Homework assigned Friday, January 13.

We will be using Green's theorem.

Theorem 1 (Green's Theorem)**.** *Let D be a bounded domain in $\mathbf{C}$ with a nice boundary ∂D. Then if $P(x, y)$ and $Q(x, y)$ are functions on the closure of D that have continuous partial derivatives then*

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D (-P_y + Q_x)\,dx\,dy.$$

□

Problem 1 (*To be collected in the form of a quiz next week*). Have the statement of Green's theorem memorized. □

We now want to understand why Green's Theorem is true. Recall that the fundamental theorem of calculus is that for a continuously differentiable function

$$f(b) - f(a) = \int_a^b f'(x)\,dx.$$

Problem 2*.* Use this to show that if $Q(x, y)$ has continuous partial derivatives that

$$Q(b, y) - Q(a, y) = \int_a^b Q_x(x, y)\,dx.$$

Hint: View y as constant and let $f(x) = Q(x, y)$. Then $f'(x) = Q_x(x, y)$. □

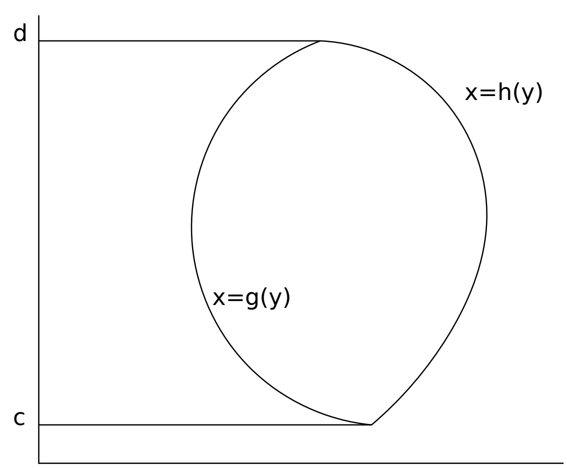

FIGURE 1

We now show half of Green's theorem: $\int_{\partial D} Q\,dy = \iint_D Q_x\,dx\,dy$ (we did the other half in class).

Problem 3*.* Let D be the region bounded by the curves $x = g(y)$ and $x = h(y)$ as in Figure 1.

(a) Show that

$$\int_{\partial D} Q\,dy = \int_c^d (Q(h(y),y) - Q(g(y),y))\,dy.$$

(b) Apply Problem 2 with $a = g(y)$ and $b = h(y)$ to the integrand in part (a) to conclude that

$$\int_{\partial D} Q\,dy = \int_c^d \int_{g(y)}^{h(y)} Q_(x,y)\,dx\,dy = \iint_D Q_x\,dx\,dy.$$

which finishes the proof. □

Problem 4. This is a bit of practice in the use of Green's theorem. Let D be a bounded region with nice boundary.

(a) Letting $z = x + iy$ so that $dz = dx + idz$ show that

$$\int_{\partial D} \overline{z}\,dz = \int_{\partial D} (x\,dx + y\,dy) + i\int_{\partial D} (-y\,dx + x\,dy)$$

(b) Now use Green's theorem to conclude that

$$\int_{\partial D} \overline{z}\,dz = 2i\,\mathrm{Area}(D).$$

□

One the most important results in complex analysis is the following:

Theorem 2 (Cauchy's Theorem)**.** *Let D be a bounded domain with nice boundary. Then if $f(z)$ is analytic on the closure of D*

$$\int_{\partial D} f(z)\,dz = 0.$$

Problem 5. Prove this. *Hint:* We did this in class. Here is the basic idea, let $f(z) = u+iv$ and $dz = dx+idy$. Then multiply out $f(z)\,dz$ in $\int_{\partial D} f(z)\,dz$ in terms of u, v, dx, dy, separate into real and imaginary parts, use Green's theorem and the Cauchy-Riemann equations. □

Remark 3. I expect you to know the proof of the Cauchy theorem from memory. □