Catalan Numbers and Grouping with Parenthesis.

Here is the correct version of how many ways to group n factors with parenthesis. Call this number P_n. We set

$$P_1 = 1$$

just because it makes things work out nicely (rather like setting $0! = 1$). $P_2 = 1$ as there is only one way to do the grouping:

$$(ab).$$

$P_3 = 2$ as there are two groupings:

$$(ab)c, \quad a(bc).$$

$P_4 = 5$ as there are 5 groupings:

$$a(b(cd)), \quad a((bc)d), \quad (ab)(cd), \quad (a(bc))d, \quad ((ab)c)d.$$

$P_5 = 14$. This can be seen as without writing all the groupings down by noting that if we have a product of 5 factors *abcde* then it will be one of the forms

$$w_1(w_4), \quad (w_2)(w_3), \quad (w_3)(w_2), \quad (w_4)w_1$$

where w_j will be an a product of j factors. Then there are $P_1 \times P_4 = 1 \times 5 = 5$ groupings of the form $w_1(w_4)$, there are $P_2 \times P_3 = 1 \times 2 = 2$ of the form $(w_2)(w_3)$, $P_3 \times P_2 = 2 \times 1 = 2$ of the form $(w_3)(w_2)$, and $P_4 \times P_1 = 5 \times 1 = 5$ of the form $(w_4)w_1$. Adding these up gives a total of

$$P_5 = P_1P_4 + P_2P_3 + P_3P_2 + P_4P_1 = 5 + 2 + 2 + 5 = 14.$$

This same argument shows that

$$\begin{aligned} P_6 &= P_1P_5 + P_2P_4 + P_3P_3 + P_4P_2 + P_5P_1 \\ &= (1)(14) + (1)(5) + (2)(2) + (5)(1) + (14)(1) = 42 \end{aligned}$$

and in general

$$P_n = P_1P_{n-1} + P_2P_{n-2} + \cdots + P_{n-1}P_1 = \sum_{k=1}^{n-1} P_kP_{n-k} \tag{1}$$

which can be used to compute P_n once we have computed $P_1, P_2, \ldots, P_{n-1}$. The first several are

n	2	3	4	5	6	7	8	9	10	11	12	13
P_n	1	2	5	14	42	132	429	1,430	4,862	16,796	58,786	208,012

The recursive formula (1) and induction can be used to show

$$P_n = \frac{1}{n}\binom{2(n-1)}{n-1}.$$

However this is not a trivial proof.

The n-th Catalan number is defined to be

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

and therefore the formula I should have given in class is:

$$P_n = C_{n-1}.$$