Mathematics 552 Homework due Wednesday, April 12, 2006.

There will be a quiz on Wednesday. Here is what will be on it (view this in part as practice for the test next Monday). Know the following statements.

1. **Existence of Laurent expansions:** If $0 \leq r < |z - z_0| < R \leq \infty$ and the function $f(z)$ is analytic in the annulus $A = \{ z : r < |z - z_0| < R \}$ then $f(z)$ has a convergent Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

in $A$.

2. **Definition of singularity:** The function $f(z)$ has a singularity at $z_0$ iff for some $R > 0$ we have that $f(z)$ is analytic in the punctured disk $A = \{ z : 0 < |z - z_0| < R \}$. (We will often also that $f(z)$ has an an isolated singularity at $z_0$.)

3. **Existence of Laurent expansions about isolated singularities.** If $f(z)$ has an isolated singularity at $z_0$ then for some $R > 0$ there is a convergent Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

which holds for $0 < |z - z_0| < R$.

4. **Classification of singularities:** If $f(z)$ has an isolated singularity at $z_0$ with Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$$

then the singularity is

(a) **removable** iff $a_n = 0$ for all $n \leq -1$. In this case $f(z)$ extends to an analytic function on the disk $\{ z : |z - z_0| < R \}$ with power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$.

(b) **a pole** iff there is a $k \leq -1$ so that $a_n = 0$ for $n < k$ but $a_k \neq 0$. In this case $z_0$ is a pole of order $-k$. In the case of $k = -1$ we also call $z_0$ a simple pole.

Examples: The $f(z) = \frac{1}{z}$ has a simple pole (i.e. A pole of order one) at $z_0 = 0$, the function $f(z) = \frac{1}{z^k}$ has a pole of order $k$ at $z_0 = 0$, and if $h(z_0) \neq 0$ then $f(z) = \frac{h(z)}{(z - z_0)^k}$ has a pole of order $k$ at $z_0$.

(c) **essential singularity** iff there are infinitely many $n \leq -1$ with $a_n \neq 0$.

5. **Characterization of removable singularities:** Let $f(z)$ have an isolated singularity at $z_0$. Then the following are equivalent.

(a) $f(z)$ has a removable singularity at $z_0$.

(b) $f(z)$ is bounded near $z_0$. That is there is an $R > 0$ and a constant $M > 0$ such that $|f(z)| \leq M$ for $0 < |z - z_0| < R$. 

(6) **Characterization of poles:** Let \( f(z) \) have an isolated singularity at \( z_0 \). Then the following are equivalent.
   (a) \( f(z) \) has a pole at \( z_0 \).
   (b) \( \lim_{z \to z_0} |f(z)| = \infty \).

(7) **Structure of poles of order \( k \):** Let \( f(z) \) have an isolated singularity at \( z_0 \). Then the following are equivalent.
   (a) \( z_0 \) is a pole of \( f(z) \) of order \( k \).
   (b) There is an analytic function \( h(z) \) defined in a neighborhood of \( z_0 \) with
   \[
   f(z) = \frac{h(z)}{(z - z_0)^k}, \quad \text{and} \quad h(z_0) \neq 0.
   \]

(8) **Definition of residue:** If \( f(z) \) has an isolated singularity at \( z_0 \) and the Laurent expansion of \( f(z) \) about \( z_0 \) is
   \[
   f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n,
   \]
   then the **residue of \( f(z) \) at \( z_0 \)** is
   \[
   \text{Res}(f, z_0) = a_{-1}.
   \]
   That is the residue of \( f(z) \) at \( z_0 \) is the coefficient of \((z - z_0)^{-1}\) in the Laurent expansion of \( f(z) \) at \( z_0 \).

(9) **Integrals in circles about singularities:** If \( f(z) \) is analytic in \( \{ z : 0 < |z - z_0| < R \} \) and \( 0 < r < R \), then
   \[
   \int_{|z-z_0|=r} f(z) \, dz = 2\pi i \text{Res}(f, z_0).
   \]