1. (20 points) Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear map 
   \[ T(x, y) = (2x - 3y, x - 2y). \]
   (a) Find the matrix of $T$ in the standard basis.
   (b) Find the matrix of $T$ in the ordered basis $(0, 3), (1, -1)$.
   (c) Find the inverse of $T$. (The answer should be expressed in
       the same form as $T$ and not as a matrix.)
   (d) Find a basis for both the range and null space of the linear
       transformation $T - I$.
   (e) Find $T^2$.
   (f) Find $T^{1995}$

2. (10 points) Give an example of linear maps $T : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ and
   $S : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ so that $ST$ is invertible.

3. (10 points) Let $T : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ and $S : \mathbb{R}^2 \rightarrow \mathbb{R}^4$ be linear maps. Show
   $\text{nullity}(ST) \geq 2$.

4. (10 points) If $u, v, w$ are linearly independent vectors in the
   vector space $V$ show that $u, u + v$ and $u + v + w$ are also linearly
   independent.

5. (10 points) Let $v_1 = (1, 0, 0), v_2 = (2, 1, 0), v_3 = (3, 2, 1)$. Then
   $v_1, v_2, v_3$ is a basis of $\mathbb{R}^3$. Find the basis of $\mathbb{R}^3^*$
   dual to $v_1, v_2, v_3$.

6. (10 points) Let $U$ and $W$ be subspaces of the finite dimensional
   vector space $V$. Assume that $\dim U + \dim W = \dim V$ and $U + W = V$. Then show $V = U \oplus W$.

7. (10 points) Let $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$ be the complex numbers. Then $\mathbb{C}$ is a two dimensional
   vector space over the real numbers $\mathbb{R}$. Let $T : \mathbb{C} \rightarrow \mathbb{C}$ be multiplication by $a + bi$, that is
   $Tz = (a + bi)z$ Then find the matrix of $T$ in basis $1, i$ of $\mathbb{C}$.

8. (10 points) Find explicitly a linear map $S : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ so that
   $S(2, 3) = (1, 2, 3)$ and $S(1, 2) = (4, 5, 6)$.

9. (10 points) Gives examples of
   (a) Matrices $A$ and $B$ with $AB \neq BA$.
   (b) A matrix with $A^3 = 0$ but $A^2 \neq 0$. 


Mathematics 700 Test 1

This is to be done in three hours in one setting. I would prefer that it is closed book, but if you feel that you have to look up something write me a note like “I looked up the definition of rank and used it in problems numbers 2 and 5”. I will then take a little off on these problems, say 20%. This is due in class next Tuesday. Good luck.