Mathematics 552 Test #3

Show your work! Answers that do not have a justification will receive no credit. Also put your answers in some standard “human” form such as \( x + iy \) or polar form, and not some specialized form used by your calculator.

(1) (20 points)
   (a) Define entire function.

(b) State Liouville’s Theorem.

(c) State the fundamental theorem of algebra.

(d) Define what it means for \( z_0 \) to be an isolated singularity of \( f(z) \).

(e) State the interior maximum modulus principle.

(f) If \( z_0 \) is an isolated singularity of \( f(z) \), then define the residue of \( f(z) \) at \( z_0 \).

(g) State the Residue Theorem.
(2) (20 points) Let $f(z)$ have an isolated singularity at $z_0$.
(a) State theorem on the existence of Laurent expansions of $f(z)$ about $z_0$.

(b) Define what it means for $z_0$ to be a removable singularity of $f(z)$.

(c) Define what it means for $z_0$ to be a pole of $f(z)$.

(d) Define what it means for $z_0$ to be an essential singularity of $f(z)$.

(e) State the theorem that characterizes when $z_0$ is a removable singularity of $f(z)$.

(f) State the theorem that characterizes when $z_0$ is a pole of $f(z)$.

(g) State the structure theorem for poles of order $k$. 
(3) (15 points)
  (a) State the mean value property theorem for analytic functions.

(b) Derive mean value property theorem for analytic functions from the Cauchy integral formula.
(4) (15 points)
(a) Define what it means for $u(z) = u(x, y)$ to be harmonic in a domain $D$.

(b) Show that if $u$ is harmonic then $f(z) = u_x - iu_y$ is analytic in $D$. **HINT:** Cauchy-Riemannian equations.
(5) (10 points) Show that if \( f(z) \) has a simple pole at \( z_0 \), then \( g(z) = (z - z_0)f(z) \) has removable singularity at \( z_0 \). HINT: Structure theorem for poles of order \( k \) (in this case \( k = 1 \)).

(6) (5 points) Show that if \( f(z) \) is an entire function such that \( |f(z) - 3i| \geq 2 \) for all \( z \), then \( f(z) \) is constant.
(7) (10 points) Let $D$ be the unit disk $\{z : |z| < 1\}$ and let $u$ be a harmonic function on $D$ such that $u(z) = x^2 - y^2$ for all $z \in \partial D$ where $z = x + iy$. Show that $u(z) = x^2 - y^2$ for all $z \in D$. HINT: Is $v(z) = x^2 - y^2$ harmonic?