Mathematics 700 Final
Name:

Show your work to get credit. An answer with no work will not get credit.

1. Define the following:
   (a) eigenvalue.
   (b) eigenvector.
   (c) coordinates of a vector relative to a basis.
   (d) the three elementary row operations.
   (e) the Smith Normal Form of a matrix over a Euclidean domain.
   (f) $V$ is the direct sum of $W_1, \ldots, W_k$.

2. Let $V$ be a vector space and let $v_1, \ldots, v_n \in V$ be linearly independent vectors. Let $v \in V$ be a vector so that $v \notin \text{Span}\{v_1, \ldots, v_n\}$. Show that $\{v_1, \ldots, v_n, v\}$ is a linearly independent set.

3. Let $M_{k \times k}(\mathbb{R})$ be the vector space of $k \times k$ matrices over the real numbers. $T: M_{3 \times 3}(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R})$ be linear. Show there is a nonzero symmetric matrix, $A$, so that $T(A) = 0$. (Recall that $A$ is symmetric iff $A^t = A$ where $A^t$ is the transpose of $A$.)

4. What is the Jordan canonical form, over the complex numbers, of a matrix that has elementary divisors $x - 1$, $(x - 1)^2$, $x^2 - 6x + 25$, $(x^2 - 6x + 25)^2$?

5. Let
   $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & -1 \\ 3 & 4 & 1 & 1 \end{bmatrix}$$

   Then for $A$ find
   (a) The invariant factors.
   (b) The elementary divisors.
   (c) The minimal polynomial.
   (d) The rational canonical form over the reals.

6. Let $V$ be a finite dimensional vector space over the real numbers $\mathbb{R}$ and let $W$ be a subspace of $V$. Assume that there are vectors $v_1, v_2 \in V$ so that
   $$v_1 \notin W \quad \text{and} \quad v_2 \notin \text{Span}(\{v_1\} \cup W).$$

   There show there is a linear functional $f \in V^*$ so that $f(w) = 0$ for all $w \in W$, $f(v_1) = 1$ and $f(v_2) = 2$.

7. Let $V$ be a finite dimensional vector space over the complex numbers and $P: V \to V$ a linear map so that $P^2 = P$. Show that trace $P = \text{rank} P$.

8. Let $\mathcal{P}_2 = \text{Span}\{1, x, x^2\}$ be the real polynomials of degree $\leq 2$. Define $T: \mathcal{P}_2 \to \mathcal{P}_2$ by
   $$T(p)(x) = p(x + 1).$$

   Let $\mathcal{P}_2^*$ be the dual space to $\mathcal{P}_2$ and let $\Lambda \in \mathcal{P}_2^*$ be the functional
   $$\Lambda(p) = p(-3).$$

   Then compute $\langle x^2, T^*\Lambda \rangle$.

9. A $5 \times 5$ real matrix $A$ has minimal polynomial $\min_A(x) = (x - 1)^2(x - 2)(x - 3)$ and trace$(A) = 10$. What is the rational canonical form for $A$?

10. Let $V$ and $W$ be vector spaces over the field $\mathbb{F}$ and let $v_1, \ldots, v_n \in V$ and $w_1, \ldots, w_n \in W$ so the following tow conditions hold
    (a) $\text{Span}\{v_1, \ldots, v_n\} = V$ (but we do not assume $\{v_1, \ldots, v_n\}$ is linearly independent),
    (b) For any scalars $c_1, \ldots, c_n \in \mathbb{F}$
    $$\sum_{k=1}^n c_k v_k = 0 \quad \text{implies} \quad \sum_{k=1}^n c_k w_k = 0.$$

   Show that there is a unique linear map $T: V \to W$ so that $Tv_k = w_k$ for $1 \leq k \leq n$. 