Mathematics 552 Homework due Wednesday, February 1, 2006

(1) Assuming that you can differentiate term by term find the sum of

\[ S = 1 + 2r + 3r^2 + 4r^3 + \cdots = \sum_{n=1}^{\infty} nr^{n-1} \]

by taking the derivative of the series

\[ 1 + r + r^2 + r^3 + r^4 + \cdots = \sum_{n=0}^{\infty} r^n = \frac{1}{1 - r} \]

(2) We know that

\[ S = 1 + r + r^2 + \cdots + r^n = \sum_{k=0}^{n} r^k = \frac{1}{1 - r} - \frac{r^{n+1}}{1 - r} \]

(a) Take the derivative of this to get a formula for the sum

\[ S' = 1 + 2r + 3r^2 + \cdots + nr^{n-1} = \sum_{k=0}^{n} kr^{k-1}. \]

(b) Take another derivative to get a formula for the sum

\[ S'' = 2r + 2 \cdot 3r^2 + +3 \cdot 4r^2 \cdots + (n-1)nr^{n-2} = \sum_{k=0}^{n} (k-1)kr^{k-2}. \]

(c) Use this formulas, or any other method you like, to show that if \(|r| < 1\) the sums

\[ \sum_{k=0}^{\infty} kr^{k-1} \quad \text{and} \quad \sum_{k=0}^{\infty} (k-1)kr^{k-2}. \]

both converge.

Quiz on Wednesday: Have the follow series memorized. The binomial expansion:

\[ (a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}. \]

The series for \(e^x\)

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \]

The series for a finite geometric series

\[ a + ar + ar^2 + \cdots + ar^n = \sum_{k=0}^{n} ar^k = \frac{a - ar^{n+1}}{1 - r}. \]

The series for \(\sin(x)\) and \(\cos(x)\).

\[ \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}. \]
\[ \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}. \]