1. What is the span of \((1,0,1), (0,1,1)\) and \((1,1,1)\) in \(\mathbb{R}^3\)?
2. What is the span of \((1,0,1), (0,1,1)\) and \((2,3,-3)\) in \(\mathbb{R}^3\)?
3. Give an example of three vectors in \(\mathbb{R}^2\). What is the span of \((1,1)\)?
4. Let \(\mathcal{P}_n\) the real polynomials of degree \(\leq n\). What is the dimension of \(\mathcal{P}_n\). Prove your result by finding a basis of \(\mathcal{P}_n\).
5. Let \(z_1, z_2, z_3, z_4 \in \mathbb{C}\) be distinct complex numbers. Then when are the vectors
   \[
   v_1 = (1, 1, 1, 1),
   v_2 = (z_1, z_2, z_3, z_4),
   v_3 = (z_1^2, z_2^2, z_3^2, z_4^2),
   v_3 = (z_1^3, z_2^3, z_3^3, z_4^3)
   \]
   linearly independent? Prove your result. (Hint: It is not hard to do this problem by direct calculation, but here is a less messy method. If \(c_1, \ldots, c_4 \in \mathbb{C}\) so that \(c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0\) then let \(p(x)\) be the polynomial \(p(x) = c_1 + c_2 x + c_3 x^2 + c_4 x^3\). Then \(p(z_k) = 0\) for \(k = 1, \ldots, 4\). How many roots does a polynomial of degree \(\leq 3\) have?)
6. Let \(\mathcal{P}_3\) be the polynomials of degree \(\leq 3\) over the field \(\mathbb{F}\). Let \(a_1, a_2, a_3, a_4 \in \mathbb{F}\) be distinct. Let \(\ell_i(x)\) for \(i = 1, \ldots, 4\) be the polynomials
   \[
   \ell_1(x) = \frac{(x-a_2)(x-a_3)(x-a_4)}{(a_1-a_2)(a_1-a_3)(a_1-a_4)},
   \ell_2(x) = \frac{(x-a_1)(x-a_3)(x-a_4)}{(a_2-a_1)(a_2-a_3)(a_2-a_4)},
   \ell_3(x) = \frac{(x-a_1)(x-a_2)(x-a_4)}{(a_3-a_1)(a_3-a_2)(a_3-a_4)},
   \ell_4(x) = \frac{(x-a_1)(x-a_2)(x-a_3)}{(a_4-a_1)(a_4-a_2)(a_4-a_3)}.
   \]
   These are the Lagrange interpolation polynomials with nodes at \(a_1, \ldots, a_4\). (You are not responsible for this terminology, but it is standard and you will likely see it again in other classes.)
   (a) Show that \(\ell_i(a_j) = \delta_{ij}\) where \(\delta\) is the Kronecker delta function:
   \[
   \delta_{ij} := \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases}
   \]
   (b) Show that \(\ell_1, \ell_2, \ell_3, \ell_4\) is a basis of \(\mathcal{P}_3\). (Hint: To show linear independence set a linear combination of \(\ell_1, \ell_2, \ell_3, \ell_4\) to 0 and evaluate this linear combination at \(a_1, a_2, a_3, a_4\) and use that \(\ell_i(a_j) = \delta_{ij}\).)
   (c) Let \(b_1, b_2, b_3, b_4 \in \mathbb{F}\). Let
   \[
   p(x) = b_1 \ell_1(x) + b_2 \ell_2(x) + b_3 \ell_3(x) + b_4 \ell_4(x).
   \]
   Show that \(p(a_i) = b_i\) for \(i = 1, \ldots, 4\).
(d) **Optional extra credit:** Extend these results to $P_n$ for $n \geq 1$.

7. Let $P_2$ be the vector space of all real polynomials of degree $\leq 2$.
   (a) Letting $P_1 \subset P_2$ in the natural way, show that $P_1$ is a two dimensional subspace of $P_2$.
   (b) For any $a \in \mathbb{R}$ show that $Z(a) := \{p(x) \in P_2 : p(a) = 0\}$ is a two dimensional subspace of $P_2$ by giving a basis of $Z(a)$.
   (c) True or False: If $V$ is a two dimensional subspace of $P_2$ then either $V = P_1$ or $V = Z(a)$ for some $a \in \mathbb{R}$? Prove your answer is correct.

8. Let $V$ be a vector space with $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_m\}$ subsets of $V$ with the same finite number of elements. Assume that $\{u_1, \ldots, u_m\}$ is linearly independent and that

   \[ \text{Span}\{u_1, \ldots, u_m\} \subseteq \text{Span}\{v_1, \ldots, v_m\}. \]

   Then show $\text{Span}\{u_1, \ldots, u_m\} = \text{Span}\{v_1, \ldots, v_m\}$ and that $\{v_1, \ldots, v_m\}$ is linearly independent.