this means you need to show

$$[v_1]_W = [v_1']_W \quad \text{and} \quad [v_2]_W = [v_2']_W \qquad \text{implies} \qquad [v_1 + v_2]_W = [v_1' + v_2']_W$$

and

$$[v]_W = [v']_W \qquad \text{implies} \qquad [cv]_W = [cv']_W.$$

Proposition 1. *With these operations V/W is a vector space.* □

Problem 5. Prove this. □

Define a map $\pi\colon V \to V/W$ by $\pi(v) = [v]_W$. (Or in slightly different notation $\pi v = V + W$.) This is the ***natural projection*** or ***canonical projection*** of V onto the quotient space V/W.

Problem 6. The natural projection $\pi\colon V \to V/W$ is a linear map and $\ker(\pi) = W$. □

Problem 7. If V is finite dimensional then what is the dimension of V/W in terms of $\dim V$ and $\dim W$? Prove your answer is correct. (HINT: Rank plus nullity.) □

Problem 8. In the example of Problem 3 draw some pictures of cosets $v_2 + W$ and $v_2 + W$ what their sum $(v_1 + W) + (v_2 + W)$ and the linear combination $2(v_1 + W) - 3(v_2 + W)$ for a few choices of v_1 and v_2. □

The following are review:

Problem 9. If the vectors v_1, v_2, v_3 in the complex vector space V are linearly independent then show

1. The vectors $u_1 = v_1$, $u_2 = v_1 + v_2$, $u_3 = v_1 + v_2 + v_3$ are also linear independent.
2. The vectors

$$\begin{aligned} w_1 &= v_1 + 2v_2 + 3v_3 \\ w_2 &= 4v_1 + 5v_2 + 6v_3 \\ w_3 &= 7v_1 + 8v_2 + 9v_3 \end{aligned}$$

are linearly dependent. □

Problem 10. Let $M_{3\times 3}$ the three by three real matrices. Define two subspaces of $M_{3\times 3}$ by

$$\begin{aligned} \mathcal{S} &:= \{A \in M_{3\times 3} : A^t = A\} \\ \mathcal{A} &:= \{A \in M_{3\times 3} : A^t = -A\} \end{aligned}$$

where A^t is the transpose of A. (That is if $A = [a_{ij}]$, then $A^t = [a_{ji}]$.) Elements of $\mathcal{S}$ are called ***symmetric*** matrices and elements of $\mathcal{A}$ are called ***skew symmetric*** or ***anti-symmetric*** matrices. Show the following

1. $\dim \mathcal{S} = 6$,
2. $\dim \mathcal{A} = 3$,
3. $M_{3\times 3} = \mathcal{S} \oplus \mathcal{A}$.
4. Show that any four dimensional subspace of $M_{3\times 3}$ contains at least one nonzero symmetric matrix.
5. Give an example of a four dimensional subspace of $M_{3\times 3}$ that does not contain any nonzero skew symmetric matrix. □