Here are some fun and games with series. First let
\[
    f(z) = \sum_{k=0}^{\infty} a_k z^k
\]
and assume that this series has a positive radius of convergence \( R \). Then we have seen that \( f(z) \) is analytic in the disk \( |z| < R \) and that its derivative is given by
\[
    f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} = \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k
\]
and that this series also has radius of convergence \( R \). Therefore \( f'(z) \) is also analytic in the disk \( |z| < R \) and so we can take its derivative to get
\[
    f''(z) = \sum_{k=0}^{\infty} k(k-1) a_k z^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} z^k.
\]

**Problem 1.** Show, for example by using mathematical induction, that we can continue in this manner and that the \( n \)-th derivative \( f^{(n)}(z) \) of \( f(z) \) exists in the disk \( |z| < R \) and is given by the series
\[
    f^{(n)}(z) = \sum_{k=0}^{\infty} k(k-1)(k-2) \ldots (k-(n-1)) a_k z^{k-n} = \sum_{k=n}^{\infty} (k-n)(k-n-1)(k-n-2) \ldots (k+1) a_{k-n} z^k.
\]

**Problem 2.** If \( f(z) \) is given by the series (1) we can let \( z = 0 \) to find that
\[
    f(0) = a_0.
\]
Therefore \( a_0 = f(0) \). Letting \( z = 0 \) in the formula (2) for \( f'(z) \) gives
\[
    f'(0) = a_1.
\]
Which gives \( a_1 = f'(0) \). Letting \( z = 0 \) in the formula (3) for \( f''(z) \) gives
\[
    f''(0) = 2a_2.
\]
Giving \( a_2 = \frac{f''(0)}{2} \). Now let \( z = 0 \) in (4) to find a formula for \( a_n \) in terms of \( f^{(n)}(0) \). **Remark:** If you want to check your formula, you can find the it in the powers series section of your calculus book, or in our text.

**Problem 3.** Use your solution to problem 2 to find series for the following
(a) \( (1+z)^{-2} \). That is in the expansion
\[
    (1+z)^{-2} = \sum_{k=0}^{\infty} a_k z^k
\]
find a formula for \( a_k \).
(b) It is known that \( \sqrt{1+z} = (1+z)^{1/2} \) is analytic in the disk \( |z| < 1 \). Find the series expansion for \( \sqrt{1+z} \).
More generally let \( \alpha \) be a complex number. We will see later that we can define \( f(z) = (1 + z)^\alpha \) in such a way that it has a power series \( (1 + z)^\alpha = \sum_{k=0}^{\infty} a_k z^k \) that converges for \( |z| < 1 \). Find coefficients \( a_k \), in this expansion. REMARK: If you do this problem first, then you get the solutions to (a) and (b) let letting \( \alpha = -2 \) and \( \alpha = 1/2 \) respectively.

**Problem 4.** Use multiplication and long division of series to find the first four (that is up to degree three terms) of the series

(a) \( \frac{\sin(z)}{z} \).
(b) \( \tan(z) = \frac{\sin(z)}{\cos(z)} \).
(c) \( \frac{e^{2z} - 1}{z} \).
(d) \( (1 + z)e^{2z} \).
(e) \( \frac{e^{3z}}{1 - 2z} \).

Some extra credit.

Assume that the series

\[
f(z) = \sum_{k=0}^{\infty} a_k z^k, \quad g(z) = \sum_{j=0}^{\infty} b_j z^j
\]

converge in the disk \( |z| < R \). Let \( p(z) \) be the series obtained by multiplying this together formally. That is

\[
p(z) = f(z)g(z) = \left( \sum_{k=0}^{\infty} a_k z^k \right) \left( \sum_{j=0}^{\infty} b_j z^j \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) z^k = \sum_{n=0}^{\infty} c_n z^n
\]

where

\[
c_n = \sum_{k=0}^{\infty} a_k b_{n-k}.
\]

We would like to show that the series for \( p(z) \) also converges for \( |z| < R \). So let \( |z| < R \) and choose an \( r \) with \( |z| < r < R \). Then the series for \( f(r) \) and \( g(r) \) both converge.

**EC 1.** Explain why this implies there is a constant \( M > 0 \) such that \( |a_k r^k|, |b_k r^k| \leq M \). This implies that

\[
|a_k|, |b_k| \leq \frac{M}{r^k}.
\]

**EC 2.** Use your solution to EC 1 to show that

\[
|c_n| \leq (n + 1) \frac{M^2}{r^n},
\]

**EC 3.** Thus if \( \rho = \frac{|z|}{r} \), which satisfies \( \rho < 1 \) as \( |z| < r \), then use EC 2 to show that

\[
|p(z)| \leq \sum_{n=0}^{\infty} |c_n z^n| \leq \sum_{n=0}^{\infty} (n + 1) M^2 \rho^n = M^2 \sum_{n=0}^{\infty} (n + 1) \rho^n
\]

and that the series \( \sum_{n=0}^{\infty} (n + 1) \rho^n \) converges. This shows that the series for \( p(z) \) is absolutely convergent and completes the proof.