Take home part of Test 3.

This will be due at the beginning of class on Monday. Write you solutions as if you the reader was someone who did not know how to do the problem and wanted to understand it. In particular this means including a English explanations of what you are doing. And problem that only has formulas and no English will lose points. I don't know if this idea will help you in writing things up, but I do in writing up solutions is pretend that write a solution for a younger version of myself this is just leaning the material and try to give enough detail younger me could follow things easily.

You may use your notes and other non-human sources (for example to look up trig identities). I do not mind if you talk to each other a bit about the basic ideas, but do not ask for information from people not in the class.

Here you will work out some of the theory for reflections in lines. Recall that if $\vec{u}$ is a unit vector, that is $\|\vec{u}\| = 1$, then the ***reflection*** in $\vec{u}^{\perp}$ (that is the line perpendicular to $\vec{u}$) is given by

$$M_{\vec{u}}(\vec{x}) = \vec{x} - 2(\vec{x} \cdot \vec{u})\vec{u}.$$

As a warm up:

1. (5 points) Show that for all $\vec{x}$ that

$$\|M_{\vec{u}}(\vec{x})\| = \|\vec{x}\|.$$

Hint: Here is start on it:

$$\begin{aligned}\|M_{\vec{u}}(\vec{x})\|^2 &= (\vec{x} - 2(\vec{x} \cdot \vec{u})\vec{u}) \cdot (\vec{x} - 2(\vec{x} \cdot \vec{u})\vec{u}) \\ &= \|\vec{x}\|^2 - 4(\vec{x} \cdot \vec{u}) \cdot (\vec{x} \cdot \vec{u}) + 4(\vec{x} \cdot \vec{u})^2\|\vec{u}\|^2\end{aligned}$$

□

2. (5 points) Show that $M_{\vec{u}}$ is linear. That is show

$$M_{\vec{a}}(\vec{x} + \vec{y}) = M_{\vec{a}}(\vec{x}) + M_{\vec{a}}(\vec{y})$$

holds for all $\vec{x}, \vec{y} \in \mathbb{R}^2$ and that for all scalars c

$$M_{\vec{a}}(c\vec{x}) = cM_{\vec{a}}(c\vec{x}).$$

Hint: You can use the properties $(\vec{x}+\vec{y})\cdot\vec{u} = \vec{x}\cdot\vec{u}+\vec{y}\cdot\vec{u}$ and $(c\vec{x})\cdot\vec{u} = c(\vec{x}\vec{u})$ of inner products. □

3. (5 points) Use the last two problems to show that $M_{\vec{u}}$ is a rigid motion. That is show

$$\|M_{\vec{u}}(\vec{y}) - M_{\vec{u}}(\vec{x})\| = \|\vec{y} - \vec{x}\|$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^2$. □

If $L\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map the matrix, M, of L is the map with columns $L\vec{e}_1$ and $L\vec{e}_2$. That is

$$M = [L\vec{e}_1, L\vec{e}_2]$$

and

$$\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \qquad \vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$$

For example if

$$L\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3x+2y\\-4x+5y\end{bmatrix}$$

then

$$L\vec{e}_1 = L\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}3(1)+2(0)\\-4(1)+5(0)\end{bmatrix} = \begin{bmatrix}3\\-4\end{bmatrix}, \qquad L\vec{e}_1 = L\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3(0)+2(1)\\-4(0)+5(1)\end{bmatrix} = \begin{bmatrix}2\\5\end{bmatrix}$$

and therefore the matrix of L is

$$M = [L\vec{e}_1, \vec{e}_2] = \begin{bmatrix} 2 & 2 \\ -4 & 5 \end{bmatrix}.$$

We have seen that the matrix for the rotation by α is

$$R_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

and in fact have use notation R_α for both the rotation and the matrix.

Let α be a real number and let

$$\vec{u}(\alpha) = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}.$$

4. (5 points) Show this is a unit vector and draw a picture to showing that it is the unit vector that makes an angle of α with the positive x-axis. ☐

5. (5 points) Let M_α be the matrix of $R_{\vec{u}(\alpha)}$. Find M_α. *Hint:* To start note

$$R_{\vec{u}(\alpha)}\vec{e}_1 = \vec{e}_1 - 2(\vec{e}_1 \cdot \vec{u}(\alpha))\vec{u}(\alpha) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}\right)\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} 1 - 2\cos^2\alpha \\ -2\cos\alpha\sin\alpha \end{bmatrix}.$$

This will be the first column of M_α. So all you have to do is find the second column. ☐

6. (5 points) Use trigonometric identities to show that M_α can rewritten as

$$M_\alpha = \begin{bmatrix} -\cos(2\alpha) & -\sin(2\alpha) \\ -\sin(2\alpha) & \cos(2\alpha) \end{bmatrix}.$$

☐

7. (10 points) Show that the product $M_\alpha M_\beta$ is a rotation. What is the angle of the rotation? ☐