

Homework assigned Wednesday, February 29.

The following problems are to prepare for results that we are about to prove.

Problem 1. Let $f(z)$ be analytic in a domain D and let $a \in K$. Let $r > 0$ be so small that the disk $|z - a| \leq r$ is contained in D .

- (a) Parametrize $|z - a| = r$ by $z = a + re^{it}$ with $0 \leq t \leq 2\pi$. Use this reparameterizing to show that

$$\int_{|z-a|=r} \frac{f(z)}{z-a} dz = i \int_0^{2\pi} f(a + re^{it}) dt.$$

- (b) Use part (a) to show

$$\lim_{r \rightarrow 0^+} \int_{|z-a|=r} \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

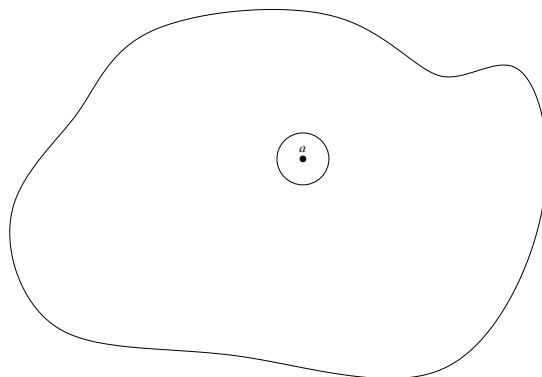


FIGURE 1

In the next problem we will be using Cauchy's theorem, which we now recall.

Theorem 1 (Cauchy's Theorem). *Let D be a bounded domain with nice boundary and $f(z)$ a function that is analytic on the closure of D . Then*

$$\int_{\partial D} f(z) dz = 0$$

where, as usual, we orient ∂D so as we move with the inside on our left. □

Problem 2. In Figure 1 we have a bounded domain with nice boundary and a point a inside. Let $f(z)$ be a function that is analytic on the closure of D . Let r be a small positive number and D_r the domain D with the inside of the circle $|z - a| = r$ removed. That is D_r is the region inside of D and outside of $|z - a| = r$.

- (a) Explain why

$$\int_{\partial D_r} \frac{f(z)}{z-a} dz = 0.$$

Hint: The function $g(z) = \frac{f(z)}{z-a}$ is analytic in D_r .

- (b) The boundary of ∂D_r has two pieces. First there is the boundary, ∂D , of the original domain and second there is the circle $|z - a| = r$. Thus

$$\int_{\partial D_r} \frac{f(z)}{z-a} dz = \int_{\partial D} \frac{f(z)}{z-a} dz - \int_{|z-a|=r} \frac{f(z)}{z-a} dz.$$

Explain why the sign on the second integral is negative. *Hint:* We always move along the boundary with the inside on our left.

(c) Combine parts (a) and (b) to conclude

$$\int_{\partial D} \frac{f(z)}{z-a} dz = \int_{|z-a|=r} \frac{f(z)}{z-a} dz.$$

(d) In the last equation take the limit as r goes to 0 and part (b) of Problem 1 to conclude

$$\int_{\partial D} \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

We have thus proven the following, which is maybe the most important result in complex analysis.

Theorem 2 (Cauchy Integral Formula). *Let D be a bounded domain with nice boundary and $f(z)$ be analytic on the closure of D . Then for any point $a \in D$*

$$f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-a} dz.$$

□

Example. Let γ be the path in Figure 2.

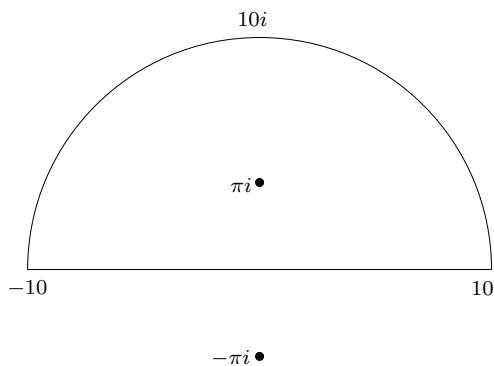


FIGURE 2

We now use the Cauchy Integral formula to evaluate

$$\int_{\gamma} \frac{e^z}{z^2 + \pi^2} dz.$$

This function is analytic except where the denominator becomes zero. That is where $z^2 + \pi^2 = 0$. Note that $z^2 + \pi^2 = (z - \pi i)(z + \pi i)$. So that the bad points are $z = \pi i$ and $z = -\pi i$. Thus our integral becomes

$$\int_{\gamma} \frac{e^z}{(z - \pi i)(z + \pi i)} dz.$$

We only need to work about the point πi as it is the only non-analytic point inside of γ . Rewrite the integral as

$$\int_{\gamma} \frac{e^z/(z + \pi i)}{(z - \pi i)} dz = \int_{\gamma} \frac{f(z)}{(z - \pi i)} dz$$

where

$$f(z) = \frac{e^z}{z + \pi i}.$$

The function $f(z)$ is analytic inside of γ . So by the Cauchy integral formula

$$\int_{\gamma} \frac{e^z}{z^2 + \pi^2} dz = \int_{\gamma} \frac{f(z)}{(z - \pi i)} dz = 2\pi i f(\pi i) = 2\pi i \frac{e^{\pi i}}{\pi i + \pi i} = e^{\pi i} = -1.$$

□

Problem 3. Let z_1 be a complex number and γ a simple closed curve that does not pass through z_1 . Show

$$\int_{\gamma} \frac{dz}{z - z_1} = \begin{cases} 2\pi i, & \text{if } z_1 \text{ is inside of } \gamma, \\ 0, & \text{if } z_1 \text{ is outside of } \gamma. \end{cases}$$

Hint: Use part (d) of Problem 2, or the Cauchy Integral Formula, with $f(z) = 1$, D the region inside of γ , and $z = z_1$.



FIGURE 3

Problem 4. Figure 3 shows the points i , $-i$, 0 , and 4 along with three paths α , β , and γ . Use either part (d) or Problem 2 or the Cauchy integral formula to

- (a) Evaluate $\int_{\alpha} \frac{2z + 1}{z(z - 4)(z^2 + 1)} dz$,
- (b) Evaluate $\int_{\beta} \frac{2z + 1}{z(z - 4)(z^2 + 1)} dz$, and
- (c) Evaluate $\int_{\gamma} \frac{2z + 1}{z(z - 4)(z^2 + 1)} dz$.