Review for test 1

We started out with some preliminary results about bijective maps. You should know the definition of ***injective***, ***surjective***, and ***bijective***.

We then gave the axioms for affine geometry, which are

Affine Geometry: Axiom 1 (Two points determine a unique line)**.** *If P and Q are distinct points of $\mathbb{A}^2$ (that is $P \neq Q$) then there is a unique line that passes through both of them.* $\square$

Affine Geometry: Axiom 2 (Parallel Axiom)**.** *If ℓ is a line of $\mathbb{A}^2$ and P is a point that is not on ℓ, then there is a unique line m that goes through P and is parallel to ℓ.* $\square$

Affine Geometry: Axiom 3. *There exists a set of four points in $\mathbb{A}^2$ such that no three of which are all on the same line.* $\square$

You should have these memorized. And you should know the definition of ***parallel*** (and that under our definition a line is parallel to itself).

Problem 1*.* If ℓ_1 and ℓ_2 are parallel lines and m is a line that intersects ℓ_1 in a single point, then it intersects ℓ_2 in a single point.

Solution. If $\ell_1 = \ell_2$, there is nothing to prove. So assume that $\ell_1 \neq \ell_2$. We first show that m intersects ℓ_2 in at least one point. For if this were not the case, then let $P = m \cap \ell_1$. Towards a contradiction assume m doe not intersect ℓ_2 in any point. Then m is parallel to ℓ_1 and we are given that ℓ_2 is parallel to ℓ_1. Thus the point P has two lines passing through it that are parallel to ℓ_2, contradicting the parallel axiom with says that there is only one.

To see that ℓ_2 intersects m in at most one point, note if it intersected m in two points by the first axiom of affine geometry we would have $m = \ell_2$ which this line is parallel to ℓ_1, (as $\ell_2 \parallel \ell_1$) and that it only only intersects ℓ_1 in one point (as m only intersects ℓ_1 in one point). This is a contradiction. $\square$

Problem 2*.* Show that if ℓ, m and n are lines with $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$.

Solution. If $\ell = m$, or $m = n$, or $\ell = n$ then the result is true. So we assume that ℓ, m and n are distinct. Towards a contradiction assume that ℓ and n are not parallel. Then they intersect at some point, call it P. Then P is not on m (as P is on ℓ and ℓ and m have no points in common as $\ell \parallel m$ and $\ell \neq m$). But then the point P has two distinct lines through it (ℓ and n) which are parallel to m which contradicts the parallel axiom. $\square$

Problem 3*.* Let ℓ be a line and m a line that intersects ℓ in exactly one point. Then there is a bijective correspondence between the points, P, on m and the lines, ℓ', parallel to ℓ. Proof by picture, provided there is enough description in English, is fine.

Solution. The correspondence is given as follows. For each point, P, on m let $\ell'(P)$ be the line through P parallel to ℓ. For each line ℓ' parallel to ℓ let $P(\ell') = \ell' \cap m$ be the point where ℓ' intersects m. From Figure 1 we see that each point P on m determines a unique line ℓ' parallel to ℓ (by the Parallel Axiom) and each line ℓ' parallel to ℓ determines a unique point $P = m \cap \ell'$ on m (by Problem 1). This gives a bijective correspondence.

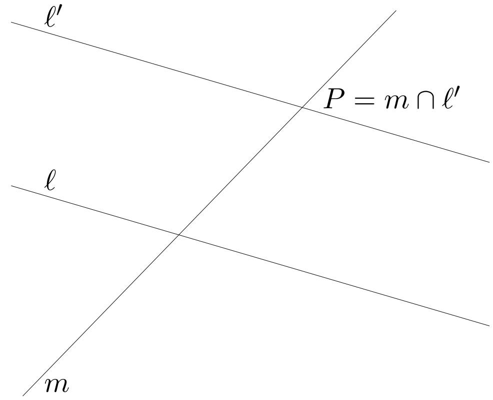

FIGURE 1

□

Problem 4. Show that any affine plane contains at least 6 lines.

Solution. You think about this one in light of Affine axiom 3. □

You should understand Theorem 14 on Homework 1.

Problem 5. In Figure 2 find the following

(a) The line through C parallel to the line e.
(b) All lines parallel to c.
(c) $\overleftrightarrow{(f \cap d)(a \cap e)}$

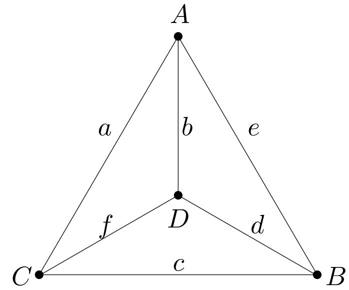

FIGURE 2

Solution. (a) f, (b) b and c, (c) $\overleftrightarrow{(f \cap d)(a \cap e)} = \overleftrightarrow{DA} = b$

Problem 6. In Figure 3

(a) What is the line through E and parallel to b?
(b) What is the line through I and parallel to g?
(c) What are the lines parallel to b?
(d) What are the lines parallel to a?

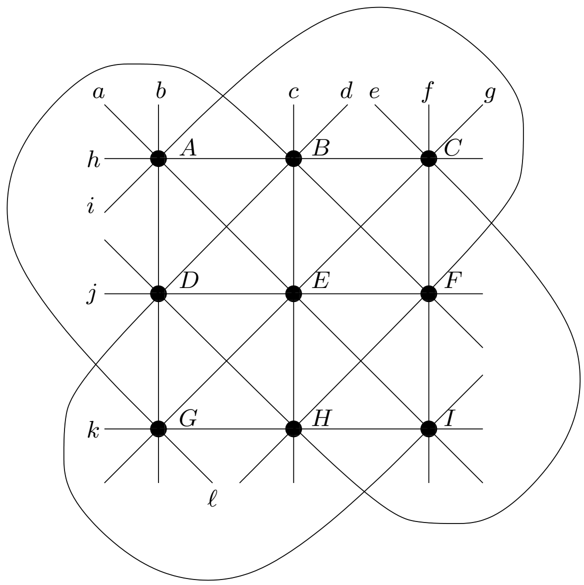

FIGURE 3

Solution. (a) c, (b) d, (c) b, c, and f, (d) a, e, and ℓ.

After giving the basic axioms, propositions, and some examples of affine geometries, we went about the project of showing that the basic coordinate plane $\mathbb{F}^2 = \mathbb{A}^{2^2}$, that is the set of ordered pairs (x, y) with $x, y \in \mathbb{F}$ we know and love from high school geometry is an example of an affine plane where we defined a line to be a set

$$L(a, b, c) = \{(x, y) : ax + by + c = 0\}$$

where a and b are not both zero. In the case that our filed is $\mathbb{F} = \mathbb{R}$ you should certainly be able to graph these lines.

Problem 7*.* Graph the following
(a) $L(1, 2, 3)$. What is the slope of this line? What is its slope intercept equation?
(b) $L(1, 0, 5)$
(c) $L(m, -1, \beta)$

Here is a particular case of showing that the first axiom of affine geometry holds.

Problem 8*.* Find the line that goes through $(3, -2)$ and $(4, 1)$.

Solution. A vector parallel to this line is $(4,1)-(3,-2) = \langle 1,3\rangle$. Thus $\vec{\text{n}} = \rangle -3, 1\langle$ is a perpendicular to this line. Therefore one equation for this line is

$$\vec{\text{n}} \cdot ((x,y)-(4,1)) = 0.$$

That is

$$-3(x-4) + 1(y-1) = 0$$

which can be rewritten as

$$-3x + y + 11 = 0.$$

As a check note

$$-3(3) + (-2) + 11 = 0 \qquad \text{and} \qquad (-3)(4) + (1) + 11 = 0.$$

So the line is $L(-3,1,11)$. This is the same as $L(3,-1,-11)$ or $L(6,-2,-22)$. □

More generally

Problem 9. If $P = (x_0, y_0)$ and $Q = (x_1, y_1)$ are distinct points of $\mathbb{A}^2$, show that there is a line $L(a,b,c)$ that passes through these points.

Solution. This works just like the last problem. The vector

$$Q - P = \langle x_1 - x_0, y_1 - y_0\rangle$$

is parallel to the line. Thus

$$\vec{\text{n}} = \langle -(y_1 - y_0), (x_1 - x_0)\rangle$$

is perpendicular to the line. Thus the equation should be

$$\langle -(y_1 - y_0), (x_1 - x_0)\rangle \cdot \langle x - x_0, y - y_0\rangle = 0.$$

This simplifies down to

$$(y_0 - y_1)x + (x_1 - x_0)y + (x_0 y_1 - x_1 y_0) = 0.$$

Thus $L(y_0 - y_1, x_1 - x_0, x_0 y_1 - x_1 y_0)$ should work. Verify by direct calculation that P and Q are on $L(y_0 - y_1, x_1 - x_0, x_0 y_1 - x_1 y_0)$ and explain why $(y_0 - y_1)$ and $(x_1 - x_0)$ are not both zero.

For anther way to do this problem see Remark 6 on Homework 2. □

Here is a particular case of showing that the second axiom of affine geometry holds.

Problem 10. Find the line through $(3,7)$ that is parallel to $L(6,-2,9)$.

Solution. The lines parallel to $L(6,-2,9)$ are of the form $L(6,-2,c)$. We want to choose c so that $(3,7)$ is on this line. That is we want

$$6(3) - 2(7) + c = 0$$

This gives $c = -18 + 14 = -4$. Thus the line is $L(6,-2,-4)$. □

Problem 11. Let $L(a,b,c)$ be a line and $P = (x_0, y_0)$ any point of $\mathbb{A}^2$. Show there is a line through P and parallel to $L(a,b,c)$.

Solution. The lines parallel to $L(a, b, c)$ are of the form $L(a, b, c')$. We wish to choose c' so that P is on the line. That is so that

$$ax_0 + by_0 + c' = 0.$$

This clearly works when $c' = -ax_0 - by_0$. So the required parallel to $L(a, b, c)$ through P is $L(a, b, -ax_0 - by_0)$. $\square$

We next started to review some linear algebra. The first topic we hit here was Cramer's rule.

Theorem 1 (Cramer's rule)**.** *If $a, b, c, \in \mathbb{F}$ with*

$$ad - bd \neq 0$$

then the system

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

has a unique solution. It is given by

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - fb}{ad - bc}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ce}{ad - bc}$$

*We call $ad - bc$ the **determinant of the system**.*

A special case that we have use repeatedly is

Theorem 2 (Cramer's rule for homogeneous equations)**.** *If $a, b, c, \in \mathbb{F}$ with*

$$ad - bd \neq 0$$

then the system

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

only has the solution

$$x = 0 \qquad y = 0.$$

Some basic definitions are

Definition 3. The two vectors $\vec{a}$ and $\vec{b}$ are ***linearly dependent*** iff there are scalars α and β with at least one of them nonzero with

$$\alpha\vec{a} + \beta\vec{b} = \vec{0}.$$

Thus to show two vectors $\vec{a}$ and $\vec{b}$ are linearly dependent, your job is to find scalars α and β, not both zero such that $\alpha\vec{a} + \beta\vec{b} = \vec{0}$.

Problem 12. Show that the vector $\vec{a} = \langle 2, 4 \rangle$ and $\vec{b} = \langle 3, 6 \rangle$ are linearly dependent.

Solution. Note

$$3\vec{a} + (-2)\vec{b} = 3\langle 2, 4\rangle - 2\langle 3, 6\rangle = \langle 0, 0\rangle = \vec{0}.$$

Therefore if $\alpha = 3$ and $\beta = -2$ we have $\alpha\vec{a} + \beta\vec{b} = \vec{0}$ with and $\alpha \neq 0$. Thus $\vec{a}$ and $\vec{b}$ are linearly dependent. (Note this is not the only possible choice of α and β that works. For example $\alpha = 15$ and $\beta = -10$ would work. But to show vectors are linearly dependent, you don't have to find all α and β that work, you only to find one pair of α and β that work.) $\square$

Problem 13. For any vector $\vec{c}$ show that $\vec{a}4\vec{c}$ and $\vec{b} = 5\vec{c}$ are linearly dependent.

Solution. Just note that

$$5\vec{a} + (-4)\vec{b} = 20\vec{c} - 20\vec{c} = \vec{0}.$$

Thus for $\alpha = 5$ and $\beta = -4$ we have $\alpha\vec{a} + \beta\vec{b} = \vec{0}$ and $\beta \neq 0$. So the vectors are linearly dependent. $\square$

Definition 4. The vectors $\vec{a}$ and $\vec{b}$ are ***linearly independent*** iff

$$\alpha\vec{a} + \beta\vec{b} = \vec{0}$$

for scalars α and β implies $\alpha = \beta = 0$.

Problem 14. Show the vectors $\vec{a} = \langle 1, 2\rangle$ and $\vec{b} = \langle 3, -1\rangle$ are linearly independent.

Solution. If $\alpha\vec{a} + \beta\vec{b} = \vec{0}$ we have

$$\alpha\langle 1, 2\rangle + \beta\langle 3, -1\rangle = \langle \alpha + 3\beta, 2\alpha - \beta\rangle = \langle 0, 0\rangle.$$

This gives the system

$$\begin{aligned} \alpha + 3\beta &= 0 \\ 2\alpha - \beta &= 0 \end{aligned}$$

The determinant of this system is $(1)(-1) - (3)(2) = -7 \neq 0$. Thus by Cramer's rule $\alpha = \beta = 0$. Which is just what we needed to show that the vectors are linearly independent. $\square$

You should look at Homework 3 and understand the basic properties of the ***determinant***, $\det(\vec{a}, \vec{b})$ of two vector $\vec{a}$ and $\vec{b}$. It would not hurt to be able to prove the properties of det given in Proposition 3 of Homework 3.

Problem 15. Let $\vec{a}$ and $\vec{b}$ be vectors with $\det(\vec{a}, \vec{b}) \neq 0$. Then show that $\vec{a}$ and $\vec{b}$ are linearly independent.

Solution. Let $\vec{a} = \langle a_1, a_2\rangle$ and $\vec{b} = \langle b_1, b_2\rangle$. We are assuming that

$$\det(\vec{a}, \vec{b}) = a_2 b_2 - a_2 b_1 \neq 0.$$

If α and β are scalars such that

$$\alpha\vec{a} + \beta\vec{b} = \vec{0}$$

then

$$\alpha\vec{a} + \beta\vec{b} = \langle a_1\alpha + b_1\beta, a_2\alpha + b_2\beta\rangle = \langle 0, 0\rangle.$$

This gives the system

$$\begin{aligned} a_1\alpha + b_1\beta &= 0 \\ a_2\alpha + b_2\beta &= 0 \end{aligned}$$

for α and β. By assumption the determinant of this system is not zero, and therefore by Cramer's rule we have that $\alpha = \beta = 0$. Which is what is required to show that $\vec{a}$ and $\vec{b}$ are linearly independent. $\square$

This is enough sample problems. Here are some other topics that you should review.

(1) The definition of affine linear combination of two or three points. Review the quiz.

(2) Be prepared for surprise mystery questions.