Recall that if \( f(x, y) \) is a function of two variables, then its **gradient** is the vector field \( \nabla f(x, y) = (f_x, f_y) \). A standard fact is that the gradient is perpendicular to the curves defined by \( f(x, y) = C \) where \( C \) is a constant.

**Problem 1.** Let \( f = u + iv \) be an analytic function in a domain \( U \).

(a) Use the Cauchy-Riemann equation so show that at each point of \( U \) that
\[
\|\nabla u\| = \|\nabla v\| \quad \text{(that is at any point of \( U \) the gradients of \( u \) have the same length)} \quad \text{and that \( \nabla u \) and \( \nabla v \) are always perpendicular. (That is the dot product \( \nabla u \cdot \nabla v = 0 \).)}
\]

(b) Use that \( \nabla u \) and \( \nabla v \) are always perpendicular to explain why for any constants \( a \) and \( b \) the curves \( u = a \) and \( v = b \) meet at right angles. (At least if the curves meet at a point where \( f'(z) \neq 0 \).)

(c) Let \( f(z) = z^2 \). Find \( u \) and \( v \) and graph some of the curves \( u = a \) and \( v = b \).

Shortly we will need to know how to expand some rational function into series. Recall that if \( w \) is a complex number with \( |w| < 1 \) that
\[
\frac{1}{1-w} = 1 + w + w^2 + w^3 + \cdots = \sum_{k=0}^{\infty} w^k. \tag{1}
\]

**Problem 2.** If \( z, z_0 \), and \( \zeta \) are complex numbers with \( |z - z_0| < |\zeta - z_0| \) show that
\[
\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \left( \frac{1}{\zeta - z_0} \right) \left( \frac{1}{1 - \left( \frac{z - z_0}{\zeta - z_0} \right)} \right) = \sum_{k=0}^{\infty} \frac{(z - z_0)^k}{(\zeta - z_0)^{k+1}}.
\]

**HINT:** Start with
\[
\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \left( \frac{1}{\zeta - z_0} \right) \left( \frac{1}{1 - \left( \frac{z - z_0}{\zeta - z_0} \right)} \right)
\]
and use (1).

**Problem 3.** If \( z, z_0 \), and \( \zeta \) are complex numbers with \( |\zeta - z_0| < |z - z_0| \) show that
\[
\frac{1}{\zeta - z} = \frac{-1}{(z - z_0)} \left( 1 + \left( \frac{\zeta - z_0}{z - z_0} \right) + \left( \frac{\zeta - z_0}{z - z_0} \right)^2 + \left( \frac{\zeta - z_0}{z - z_0} \right)^3 + \cdots \right) = -\sum_{k=1}^{\infty} \frac{(\zeta - z_0)^{k-1}}{(z - z_0)^k}.
\]

**HINT:** This time start with
\[
\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \left( \frac{-1}{z - z_0} \right) \left( \frac{1}{1 - \left( \frac{\zeta - z_0}{z - z_0} \right)} \right)
\]
and use (1).