1. (10 points) State and prove the rank plus nullity theorem.

2. (20 points) Let $\mathcal{P}_3$ be the vector space of real polynomials of degree $\leq 3$. Define a linear map $T : \mathcal{P}_3 \to \mathcal{P}_3$ by
   \[(Tf)(x) = x(f(x + 1) - f(x))\]
   (a) Find the null space of $T$.
   (b) Find the rational canonical form of $T$.
   (c) Find the Jordan canonical form of $T$.
   (d) Find the range of $T$.

3. (10 points) Let $\mathbb{R}^3^*$ be the dual space to $\mathbb{R}^3$. Then find the basis of $\mathbb{R}^3^*$ dual to the basis $(1, 0, 0), (1, 1, 0), (0, 1, 1)$ of $\mathbb{R}^3$.

4. (10 points) Let $V$ be a finite dimensional vector space over the a field $\mathbf{F}$ and let $T \in L(V, V)$. Then state what the primary decomposition of $V$ with respect to $T$ is.

5. (10 points) Let $P \in L(V, V)$ where $V$ is a finite dimensional vector space. Assume that $P^2 = P$ and show that $\text{rank}(P) = \text{trace}(P)$.

6. (10 points) Let $A = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}$. Then find a basis of $\mathbb{R}^2$ that makes $A$ diagonal or show that no such basis exists.

7. (10 points) Let $V$ be a finite dimensional vector space and $S$ a linear operator on $V$. Let $W \subseteq V$ be a subspace that is invariant under $S$. Choose a basis $w_1, \ldots, w_k$ of $W$ and extend it to a basis $w_1, \ldots, w_k, v_{k+1}, \ldots, v_n$ of $V$. Let $A$ be the matrix of $V$ in this basis. Then what can you say about the form of $A$?

8. (5 points) Let $f(x) = x^3 + ax^2 + bx + c$. Then find a $3 \times 3$ matrix that has $f(x)$ as a minimal polynomial.

9. (5 points) Give an of a $5 \times 5$ matrix $A$ with minimal polynomial $(x - 1)^2(x - 2)(x - 3)$ and $\text{det} A = 12$.

10. (5 points) Let $A$ be a $2 \times 2$ matrix with trace $A = 0$. Then show $A^2 = -\text{det}(A)I$.

11. (5 points) Let $U, V \subset \mathbb{R}^7$ be subspaces with dim $U = 5$ and dim $V = 6$. Then what can you say about dim$(U \cap V)$?