Modern Geometry Homework.

1. Cramer's rule.

We now wish to become experts in solving two linear equations in two unknowns over our field $\mathbb{F}$. We first recall that a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & b \end{bmatrix}$$

its ***determinant*** is

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

We will also use the notation

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

The following generalizes to linear systems of n equations in n unknowns, but we will only need the case of two equations in two unknowns.

Theorem 1. *If $a, b, c, \in \mathbb{F}$ with*

$$ad - bd \neq 0$$

then the system

$$ax + by = e \tag{1}$$
$$cx + dy = f \tag{2}$$

has a unique solution. It is given by

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{ed - fb}{ad - bc}$$

$$y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{af - ce}{ad - bc}$$

*We call $ad - bc$ the **determinant of the system**.*

Proof. This is done by the usual method of Gaussian elimination. If we multiply equation (1) by d and equation (2) by b and subtract the y terms cancel out and we left with

$$(ad - bc)x = ed - fb.$$

Dividing by $ad - bc$, which were are assuming is not zero, gives the required formula for x.

Likewise we can multiply equation (2) by d and substract a times equation 1 to cancel out the x terms. The result is

$$(ad - bc)y = (af - ce)$$

and we can again divide by $ad - bc$ to get the formula for y.

We are not quite done. What we have done shows that if there are solutions, then they must be given by the formula we have just derived for x and y. We still need to show that solutions exist. This is done by checking that the values for x and y we have just given are solutions. For these values of x and y

$$\begin{aligned}
ax + by &= a\left(\frac{ed - fb}{ad - bc}\right) + b\left(\frac{af - ce}{ad - bc}\right) \\
&= \frac{aed - afb + baf - bce}{ad - bc} \\
&= \frac{aed - bce}{ad - bc} \\
&= \frac{e(ad - bc)}{ad - bc} \\
&= e \\
cx + dy &= c\left(\frac{ed - fb}{ad - bc}\right) + d\left(\frac{af - ce}{ad - bc}\right) \\
&= \frac{ced - cfd + daf - cde}{ad - bc} \\
&= \frac{-cfd + daf}{ad - bc} \\
&= \frac{f(ad - bc)}{ad - bc} \\
&= f.
\end{aligned}$$

Which shows they are solutions. □

2. Linearly independent vectors.

Given two vectors

$$\vec{a} = (a_1, b_2), \qquad \vec{b} = (b_1, b_2).$$

we ask when we can write all vector $\vec{v} = (v_1, v_2)$ as linear combinations of these two vectors. That is when can we express all vectors $\vec{v}$ in the form

$$\vec{v} = \alpha\vec{a} + \beta\vec{b},$$

where $\alpha, \beta \in \mathbb{F}$. We call such sums ***linear combinations*** of $\vec{a}$ and $\vec{b}$. Rewriting $\vec{v} = \alpha\vec{a} + \beta\vec{b}$ in terms of components gives

$$(\alpha a_1 + \beta b_1, \alpha a_2 + \beta b_2) = (v_1, v_2).$$

This is the same as the system

$$\begin{aligned} a_1\alpha + b_1\beta &= v_1 \\ a_2\alpha + b_2\beta &= v_2 \end{aligned}$$

where we want to solve for α and β. By Cramer's rule (with α and β in the roles of x and y) we find this will have a solution whenever

$$a_1b_2 - a_2b_1 \neq 0.$$

Definition 2. If $\vec{a}, \vec{b} \in \mathbb{F}$ define

$$\det(\vec{a}, \vec{b}) = a_1b_2 - a_2b_1.$$

We call this the ***determinant*** of $\vec{a}$ and $\vec{b}$. Note it is the same as the determinant of the matrix

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}.$$

Proposition 3. *The determinant of two vectors as just defined has the following properties*

(a) $\det(\vec{a}, \vec{a}) = 0$ *for all* $\vec{a}$*.*
(b) $\det(\vec{b}, \vec{a}) = -\det(\vec{a}, \vec{b})$*.*
(c) $\det(\vec{a} + \vec{a}\,', \vec{b}) = \det(\vec{a}, \vec{b}) + \det(\vec{a}\,', \vec{b})$*.*
(d) $\det(\vec{a}, \vec{b} + \vec{b}\,') = \det(\vec{a}, \det b) + \det(\vec{a}, \vec{b}\,')$*.*
(e) If $\alpha \in \mathbb{F}$ *we have* $\det(\alpha\vec{a}, \vec{b}) = \alpha\det(\vec{a}, \vec{b})$ *and* $\det(\vec{a}, \alpha\vec{b}) = \alpha\det(\vec{a}, \vec{b})$*.*

Problem 1. Prove this. *Hints:* The first two are straightforward. That is let $\vec{a} = (a_1, a_2)$, and $\vec{b} = (b_1, b_2)$ plug these in to the equations and see that they work out. For the third, I don't know any way that does not just involve doing the calculation that shows that both sides reduce to the same thing. (That is let $\vec{a} = (a_1, a_1)$, $\vec{a}\,' = (a_1', a_2')$ and $\vec{b} = (b_1, b_2)$ and plug and chug.) Now (d) can be reduced to (c) using (b). To give you a start

$$\det(\vec{a}, \vec{b} + \vec{b}\,') = -\det(\vec{b} + \vec{b}\,', \vec{a}) = -\det(\vec{b}, \vec{a}) - \det(\vec{b}\,', \vec{a})$$

Finally (e) is more straightforward plugging and chugging. $\square$

Proposition 4 (Cramer's rule revisted)**.** *If* $\vec{a}$ *and* $\vec{b}$ *are vectors with* $\det(\vec{a}, \vec{b}) \neq 0$*, and*

$$\vec{v} = \alpha\vec{a} + \beta\vec{b}.$$

Then

$$\alpha = \frac{\det(\vec{v}, \vec{b})}{\det(\vec{a}, \vec{b})}, \qquad \beta = \frac{\det(\vec{a}, \vec{v})}{\det(\vec{a}, \vec{b})}.$$

Problem 2. Prove this using Proposition 3. *Hint:* To get the formula for α justify the following

$$\begin{aligned}
\det(\vec{v},\vec{b}) &= \det(\alpha\vec{a}+\beta\vec{b},\vec{b}) && (\text{as } \vec{v}=\alpha\vec{a}+\beta\vec{b})\\
&= \det(\alpha\vec{a},\vec{b})+\det(\beta\vec{b},\vec{b}) && (\text{Give a reason from Prop. 3})\\
&= \alpha\det(\vec{a},\vec{b})+\beta\det(\vec{b},\vec{b}) && (\text{Give a reason from Prop. 3})\\
&= \alpha\det(\vec{a},\vec{b})+0 && (\text{Give a reason from Prop. 3})\\
&= \alpha\det(\vec{a},\vec{b}).
\end{aligned}$$

Now divide by $\det(\vec{a},\vec{b})\neq 0$ to get the formula for α. To get the formula for β start with

$$\det(\vec{a},\vec{v}) = \det(\vec{a},\alpha\vec{a}+\beta\vec{b})$$

and do a similar calculation. □

Definition 5. The vectors $\vec{a}$ and $\vec{b}$ are ***linearly dependent*** iff there are scalars $\alpha,\beta\in\mathbb{F}$ not both zero such that

$$\alpha\vec{a}+\beta\vec{b}=\vec{0}.$$

□

Problem 3. Show that if $\vec{0}=(0,0)$ is the zero vector, then $\vec{0}$ and $\vec{b}$ are linearly dependent for any vector $\vec{b}$. (Likewise $\vec{a}$ and $\vec{0}$ are linearly dependent for any $\vec{a}$.) □

Proposition 6. *The two vector $\vec{a}$ and $\vec{b}$ are linearly dependent if and only if* $\det(\vec{a},\vec{b})=0$.

Problem 4. Prove this. *Hint:* First assume that the two vectors are linearly dependent. Then there are α and β, not both zero, such that

$$\alpha\vec{a}+\beta\vec{b}=\vec{0}.$$

Assume that $\alpha\neq 0$ justify the following

$$\begin{aligned}
0=\det(\vec{0},\vec{b}) &= \det(\alpha\vec{a}+\beta\vec{b},\vec{b})\\
&= \alpha\det(\vec{a},\vec{b})+\beta\det(\vec{b},\vec{b})\\
&= \alpha\det(\vec{a},\vec{b})+0\\
&= \alpha\det(\vec{a},\vec{b}).
\end{aligned}$$

But $\alpha\neq 0$, so this implies $\det(\vec{a},\vec{b})=0$. Do the corresponding calculation in the case $\beta\neq 0$.

Conversely assume that

$$\det(\vec{a},\vec{b}) = a_1b_2-a_2b_1=0.$$

If $\vec{a}=\vec{0}$, then $\vec{a}$ and $\vec{b}$ are linearly dependent by Problem 3. So we can assume that $\vec{a}=(a_1,a_2)\neq(0,0)$. Thus either $a_1\neq 0$ or $a_2\neq 0$.

Case 1: $a_1 \neq 0$. I will do this case. Using $a_1 \neq 0$. Then we can solve for b_2 in

$$a_1 b_2 - a_2 b_1 = 0$$

to get

$$b_2 = \frac{a_2 b_1}{a_1}.$$

Therefore

$$\vec{b} = (b_1, b_2) = \left(b_1, \frac{a_2 b_1}{a_1}\right) = \frac{b_1}{a_1}(a_1, a_2) = \alpha \vec{a}$$

where

$$\alpha = \frac{b_1}{a_1}.$$

Then if we let $\beta = -1 \neq 0$ we have

$$\alpha \vec{a} + \beta \vec{b} = \vec{b} + (-1)\vec{b} = \vec{0}$$

and thus $\vec{a}$ and $\vec{b}$ are linearly dependent.

Case 2: $a_2 \neq 0$. This case is up up to you. □

Definition 7. The two vectors $\vec{a}$ and $\vec{b}$ are ***linearly independent*** iff they are not linearly dependent. Explicitly this means the only way that a linear combination can vanish, that is

$$\alpha \vec{a} + \beta \vec{b} = \vec{0}$$

is if

$$\alpha = \beta = 0.$$

□

Here is an example. Show the two vectors $\vec{a} = (1, 2)$ and $\vec{b} = (-2, 3)$ are linearly independent from the definition. That is we need to show that if

$$\alpha \vec{a} + \beta \vec{b} = \vec{0}$$

then $\alpha = \beta = 0$. Note

$$\alpha \vec{a} + \beta \vec{b} = \alpha(1, 2) + \beta(-2, 3) = (\alpha - 2\beta, 2\alpha + 3\beta) = (0, 0).$$

This leads to the system

$$\begin{aligned} \alpha - 2\beta &= 0 \\ 2\alpha + 3\beta &= 0 \end{aligned}$$

and the only solution to this is $\alpha = \beta = 0$ (which can be seen by Cramer's rule).

We have seen that $\vec{a}$ and $\vec{b}$ are linearly dependent if and only if $\det(\vec{a}, \vec{b}) = 0$ (Proposition 6.) This implies

Theorem 8. *The two vectors $\vec{a}$ and $\vec{b}$ are linearly independent if and only if* $\det(\vec{a}, \vec{b}) \neq 0$. □

One reason that linearly independent vectors are important is the following:

Theorem 9. *If the two vectors $\vec{a}$ and $\vec{b}$ of $\mathbb{F}$ are linearly independent, then every vector $\vec{v}$ can uniquely written as a linear combination of $\vec{a}$ and $\vec{b}$. That is there are unique scalars $\alpha, \beta \in \mathbb{F}$ such that*

$$\vec{v} = \alpha\vec{a} + \beta\vec{b}.$$

Moreover α and β are given by the following form of Cramer's rule.

$$\alpha = \frac{\det(\vec{v}, \vec{b})}{\det(\vec{a}, \vec{b})}, \qquad \beta = \frac{\det(\vec{a}, \vec{b})}{\det(\vec{a}, \vec{b})}.$$

Problem 5. Prove this. □

Problem 6. (a) Write $\vec{v} = (5, 6)$ as a linear combination of $\vec{a} = (3, 4)$ and $\vec{b} = (-3, 2)$.

(b) Write $\vec{v} = (1, 2)$ as a linear combination of $\vec{a} = (1, -1)$ and $\vec{b} = (1, 1)$.

(c) Write $\vec{v} = (v_1, v_2)$ as a linear combination of $\vec{a} = (\cos t, \sin t)$ and $\vec{b} = (-\sin t, \cos t)$.

3. Matrices

From now on we will write vectors as column vectors. That is a vector will be

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

The reason for this is that is works better with matrix multiplication. Let us review a bit about 2×2 matrices. This is a square array of numbers of the form

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

If B is

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

then the sum of A and B is

$$A + B = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}.$$

The product is

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b21 + a_{22}b_{22} \end{bmatrix}.$$

And we multiply a matrix A by a scalar in the reasonable way:

$$\alpha A = \alpha \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} \\ \alpha a_{21} & \alpha a_{22} \end{bmatrix}.$$

We multiply a vector by a matrix by the rule

$$A\vec{v} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{bmatrix}.$$

Problem 7. This is a problem to review doing matrix operations. Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 5 \end{bmatrix}, \qquad B = \begin{bmatrix} 6 & 4 \\ 2 & -1 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

then find the following

(a) $A + B$
(b) AB
(c) $2A - 4B$
(d) $A\vec{v}$
(e) $B\vec{v}$
(f) $AB - BA$

The ***identity matrix*** is

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The reason it is called the identity is

Proposition 10. *If A is any 2×2 matrix and $\vec{v}$ is any vector, then*

$$AI = IA = A$$

and

$$I\vec{v} = \vec{v}.$$

Problem 8. Prove this. *Hint:* Here is part of it:

$$AI = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11}1 + a_{12}0 & a_{11}0 + a_{12}1 \\ a_{21}1 + a_{22}0 & a_{21}0 + a_{22}1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A.$$

□

There are other properties of matrices we will use, for example the associative law

$$A(BC) = (AB)C$$

and the distributive laws

$$A(B + C) = AB + AC, \qquad (A + B)C = AC + BC,$$

and the commutative law for addition

$$A + B = B + A.$$

But you should keep in mind that the commutative law for multiplication does not hold. That is for many matrices

$$AB \neq BA.$$

Problem 9. Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Show $AB \neq BA$. □

Definition 11. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

with $\det(A) = ad - bc \neq 0$. Set

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

□

Proposition 12. *With this definition show*

$$AA^{-1} = A^{-1}A = I$$

where I is the identity matrix. We call A^{-1} the ***inverse*** *of A. (Note that the inverse of A is only defined when* $\det(A) \neq 0$*.)* □

Problem 10. Prove this. *Hint:* Just multiply out each of AA^{-1} and $A^{-1}A$ long hand and show that the result is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. □

The ***standard basis*** if $\mathbb{F}^2$ is the pair of vectors

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

In other classes you may have used the notation $\vec{i} = \vec{e}_1$ and $\vec{j} = \vec{e}_2$.

Proposition 13. *Let $\vec{a}$ and $\vec{b}$ be any two vectors. Then there is a matrix A such that*

$$A\vec{e}_1 = \vec{a}, \qquad A\vec{e}_2 = \vec{b}.$$

Explicitly if

$$\vec{a} = \begin{bmatrix} a_1 \\ a_1 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

then the matrix

$$A = [\vec{a}, \vec{b}] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

does the job.

Problem 11. Prove this. *Hint:* If you start with

$$A\vec{e}_1 = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

the rest of the proof should take care of itself. □

While it does not look like it here, the following will later be rewritten to be one of our main theorems about affine geometry.

Theorem 14. *Let $\vec{a}$ and $\vec{b}$ be linearly independent vectors and let $\vec{a}\,'$ and $\vec{b}\,'$ be any vectors. Then there is a matrix A such that*

$$A\vec{a} = \vec{a}\,' \qquad \textit{and} \qquad A\vec{b} = \vec{b}\,'.$$

Problem 12. Prove this along the following lines.

(a) There is a matrix B such that

$$B\vec{e}_1 = \vec{a} \qquad \text{and} \qquad B\vec{e}_2 = \vec{b}. \tag{3}$$

Hint: Proposition 13.

(b) There is a matrix C with

$$C\vec{e}_1 = \vec{a}\,' \qquad \text{and} \qquad B\vec{e}_2 = \vec{b}\,'.$$

(c) $\det(B) \neq 0$ and therefore B^{-1} exists. *Hint:* This follows from the form of B given in Proposition 13 (which gives $B = [\vec{a}, \vec{b}] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$) and that $\vec{a}$ and $\vec{b}$ are linearly independent.

(d) Multiply the equations of (3) by B^{-1} to get

$$B^{-1}\vec{a} = \vec{e}_1 \qquad \text{and} \qquad B^{-1}\vec{b} = \vec{e}_2.$$

(e) Finally set

$$A = CB^{-1}$$

and show that A does what is required. $\square$

4. Affine maps

We start by giving anther description of lines.

Proposition 15. *Let $L(a, b, c)$ be a line in $\mathbb{A}^2$ with P_0 and P_1 distinct points on $L(a, b, c)$. Let*

$$P(t) = (1 - t)P_0 + tP_1.$$

*Then as t varies over $\mathbb{F}$, the point $P(t)$ varies all the points of $L(a, b, c)$. We call the function $P(t)$ an **affine parameterization** of $L(a, b, c)$.*

Proof. Let $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$. As these points are on $L(a, b, c)$ we have

$$ax_0 + by_0 + c = 0, \qquad ax_1 + by_1 + c = 0.$$

The point $P(t)$ is

$$P(t) = (1 - t)(x_0, y_0) + t(x_1, y_1) = ((1 - t)x_0 + tx_1, (1 - t)y_0 + ty_1)).$$

Let $(x, y) = ((1 - t)x_0 + tx_1, (1 - t)y_0 + ty_1))$ we see

$$\begin{aligned} ax + by + c &= a(1 - t)x_0 + tx_1) + b((1 - t)y_0 + ty_1)) + c \\ &= (1 - t)(ax_0 + by_0 + c) + t(ax_1 + by_1 + c) \\ &= (1 - t)0 + t0 \\ &= 0. \end{aligned}$$

Therefore $P(t)$ is on $L(a, b, c)$.

We still need to show that all points on $L(a, b, c)$ are of the form $P(t)$ for some $t \in \mathbb{F}$. We first consider the case where $b \neq 0$. Then in $ax + by + c = 0$ we can solve for y to get

$$y = \frac{-1}{b}(ax + c).$$

This shows that if P and Q are points on $L(a, b, c)$ with the same x coordinates, then $P = Q$. Let $Q = (x_2, y_2)$ be a point on $L(a, b, c)$. As P_0 and P_1 are distinct points on $L(a, b, c)$ and $b \neq 0$ we see that $x_0 \neq x_1$. Thus there is a t_2 such that

$$x_2 = (1 - t_2)x_0 + t_2 x_1$$

(you can check that $t_2 = (x_2 - x_0)/(x_1 - x_0)$ works). Thus the point $P(t_2)$ is on $L(a, b, c)$. But $P(t_2)$ and Q are on $L(a, b, c)$ and both have the same x coordinate. Therefore $Q = P(t_2)$ with shows that Q is of the required form.

This only leaves the case where $b = 0$. In that case $a \neq 0$ and a similar argument works (just reverse the roles of a and b and of x and y). □

Let $t = 1/2$ in affine parameterization of the line through P and Q gives the following:

Definition 16. Let P and Q be points of $\mathbb{A}^2$. Then the midpoint of P and Q is the point

$$M = \frac{1}{2}P + \frac{1}{2}Q.$$

□

Here is a nice geometric property of midpoints (and the first result we have had in a while that really looks like a geometry result).

Proposition 17. *Let A, B, C, and D be point of $\mathbb{A}^2$. Let*

$$\begin{aligned} M_1 &= \textit{midpoint of } A \textit{ and } B \\ M_2 &= \textit{midpoint of } B \textit{ and } C \\ M_3 &= \textit{midpoint of } C \textit{ and } D \\ M_4 &= \textit{midpoint of } D \textit{ and } A. \end{aligned}$$

Then

$$\overleftrightarrow{M_1M_2} \parallel \overleftrightarrow{M_3M_4} \qquad \textit{and} \qquad \overleftrightarrow{M_1M_3} \parallel \overleftrightarrow{M_2M_4}.$$

Put somewhat informally this says that for any quadrilateral $ABCD$ that the midpoints of the sides of $ABCD$ always form a parallelogram. (See Figure 1)

Problem 13. Prove this. *Hint:* It is enough to show that the following equalities of vectors.

$$\begin{aligned} M_4\vec{M}_1 &= M_3\vec{M}_2 \\ M_3\vec{M}_4 &= M_2\vec{M}_1 \end{aligned}$$

which should not be hard. □

Definition 18. A map $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ is an ***affine map*** iff for all $P_0, P_1 \in \mathbb{A}^2$ and all scalars $\alpha, \beta \in \mathbb{F}$ the equality

$$f(\alpha P_0 + \beta P_1) = \alpha f(P_0) + \beta f(P_1)$$

holds. □

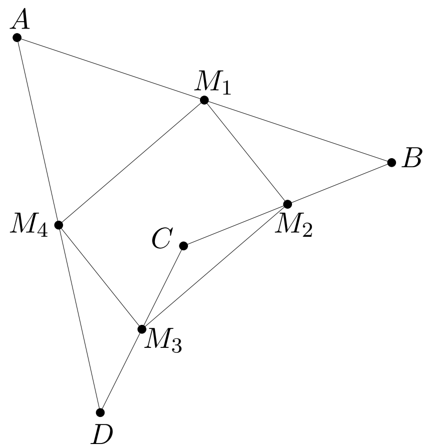

FIGURE 1. In any quadrilateral the midpoints of the sides form a parallelogram. More precisely $\vec{M_4M_1} = \vec{M_3M_2}$ and $\vec{M_3M_4} = \vec{M_2M_1}$.

Let $\beta = t$, so that $\alpha = 1 - \beta = 1 - t$ we see that the definition of f being affine is the same as requiring

$$f((1-t)P_0 + tP_1) = (1-t)f(P_0) + tf(P_1)$$

for all $t \in \mathbb{F}$. This implies that f maps lines to lines.

Proposition 19. *Let $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ be an affine map and let P_0 and P_1 be distinct points of $\mathbb{A}^2$. Then f maps the point on the line through P_0 and P_1 onto the points on the line through $f(P_0)$ and $f(P_1)$.*

Problem 14. Prove this along the following line. Let ℓ be the line through P_0 and P_1. Let ℓ' be the line through $f(P_0)$ and $f(P_1)$. We wish to show that f maps the points of ℓ to the points of ℓ'.

(a) Show the points of ℓ are just the points of the form

$$P(t) = (1-t)P_0 + tP_1$$

with $t \in \mathbb{F}$. *Hint:* You just have to say that Proposition 15 applies.

(b) Show the points of ℓ' are all of the form

$$P'(t) = (1-t)f(P_0) + t(P_1)$$

with $t \in \mathbb{F}$. *Hint:* Again it is enough to just quite Proposition 15.

(c) Use that affine maps satisfy $f((1-t)P_0 + tP_1) = (1-t)f(P_0) + tf(P_1)$ to show

$$f(P(t)) = P'(t)$$

and explain why this shows that f maps the points of ℓ bijectively onto the points of ℓ'. □

Problem 15. Let $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ be an affine map such that

$$f(1,0) = (1,-1) \qquad \text{and} \qquad f(0,1) = (2,3).$$

(a) Let ℓ be the line through $(1,0)$ and $(0,1)$ and let ℓ' be the line through $f(1,0)$ and $f(0,1)$. Draw these two line on the same axis.
(b) Find $f(1/2,1/2)$ and label it on your graph. *Hint:* We have

$$(1/2,1/2) = \frac{1}{2}(1,0) + \frac{1}{2}(0,1)$$

and as f is affine

$$f(1/2,1/2) = \frac{1}{2}f(1,0) + \frac{1}{2}f(0,1).$$

(c) Find $f(3,-2)$ and label it on your graph. *Hint:* We have that $(3,-2)$ is an affine combination of $(1,0)$ and $(0,1)$ as

$$(3,-2) = 3(1,0) + (-2)(0,1)$$

and $3 + (-2) = 1$. Thus

$$f(3,-2) = 3f(1,0) - 2f(0,1).$$

(d) Find $f(-7,8)$ and label it on your graph. $\square$

Problem 16*.* Let $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ be an affine map with $f(1,2) = (3,4)$ and $f(-1,6) = (5,1)$.
(a) Draw ℓ, the line through $(1,2)$ and $(-1,6)$ and the line, ℓ', trough $f(1,2)$ and $f(-1,6)$ on the same axis.
(b) Find $f(0,4)$. Label $(0,4)$ (which should be on ℓ) $f(0,4)$ on your graph.
(c) Find $f(-7,18)$. Label $(-7,18)$ (which should be on ℓ) $f(-7,18)$ on your graph. $\square$

Proposition 20. *Let P and Q be points of $\mathbb{A}^2$ and M their midpoint. Then for any affine map $f\colon \mathbb{A}^2 \to \mathbb{A}^2$*

$$f(M) = \textit{midpoint of } f(P) \textit{ and } f(Q).$$

Problem 17*.* Prove this. $\square$

Proposition 21. *Let $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ be an affine map and P_0, P_1, P_2 any points of $\mathbb{A}^2$. Let $\alpha, \beta, \gamma \in \mathbb{F}$ be any scalars with*

$$\alpha + \beta + \gamma = 1.$$

Then

$$f(\alpha P_0 + \beta P_1 + \gamma P_1) = \alpha f(P_0) + \beta f(P_1) + \gamma f(P_2).$$

Problem 18*.* Prove this. *Hint:* If $\alpha = \beta = \gamma = 1$, then we have the contradiction that $\alpha + \beta + \gamma = 3 \neq 1$. So at least one of α, β, or γ is $\neq 1$. By possibly relabeling we can assume that $\gamma \neq 1$. Then write

$$\alpha P_0 + \beta P_1 + \gamma P_2 = (1-\gamma)\left(\frac{\alpha}{1-\gamma}P_0 + \frac{\beta}{1-\gamma}P_1\right) + \gamma P_2 = (1-\gamma)P + \gamma P_2$$

where

$$P = \frac{\alpha}{1-\gamma}P_0 + \frac{\beta}{1-\gamma}P_1.$$

Now explain why

$$f(\alpha P_0 + \beta P_1 + \gamma P_2) = (1-\gamma)f(P) + \gamma f(P_2).$$

Then show

$$\frac{\alpha}{1-\gamma} + \frac{\beta}{1-\gamma} = 1$$

and use this to show

$$f(P) = \frac{\alpha}{1-\gamma}P_o + \frac{\beta}{1-\gamma}P_1$$

and from there you should be able to finish the proof. □

Proposition 22. *Let $P_0, P_1, \dots, P_n$ be points in $\mathbb{A}^2$ and $\alpha_0, \alpha_1, \dots, \alpha_n$ scalars such that*

$$\alpha_0 + \alpha_1 + \cdots + \alpha_n = 1.$$

Then for any affine map $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ we have

$$f(\alpha_0 P_0 + \alpha_1 P_1 + \cdots + \alpha_n P_n) = \alpha_0 f(P_0) + \alpha_1 f(P_1) + \cdots + \alpha_n f(P_n).$$

Problem 19. Prove this. *Hint:* Use induction. Problem 18 shows what the induction step looks like going from $n = 2$ to $n = 3$. □

Definition 23. Let $\vec{b}$ be a vector in $\mathbb{F}^2$. Then the ***translation*** defined by $\vec{a}$ is the map $\tau_{\vec{a}}\colon \mathbb{A}^2 \to \mathbb{A}^2$ given by

$$\tau_{\vec{a}}(P) = P + \vec{a}.$$

□

If $\vec{a} = \langle a_1, a_2 \rangle$ then in coordinates we have that the translation defined by $\vec{a}$ is

$$\tau_{\vec{a}}(x, y) = (x + a_1, y + a_2).$$

Thus $\tau_{\vec{e}_1}(x, y) = (x + 1, y)$. This can be described be saying it moves all the points of the plane one unit to the right.

Problem 20. Give descriptions of what the translations $\tau_{\vec{a}}$ when $\vec{a} = \langle 0, 1 \rangle$, and $\vec{a} = \langle 1, 2 \rangle$.

Proposition 24. *Every translation is an affine map.*

Problem 21. Prove this. □

Proposition 25. *The following hold.*

(a) The translation $\tau_{\vec{0}}$ is the identity map on $\mathbb{A}^2$.

(b) For any two vectors $\vec{a}$ and $\vec{b}$ we have

$$\tau_{\vec{a}} \circ \tau_{\vec{b}} = \tau_{\vec{a}+\vec{b}}.$$

(c) The translations $\tau_{\vec{a}}\colon \mathbb{A}^2 \to \mathbb{A}^2$ is a bijection.

Problem 22. Prove this. □

More generally if A is a 2×2 matrix and $\vec{b}$ is a vector, then define a map $FA, \vec{b}\colon \mathbb{A}^2 \to \mathbb{A}^2$ be

$$F_{A,\vec{b}}(P) = AP + \vec{b}.$$

Proposition 26. *The map $F_{A,\vec{b}}$ is an affine map.*

Proof. Let α, β be scalars with $\alpha + \beta = 1$. Then for any points P and Q we have

$$\begin{aligned}
F_{A,\vec{b}}(\alpha P + \beta Q) &= A(\alpha P + \beta Q) + \vec{b} \\
&= A(\alpha P) + A(\beta Q) + \vec{b} && \begin{pmatrix}\text{Additive property of} \\ \text{matrix multiplication.}\end{pmatrix} \\
&= \alpha AP + \beta AQ + \vec{b} && \begin{pmatrix}\text{Anther property of} \\ \text{matrix multiplication.}\end{pmatrix} \\
&= \alpha AP + \beta AQ + (\alpha + \beta)\vec{b} && (\text{As } \alpha + \beta = 1) \\
&= \alpha(AP + \vec{b}) + \beta(AQ + \vec{b}) \\
&= \alpha F_{A,\vec{b}}(P) + \beta F_{A,\vec{b}}(Q)
\end{aligned}$$

which shows $F_{A,\vec{b}}$ is affine. $\square$

5. Affinely independent sets.

Definition 27. The points $A, B, C \in \mathbb{A}^2$ are ***affinely independent*** iff they are not collinear. That is if they do not all line on the same line. $\square$

Proposition 28. *The points $A, B, C \in \mathbb{A}^2$ are affinely independent if and only if the vectors $\vec{AB}$ and $\vec{AC}$ are linearly independent.*

Proof. We will prove the equivalent statement that $A, B, C \in \mathbb{A}^2$ are affinely dependent (that is collinear) if and only if $\vec{AB}$ and $\vec{AC}$ are linearly dependent.

First assume $\vec{AB}$ and $\vec{AC}$ are linearly dependent. Then one is a scalar multiple of the other, say $\vec{AB} = \lambda\vec{AC}$. This implies

$$B - A = \lambda(C - A).$$

This can be rearranged to give

$$B = (1 - \lambda)A + \lambda C$$

and therefore B is an affine combination of A and C. Thus by Proposition 15 B is on the line $\overleftrightarrow{AC}$, which shows that the three points are collinear.

Conversely assume that $A, B, C \in \mathbb{A}^2$ are affinely dependent, that is they are collinear. Then B is on the line through A and C and so by Proposition 15 B is an affine combination of A and B, that is

$$B = (1 - \lambda)A + \lambda C$$

for some scalar λ. But then

$$B - A = \lambda(C - A),$$

which shows that $\vec{AB}$ is a scalar multiple of $\vec{AC}$ and therefore these two vectors are linearly dependent. □

Theorem 29. *Let P_0, P_1, P_2 be affinely independent points. Then every point $P \in \mathbb{A}^2$ can be uniquely expressed as an affine combination of P_0, P_1, and P_2. Explicitly this means there are unique scalars α_0, α_1, and α_2 such that such*

$$P = \alpha_0 P_0 + \alpha_1 P_1 + \alpha_2 P_2 \qquad \text{and} \quad \alpha_0 + \alpha_1 + \alpha_2 = 1.$$

Proof. We first show the existence of the scalars α_0, α_1, and α_2. By Proposition 28 the vectors $\vec{P_0P_1}$ and $\vec{P_0P_1}$ are linearly independent and therefore by Theorem 9 every vector is a linear combination of these two vectors. In particular there are scalars α_1 and α_2 such that

$$\vec{P_0P} = \alpha_1 \vec{P_0P_1} + \alpha_2 \vec{P_0P_2}.$$

Using that $\vec{P_0P} = P - P_0$, $\vec{P_0P_1} = P_1 - P_0$, and $\vec{P_0P_2} = P_2 - P_1$ the last equation can be rewritten as

$$P = (1 - \alpha_1 - \alpha_2)P_0 + \alpha_1 P_1 + \alpha_2 P_2.$$

Letting $\alpha_0 = 1 - \alpha_1 - \alpha_2$ completes the proof of existence.

To prove uniqueness let

$$P = \alpha_0 P_0 + \alpha_1 P_1 + \alpha_2 P_2 = \beta_0 P_0 + \beta_1 P_1 + \beta_2 P_2$$

where

$$\alpha_0 + \alpha_1 + \alpha_2 = \beta_0 + \beta_1 + \beta_2 = 1.$$

We can thus rewrite as

$$P = (1 - \alpha_1 - \alpha_2)P_0 + \alpha_1 P_1 + \alpha_2 P_2 = (1 - \beta_2 - \beta_2)P_0 + \beta_1 P_1 + \beta_2 P_2$$

which in turn can be rewritten as

$$(\alpha_1 - \beta_1)(P_1 - P_0) + (\alpha_2 - \beta_2)(P_2 - P_1) = \vec{0},$$

that is

$$(\alpha_1 - \beta_1)\vec{P_0P_1} + (\alpha_2 - \beta_2)\vec{P_2P_0} = \vec{0}.$$

By Proposition 28 the vectors $\vec{P_0P_1}$ and $\vec{P_0P_2}$ are linearly independent so this implies

$$(\alpha_1 - \beta_1) = (\alpha_2 - \beta_2).$$

That is $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$. Then

$$\alpha_0 = 1 - \alpha_1 + \alpha_2 = 1 - \beta_1 - \beta_2 = \beta_0$$

which completes the proof of uniqueness. □

This has a nice corollary, that is important enough to be called a theorem.

Theorem 30. *Let $f, g\colon \mathbb{A}^2 \to \mathbb{A}^2$ be affine maps and P_0, P_1, P_2 affinely independent points. If f and g agree on P_0, P_1, and P_2, that is*

$$f(P_0) = g(P_0), \qquad f(P_1) = g(P_1), \qquad f(P_2) = g(P_2)$$

then $f(P) = g(P)$ for all $P \in \mathbb{A}^2$. (Concisely: If two affine maps agree on three affinely independent points, then the maps are equal.)

Problem 23. Prove this. *Hint:* Let $P \in \mathbb{A}^2$. By Theorem 29 there scalars $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{F}$ such that

$$P = \alpha_0 P_0 + \alpha_1 P_1 + \alpha_2 P_2.$$

By Proposition 21 and the hypothesis

$$f(P) = \alpha_0 f(P_0) + \alpha)_1 f(P_1) + \alpha_2 f(P_2) = \alpha_0 g(P_0) + \alpha_1 g(P_1) + \alpha_2 g(P_2)$$

and the rest should be easy. □

We now show the existence affine maps that map affinely independent sets anywhere we want. Recall that if A is a 2×2 matrix and $\vec{b}$ is a vector, then the map $F_{A,\vec{b}}$ defined by

$$F_{A,\vec{b}}(P) = AP + \vec{b}$$

is an affine map (see Proposition 26).

Theorem 31. *Let P_0, P_1, P_2 be an affinely independent set and P_0', P_1', and P_2' any points of $\mathbb{A}^2$. Then there is a matrix and a vector $\vec{b}$ such that*

$$F_{A,\vec{b}}(P_0) = P_0', \qquad F_{A,\vec{b}}(P_1) = P_1', \qquad F_{A,\vec{b}}(P_2) = P_2'$$

Proof. We are looking for a matrix A and a vector $\vec{b}$ such that

$$AP_0 + \vec{b} = P_0' \tag{4}$$

$$AP_1 + \vec{b} = P_1' \tag{5}$$

$$AP_2 + \vec{b} = P_2' \tag{6}$$

Subtracting (4) from equations (5) and (6) gives

$$A(P_1 - P_0) = P_1' - P_0' \tag{7}$$

$$A(P_2 - P_0) = P_2' - P_0' \tag{8}$$

As the vector P_0, P_1, P_2 are linearly independent Proposition 28 yields that the vectors $\vec{P_0P_1} = (P_1 - P_0)$ and $\vec{P_2P_0} = (P_2 - P_0$ are linearly independent. Therefore by Theorem 14 there is a matrix A such that equations (7) and (8) hold. Now set

$$\vec{b} = P_0' - AP_0.$$

We now show that this matrix and vector work. First

$$F_{A,\vec{b}}(P_0) = AP_0 + (P_0' - AP_0) = P_0'.$$

Next we use equation (7)

$$F_{A,\vec{b}}(P_1) = AP_1 + P_0' - AP_0 = A(P_1 - P_0) + P_0' = P_1' - P_0' + P_0' = P_1'$$

A similar calculation shows $F_{A,\vec{b}}(P_2) = P_2'$. □

Theorem 32. *Let $f\colon \mathbb{A}^2 \to \mathbb{A}^2$. Then there is a matrix A and a vector $\vec{b}$ such that $f = F_{A,\vec{b}}$. That is every affine map $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ is one of the $F_{A,\vec{b}}$'s.*

Problem 24. Prove this. *Hint:* Let P_0, P_1, and P_2 be any affinely independent points in $\mathbb{A}^2$. Let $P_0' = f(P_0)$, $P_1' = f(P_1)$, and $P_2' = f(P_2)$. By Theorem there are A and $\vec{b}$ such that $F_{A,\vec{b}}(P_j) = P_j' = f(P_j)$ for $j = 0, 1, 2$. Now note that $F_{A,\vec{b}}$ is affine and so you can use Theorem 30 to conclude that $f = F_{A,\vec{b}}$. □

We now note that affine bijections $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ preserve all affine properties of figures. Here are some results that make this precise.

Proposition 33. *Let $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ be a bijective map affine map. Then the inverse of f is also an affine map.*

Problem 25. Prove this. *Hint:* We need to show that for any point A and B and any scalars α and β with $\alpha + \beta = 1$ that

$$f^{-1}(\alpha A + \beta B) = \alpha f^{-1}(A) + \beta f^{-1}(B)$$

holds. We know that for any P and Q that

$$f(\alpha P + \beta Q) = \alpha f(P) + \beta f(Q)$$

Holds. In this equation let $P = f^{-1}(A)$ and $Q = f^{-1}(B)$ and use $f(f^{-1}(A) = A$ and $(f^{-1}(B) = B$ to get

$$f\Big(\alpha f^{-1}(A) + \beta f^{-1}(B)\Big) = \alpha A + \beta B.$$

Now apply f^{-1} to both sides. □

Proposition 34. *Let $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ be an affine bijection. The the points A, B and C are collinear if and only if the points $f(A)$, $f(B)$, and $f(C)$ are collinear.*

Problem 26. Prove this. *Hint:* Three points are collinear if and only if one of then can be expressed as an affine combination of the other two and affine maps preserve affine combinations. □

Proposition 35. *Let ℓ and m be lines in $\mathbb{A}^2$ and let $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ be an affine bijection. Let $f[\ell] = \{f(P) : P \in \ell\}$ and $f[m] = \{f(P) : P \in m\}$. Then ℓ and m are parallel if and only if $f[\ell]$ and $f[m]$ are parallel.*

Problem 27. Prove this. □

Definition 36. If $A, B, C \in \mathbb{A}^2$ then the ***center of mass*** of these points is

$$M = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C.$$

(Note this is an affine combination of the points.)

Proposition 37. *Let A, B, C have center of mass M and $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ and affine map. Then*

$$f(M) = \textit{Center of mass of } f(A),\ f(B), \textit{ and } f(C).$$

Put a little differently if M' is the center of mass of $f(A)$, $f(B)$, and $f(C)$ then

$$f(M) = M'$$

Problem 28. Prove this. □

Problem 29. Let $P_1, P_2, \ldots, P_n$ be n points in $\mathbb{A}^2$.

(a) Define M is the center of mass of these points.
(b) Let $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ be an affine map. Give a precise statement "f maps the center of mass of set of points to the center of mass of their images under f" and prove your version. □

Problem 30. Recall that a median of a triangle is a line that goes through vertex of the triangle and the midpoint of the opposite side.

(a) Draw some pictures of medians of triangles.
(b) Show that if A, B and C are the vertices of the triangle $\triangle ABC$ and that m_1, m_2, and m_3 are the medians of this triangle, then for any bijective affine map $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ the lines $f[m_1]$, $f[m_2]$ and $f[m_3]$ are the medians of $\triangle f(A)f(B)f(C)$. □

Problem 31. Show that if there is any one triangle $\triangle ABC$ such that the medians of $\triangle ABC$ all go through a point, then for *every* triangle $\triangle PQR$ that the medians of $\triangle PQR$ all go through a point. □

Proposition 38. *Any two triangles are affinely equivalent in the sense that if $\triangle ABC$ and $\triangle PQR$ are triangles, then there is an affine bijection $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ such that $f(A) = P$, $f(B) = Q$, $f(C) = R$. (We assume that for three points to form a triangle that they are not collinear.)*

Problem 32. Prove this. *Hint:* This is more or less a straightforward consequence of Theorem 31 □

We can now put this all together to get a theorem.

Theorem 39. *In an triangle $\triangle ABC$ (with A, B, and C not colinear) the meadians of the triangle all go through the center of mass of $\triangle ABC$.*

Proof. See Figure 2. □

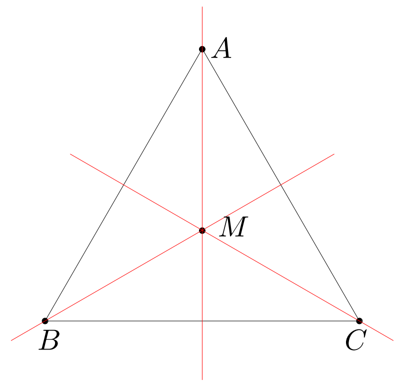

FIGURE 2. In an equilateral triangle it is more or less obvious that all the medians (in red) pass pass through the center of mass, M, of the vertices. In light of Proposition 38, Problem 30, and Proposition 37 this implies that in every triangle the medians all pass through the center of mass of the vertices.

Problem 33. We say that the points $ABCD$ form a ***parallelogram*** iff ABC are not collinear and

$$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD} \qquad \text{and} \qquad \overleftrightarrow{BC} \parallel \overleftrightarrow{AD}.$$

We denote a parallel gram by $\square ABCD$. Show that any two parallelograms are affinely equivalent in the sense it $\square ABCD$ and $\square A'B'C'D'$ are parallelograms, then there is an affine bijection $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ with

$$f(A) = A', \qquad f(B) = f(B'), \qquad f(C) = C', \qquad f(D) = D'.$$

Hint: By Theorem 31 there is an affine map $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ with $f(A) = A'$,

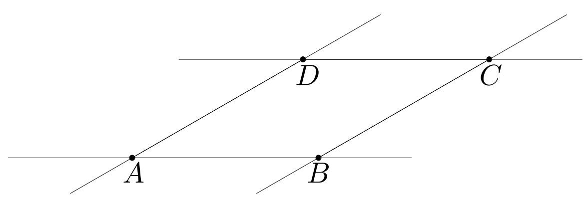

FIGURE 3

$f(B) = f(B')$, and $f(C) = C'$. Now argue that $f(D) = f(D')$. One way to start is to note that $f[\overleftrightarrow{AD}] = \overleftrightarrow{f(A)f(D)}$ is a line through $f(A)$ that is parallel to $\overleftrightarrow{f(B)f(C)} = f[\overleftrightarrow{BC}]$ and that $f[\overleftrightarrow{CD}] = \overleftrightarrow{f(C)f(D)}]$ is a line through $f(C)$ parallel to $\overleftrightarrow{f(A)f(B)} = f[\overleftrightarrow{AB}]$. $\square$

Problem 34. Consider the standard square (see Figure 4).

(a) Use that all parallelograms are affinely equivalent to the square to explain why the diagonals of any parallelogram intersect at the center of mass of the vertices.

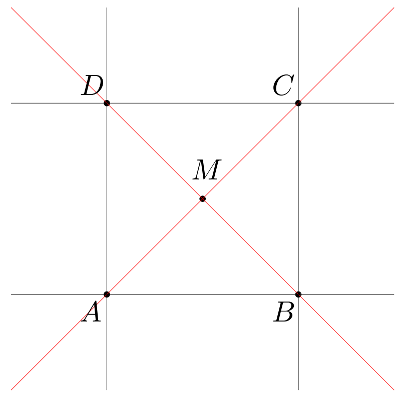

FIGURE 4. In the standard square the diagonals (in red) intersect in the center of mass, M, of the vertices. By Problem 33 all parallel grams are affinely equivalent to the square. Therefore in all parallelograms the diagonals will intersect at the center of mass of the vertices.

(b) Use that all parallelograms are affinely equivalent to the square to explain why the midpoint of a diagonal is the same as the center of mass of the vertices in every parallelogram. □

Proposition 40. *Let $f, g\colon \mathbb{A}^2 \to \mathbb{A}^2$ be affine maps. Then the composition $f \circ g\colon \mathbb{A}^2 \to \mathbb{A}^2$ is also an affine map.*

Problem 35*.* Prove this. □

Remark 41. If we let G be the set of all affine bijections $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ then Proposition 40 tells us that G is closed under composition and Proposition 33 that G is closed under taking inverses. Therefore G is a group in the sense of abstract algebra (see Math 546). In more advanced presentations of affine geometry this group plays a prominent role. □

6. Desargues' Theorem first version.

Here we are going to prove what looks like a rather special theorem, but we will see that it can be generalized to some surprising results, once we know some projective geometry. The proof we give is nice in that it will let us review much of what we have already done. We will start with the usual coordinate plane. Let x be the x-axis, y the y-axis and δ the line $y = x$. That is in our notation $\delta = L(1, -1, 0)$. We give some names to some points

$$A = (1, 0), \qquad B = (0, 1), \qquad C = (c, c)$$

where c can be any scalar.

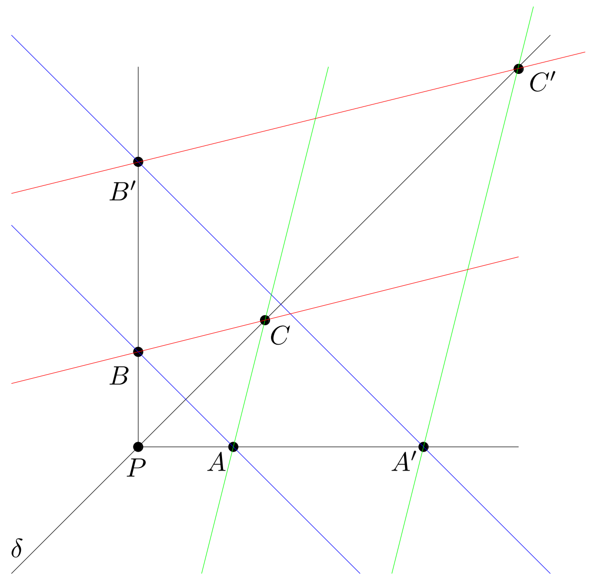

FIGURE 5. Our goal is to prove $\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'}$ and $\overleftrightarrow{AC} \parallel \overleftrightarrow{A'C'}$ implies $\overleftrightarrow{BC} \parallel \overleftrightarrow{B'C'}$. We will do this by using that two lines are parallel if and only if they have the same slope. This will let us find equations relating c, a', b' and c' and finally let us show that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ have the same slope. In this figure figure

$$\begin{aligned}
P &= (0,0) && \text{(the origin.)} \\
A &= (1,0) \\
B &= (0,1) \\
C &= (c_1, c_2) \\
A' &= (a', 0) \\
B' &= (0, b') \\
C' &= (c_1', c_2')
\end{aligned}$$

Problem 36. Referring to Figure 5 we assume

$$\overleftrightarrow{AB} \parallel \overleftrightarrow{A'B'} \qquad \text{and} \qquad \overleftrightarrow{AC} \parallel \overleftrightarrow{A'C'}.$$

and want to prove

$$\overleftrightarrow{BC} \parallel \overleftrightarrow{B'C'}.$$

(a) Using the coordinates of the points involved show

$$\text{Slope of } \overleftrightarrow{BC} = \frac{c_2 - 1}{c_1} \qquad \text{and} \qquad \text{Slope of } \overleftrightarrow{B'C'} = \frac{c_2' - b'}{c_1'}.$$

Therefore what we need to show is that

$$\frac{c_2 - 1}{c_1} = \frac{c_2' - b'}{c_2'}. \tag{9}$$

(b) Show

$$\text{Slope of } \overleftrightarrow{AB} = -1$$

and use this and $\overleftrightarrow{A'B'} \parallel \overleftrightarrow{AB}$ to show

$$a' = b'. \tag{10}$$

(c) Use that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ have the same slope to show

$$\frac{c_2}{c_1 - 1} = \frac{c_2'}{c_1' - a'}$$

(d) Use that $(0,0)$, (c_1, c_2), and (c_1', c_2') are collinear to show

$$\frac{c_2}{c_1} = \frac{c_2'}{c_1'}.$$

(e) Use Parts (c) and (d) and some algebra to show

$$\frac{c_2 - 1}{c_1} = \frac{c_2' - a'}{c_2'}. \tag{11}$$

Hint: The equation of part (d) implies $c_1 c_2' = c_1' c_2$. Cross multiply in the equation of Part (c) to get $c_2 c_1' - a' c_2 = c_2' c_1 - c_2'$ and therefore that $c_2' = a' c_2$. Then use part (c) to show $c_1' = a' c_1$. Plug these into $\frac{c_2' - a'}{c_2'}$ and cancel to complete the argument.

(f) Combine equations (10) and (11) to show that (9) holds and complete the proof that $\overleftrightarrow{AC} \parallel \overleftrightarrow{A'C'}$. □

Problem 37. In the last problem we divided by $c_1 - 1$, that is when $c_1 = 1$. So the proof does not work in that case. So find a proof for the special case where $c_1 = 1$. □

Theorem 42 (A special case of Desargues' Theorem)**.** *Let $\triangle A_0 B_0 C_0$ and $\triangle A_0' B_0' C_0'$. be so that the lines $\overleftrightarrow{A_0 A_0'}$, $\overleftrightarrow{B_0 B_0'}$, $\overleftrightarrow{C_0 C_0'}$, all go through the point P_0. Assume*

$$\overleftrightarrow{A_0 B_0} \parallel \overleftrightarrow{A_0' B_0'} \qquad \text{and} \qquad \overleftrightarrow{A_0 C_0} \parallel \overleftrightarrow{A_0' C_0'}.$$

Then

$$\overleftrightarrow{B_0 C_0} \parallel \overleftrightarrow{B_0' C_0'}.$$

(See Figure 6.)

Problem 38. Prove Theorem 42 along the lines suggested by Figure 6. □

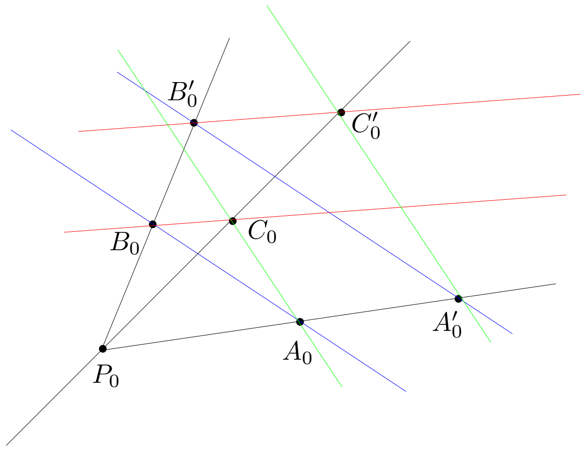

Figure 6. We are assuming $\overleftrightarrow{A_0B_0} \parallel \overleftrightarrow{A_0'B_0'}$ and $\overleftrightarrow{A_0C_0} \parallel \overleftrightarrow{A_0'C_0'}$ and wish to conclude that $\overleftrightarrow{B_0C_0} \parallel \overleftrightarrow{B_0'C_0'}$. Towards this and using the notation of Figure 5 (that is $P = (0,0)$, $A = (1,0)$, $B = (0,1)$) use Theorem 31 to find an affine bijection $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ with

$$f(P_0) = (0,0) = P, \quad f(A_0) = A, \quad f(B_0) = B.$$

Now set $C = f(C_0)$, $A' = f(A_0')$, $B' = f(B_0')$, $C' = f(C_0')$. Then the points P, A, B, C, A', B', C' are in the set up of Problem 36 and therefore we can conclude that $\overleftrightarrow{AC} \parallel \overleftrightarrow{A'B'}$. This can be transfered back to the original setup to conclude $\overleftrightarrow{B_0C_0} \parallel \overleftrightarrow{B_0'C_0'}$.