Homework assigned Wednesday, February 22.

We saw in class today that the following holds.

Theorem 1. *Let U be an open set in $\mathbf{C}$ and γ a path contained in U. Let $f(z)$ be a function defined on U that has an antiderivative $F(z)$. (That is $F(z)$ is analytic in U and $F'(z) = f(z)$.) Then*

$$\int_\gamma f(z)\, dz = F(\gamma_{\text{end}}) - F(\gamma_{\text{begin}}).$$

Problem 1. Use this to evaluate the following integrals.

(a) $\displaystyle\int_\gamma 12z^3\, dz$ where γ is parametrized by $z(t) = (1+5t-t^3)+i(6-t^4+3t^6)$; $0 \le t \le 1$.

(b) $\displaystyle\int_\alpha ze^{z^2}\, dz$ where α is the segment going from $1+i$ to $-i$.

(c) $\displaystyle\int_\beta \frac{dz}{z^2}$ where β is the upper half of the circle $|z| = 1$.

Definition 2. A path is ***closed*** iff it begins and ends at the same point. (It is allowed to cross itself.) See Figure 1 for examples of closed paths and Figure 2 for examples of non-closed paths.

Remark 3. Another way to say that a curve is closed is that $\gamma_{\text{end}} = \gamma_{\text{begin}}$.

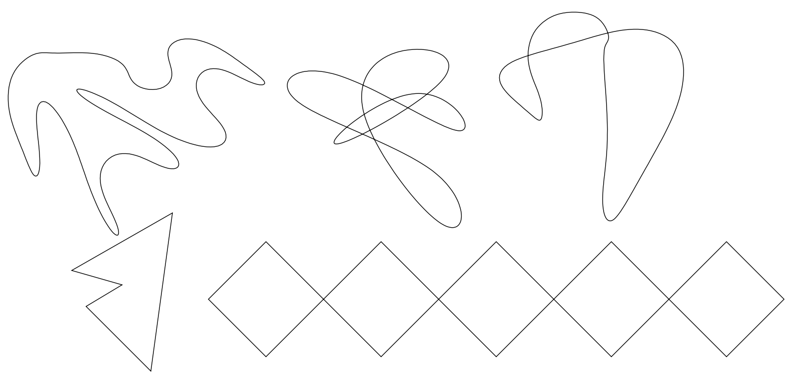

FIGURE 1. Examples of closed curves.

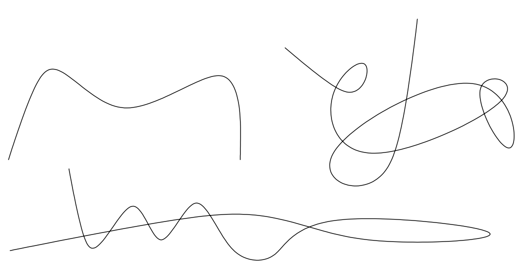

FIGURE 2. Examples of non-closed curves.

Theorem 4. *Let U be an open set, f a function that has an antiderivative, F, in U and γ a closed curve in U. Then*

$$\int_\gamma f(z)\, dz = 0.$$

Problem 2. Prove this. *Hint:* It is not hard.

Note the last theorem tells use that if a function has an antiderivative, then its integral over every closed curve is zero. The contrapositive to this is

Corollary 5. *If there is a closed curve, γ, such that*

$$\int_\gamma f(z)\,dz \neq 0$$

then $f(z)$ does not have an antiderivative in any open set containing γ.

Problem 3. Show that $f(z) = \overline{z}$ does not have an antiderivative. *Hint:* Compute $\int_\gamma f(z)\,dz$ where γ is the unit circle $|z| = 1$ traversed once in the counter-clock wise direction.

Problem 4. Show that $f(z) = \dfrac{1}{z}$ does not have an antidarivative in the set $U = \{z : z \neq 0\}$. *Hint:* Compute $\int_\gamma f(z)\,dz$ where γ is the unit circle $|z| = 1$ traversed once in the counter-clock wise direction.

Problem 5. In the last problem we have seen that $f(z) = \dfrac{1}{z}$ does not have an antiderivative, but we learned a couple of weeks ago that if $F(z) = \log(z)$ that $F'(z) = \dfrac{1}{z} = f(z)$ which makes it look like $f(z)$ does have an antiderivative. Resolve this paradox.