Homework assigned Wednesday, March 28.

Recall that the highlight of our lives in the last couple of weeks (at least as regards to what is happening in this class) is the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{\partial D}\frac{f(w)\,dz}{w-z}$$

where D is a bounded domain with nice boundary, f is analytic on the closure of D, and $z \in D$. We saw in the proof of Louisville's that being able to choose the domain D to suit our needs was useful. We now continue this theme. Define $D(z,r)$ to be the open disk of radius r centered at z. Explicitly

$$D(z,r) = \{w : |w-z| = r\}.$$

Then $\partial D(z,r)$ is the circle in the w plane defined by $|w-z| = r$.

Definition 1. Let f be the continuous on the circle $|w-z| = r$. Then the ***average value*** of f on this circle is

$$\text{Average of } f \text{ on } \partial D(z,r) = \frac{1}{2\pi r}\int_{|w-z|=r} f(w)\,|dw|.$$

(Recall that the integral with respect to $|dw|$ is just the integral with respect to arclength along $\partial D(z,r)$.)

Problem 1. Show that if $f(z) = c$ is a constant then for any disk

$$\text{Average of } f \text{ on } \partial D(z,r) = c.$$

Problem 2. Use the parametrization $w = z + re^{it}$ with $0 \le t \le 2\pi$ to show

$$\text{Average of } f \text{ on } \partial D(z,r) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})\,dt.$$

Problem 3. Let f be analytic in the closure of the disk of radius r about z. Then from the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_{|w-z|=r}\frac{f(w)\,dw}{w-z}.$$

Use the parametrization of the last problem in this integral to show

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})\,dt$$

and thus the value $f(z)$ is the average of f on the circle of radius r about z.

Here is a version of what we talked about in class today.

Proposition 2. *Let $f(z)$ be an entire function that omits some disk $D(a,r)$. Then $f(z)$ is constant.*

Problem 4. Prove this along the following lines

(a) Show that $f(z)$ omitting the disk $D(a,r)$ implies the inequality $|f(z)-a| \geq r$.

(b) Let $g(z) = \dfrac{1}{f(z)-a}$ and show that $g(z)$ is a bounded entire function.

(c) Use Louisville's theorem to conclude that $g(z)$, and therefore $f(z)$, is constant.