# Review for Test 3.

*First of all go over all the homework to be sure you understand it.*

The most recent topics we have covered is Laurent series, residues, and the Residue Theorem. Here is a list of definitions and theorems that *will* be on the

**Definition.** $f(z)$ has an ***isolated singularity*** at $z = z_0$ iff for some $r > 0$ it is analytic in the punctured disk $D = \{z : 0 < |z - z_0| < r\}$. $\square$

**Theorem** (Existence of Laurent expansion)**.** *If $f(z)$ has an isolated singularity at $z = z_0$ then in some punctured disk $D = \{z : 0 < |z - z_0| < r\}$ there is a convergent **Laurent expansion***

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

*for $f(z)$.* $\square$

**Definition.** If $f(z)$ has an isolated singularity at $z = z_0$ with Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

then the ***residue*** of $f(z)$ at $z = z_0$ is

$$\operatorname{Res}(f(z), z_0) = a_{-1}.$$

$\square$

This is important because of

**Theorem** (Residue Theorem)**.** *Let $\gamma$ be a simple closed curve and $f(z)$ a function that is analytic inside of $\gamma$ except for a finite number of singularities inside of $\gamma$. Then*

$$\int_\gamma f(z)\, dz = 2\pi i \Big( \textit{Sum of the residues of } f(z) \textit{ inside of } \gamma. \Big)$$

$\square$

That is if $z_1, \ldots, z_k$ are the residues of $f(z)$ inside of $f(z)$ then

$$\int_\gamma f(z)\, dz = 2\pi i \Big( \operatorname{Res}(f(z), z_1) + \cdots + \operatorname{Res}(f(z), z_1) \Big).$$

Here is a method for computing residues at simple poles.

**Proposition.** *Let*

$$f(z) = \frac{g(z)}{h(z)}$$

*where $g(z)$ and $h(z)$ are analytic, and $h(z_0) = 0$, but $h'(z_0) \neq 0$. Then $f(z)$ is has a simple pole at $z = z_0$ and*

$$\operatorname{Res}(f(z), z_0) = \frac{g(z_0)}{h'(z_0)}.$$

**Problem**. Use the last result to find the following residues.

(a) $f(z) = \dfrac{e^{z^2+1}}{\sin(2z)}$ at $z = 0$ and $z = \pi$. *Answer:* Thinking of $f(z) = \dfrac{g(z)}{h(z)}$ we have

$$\mathrm{Res}(f(z), 0) = \frac{g(0)}{h'(0)} = \frac{e^{0^2+1}}{2\cos(2\cdot 0)} = \frac{e}{2}.$$

and

$$\mathrm{Res}(f(z), \pi) = \frac{g(\pi)}{h'(pi)} = \frac{e^{\pi^2+1}}{2\cos(2\pi)} = \frac{e^{\pi^2+1}}{2}$$

(b) $f(z) = \dfrac{e^z}{z^2+4}$ at its poles. *Answer:* The poles are at $z = 2i$ and $z = -2i$. The residues are

$$\mathrm{Res}(f(z), 2i) = \frac{g(2i)}{h'(2i)} = \frac{e^{2i}}{4i}, \qquad \mathrm{Res}(f(z), -2i) = \frac{g(-2i)}{h'(-2i)} = \frac{e^{-2i}}{-4i}$$

(c) Use your answer to (b) to compute $\displaystyle\int_{|z-3i|=2} \frac{e^z}{z^2+4}\,dz$. *Answer:*

$$\int_{|z-3i|=2} \frac{e^z}{z^2+4}\,dz = 2\pi i\,\mathrm{Res}(f(z), 2i) = \frac{e^{2i}}{2}.$$

(d) Use your answer to (b) to compute $\displaystyle\int_{|z-i|=10} \frac{e^z}{z^2+4}\,dz$.

$$\int_{|z-i|=10} \frac{e^z}{z^2+4}\,dz = 2\pi i\Big(\mathrm{Res}(f(z), 2i) + \mathrm{Res}(f(z), -2i)\Big) = \frac{e^{2i}}{2} + \frac{e^{-2i}}{-2} = i\sin(2).$$

The last homework assignment covered most of what you would be expected to know about residues and the residue theorem.

Other important results you should know are

**Theorem** (Mean value property)**.** *Let $f(z)$ be analytic on and inside the circle $|z - z_0| = r$. Then*

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{it})\,dt.$$

□

In old fashion terminology the "mean value" is what we would now call the "average value". The content of the mean value property is that the average value of an analytic function over a circle is the same as the value of the function at the center of the circle. Even after having first seen this result over 35 years ago, I still find it remarkable.

For practice look at Problem 5.52 on page 164 of the text.

We used the mean value property to prove

**Theorem** (maximum modulus principle)**.** *Let $f(z)$ be analytic in a connected domain $D$. Assume that there is a point $z_0 \in D$ where $f(z)$ has its maximum (i.e. $|f(z)| \leq |f(z_0)|$ for all $z \in D$). Then $f(z)$ is constant.* □

Another form of this is

**Theorem** (maximum modulus principle, boundary)**.** *Let $f(z)$ be analytic on and inside a simple closed curve $\gamma$. Then the maximum of $|f(z)|$ occurs on $\gamma$.*

**Definition.** The function $f(z)$ is ***entire*** iff it is analytic on all of $\mathbf{C}$. □

**Definition.** A function, $f(z)$, is bounded iff there is a constant $M$ so that $|f(z)| \leq M$ on all of the domain of $f(z)$. □

**Theorem** (Liouville's Theorem)**.** *A bounded entire function is constant.* □

You should be able to use Liouville's theorem to prove results such as

**Problem**. Let $f(z)$ be an entire functions such that $|f(z) - i| \geq 3$ for all $z$. Then $f(z)$ is constant. *Hint:* Let $g(z) = 1/(f(z) - i)$ and show that $g(z)$ is a bounded entire function. □

Also look at Problem 5.48 page 164 of the text.

Our main application of Liouville's theorem was to showing that polynomials have roots.

**Theorem** (Fundamental Theorem of Algebra)**.** *Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$. Then there is a complex number $r$ with $p(r) = 0$.*

That is every non-constant polynomial has at least one complex root.