Mathematics 700 Final Name:______________________________

Show your work to get credit. An answer with no work will not get credit.

1. Define the following:
   (a) eigenvalue.
   (b) eigenvector.
   (c) coordinates of a vector relative to a basis.
   (d) the three elementary row operations.
   (e) the Smith Normal Form of a matrix over a Euclidean domain.
   (f) V is the direct sum of $W_1, \dots, W_k$.
2. Let V be a vector space and let $v_1, \dots, v_n \in V$ be linearly independent vectors. Let $v \in V$ be a vector so that $v \notin \operatorname{Span}\{v_1, \dots, v_n\}$. Show that $\{v_1, \dots, v_n, v\}$ is a linearly independent set.
3. Let $M_{k\times k}(\mathbf{R})$ be the vector space of $k \times k$ matrices over the real numbers. $T\colon M_{3\times 3}(\mathbf{R}) \to M_{2\times 2}(\mathbf{R})$ be linear. Show there is a nonzero symmetric matrix, A, so that $T(A) = 0$. (Recall that A is symmetric iff $A^t = A$ where A^t is the transpose of A.)
4. What is the Jordan canonical form, over the complex numbers, of a matrix that has elementary divisors $x-1$, $(x-1)^2$, $x^2-6x+25$, $(x^2-6x+25)^2$?
5. Let
$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & -1 \\ 3 & 4 & 1 & 1 \end{bmatrix}$$
   Then for A find
   (a) The invariant factors.
   (b) The elementary divisors.
   (c) The minimal polynomial.
   (d) The rational canonical form over the reals.
6. Let V be a finite dimensional vector space over the real numbers $\mathbf{R}$ and let W be a subspace of V. Assume that there are vectors $v_1, v_2 \in V$ so that
$$v_1 \notin W \qquad \text{and} \qquad v_2 \notin \operatorname{Span}(\{v_1\} \cup W\}.$$
   There show there is a linear functional $f \in V^*$ so that $f(w) = 0$ for all $w \in W$, $f(v_1) = 1$ and $f(v_2) = 2$.
7. Let V be a finite dimensional vector space over the complex numbers and $P\colon V \to V$ a linear map so that $P^2 = P$. Show that $\operatorname{trace} P = \operatorname{rank} P$
8. Let $\mathcal{P}_2 = \operatorname{Span}\{1, x, x^2\}$ be the real polynomials of degree ≤ 2. Define $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ by
$$T(p)(x) = p(x+1).$$
   Let $\mathcal{P}_2^*$ be the dual space to $\mathcal{P}_2$ and let $\Lambda \in \mathcal{P}_2^*$ be the functional
$$\Lambda(p) = p(-3).$$
   Then compute $\langle x^2, T^*\Lambda \rangle$.
9. A 5×5 real matrix A has minimal polynomial $\min_A(x) = (x-1)^2(x-2)(x-3)$ and $\operatorname{trace}(A) = 10$. What is the rational canonical form for A?
10. Let V and W be vector spaces over the field $\mathbf{F}$ and let $v_1, \dots, v_n \in V$ and $w_1, \dots, w_n \in W$ so the following tow conditions hold
    (a) $\operatorname{Span}\{v_1, \dots, v_n\} = V$ (but we do not assume $\{v_1, \dots, v_n\}$ is linearly independent),
    (b) For any scalars $c_1, \dots, c_n \in \mathbf{F}$
$$\sum_{k=1}^{n} c_k v_k = 0 \qquad \text{implies} \qquad \sum_{k=1}^{n} c_k w_k = 0.$$
    Show that there is a unique linear map $T\colon V \to W$ so that $Tv_k = w_k$ for $1 \le k \le n$.