The most basic functions in analysis other than polynomials are the functions $e^z$, $\cos(z)$, $\sin(z)$. We consider $e^z$ the most basic of these and have defined $\cos(z)$ and $\sin(z)$ by

\[
\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}.
\]

We also know that

(1) \[ \frac{d}{dz} e^{az} = ae^{az}, \quad e^{a+b} = e^a e^b. \]

In this problems you will show this the basic properties of the trigonometric functions.

(1) Use (1) to show

\[
\frac{d}{dz} \cos(z) = -\sin(z), \quad \frac{d}{dz} \sin(z) = \cos(z).
\]

(2) Use (1) to show

\[
\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b), \quad \sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b).
\]

(3) Use (1) to show

\[
\cos^2(z) + \sin^2(z) = 1.
\]

It is convenient to introduce the hyperbolic functions

\[
\cosh(z) = \frac{e^z + e^{-z}}{2}, \quad \sinh(z) = \frac{e^z - e^{-z}}{2}.
\]

(4) Show that

\[
\cosh^2(z) - \sinh^2(z) = 1.
\]

(5) Show that

\[
\frac{d}{dz} \cosh(z) = \sinh(z), \quad \frac{d}{dz} \sinh(z) = \cosh(z).
\]

(6) Show that

\[
\sin(i z) = i \sinh(z), \quad \cos(i z) = \cosh(z).
\]

(7) Let $z = x + iy$ as usual. Then use the equations (2) and (3) to expand $\cos(x + iy)$ and $\sin(x + iy)$ to find the real and imaginary parts of of $\cos(z)$ and $\sin(z)$. That is write each of $\cos(z)$ and $\sin(z)$ in the form $u(x, y) + iv(x, y)$ where $u(x, y)$ and $v(x, y)$ are real valued.

(8) Use your solution to the last problem to find formulas for $|\cos(z)|^2$ and $|\sin(z)|^2$.

(9) Here is a fun, but more challenging, problem. Find a formula for the sum

\[
\sum_{k=0}^{n} \cos(kz) = 1 + \cos(z) + \cos(2z) + \cdots + \cos(nz).
\]

**HINT:** One method is to use (1) and the sum formula for a finite geometric series.