Math 532/736I **Test 1** Name:____________________

Show your work to get credit.

1. (a) Define ℓ and m are ***parallel lines***.

Solution: Either $\ell = m$ or $\ell \cap m = \varnothing$. $\square$

(b) State the ***First Axiom of Affine Geometry***.

Solution: For any two points P and Q with $P \neq Q$ there is a unique line ℓ that is incidence with both P and Q. $\square$

(c) State the ***Second Axiom of Affine Geometry***.

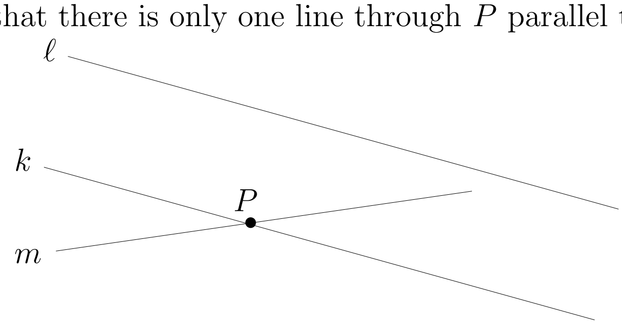
Solution: Given a line, ℓ, and a point, P, not on ℓ there is a unique line through P and parallel to ℓ. $\square$

(d) State the ***Third Axiom of Affine Geometry***.

Solution: There exist four points no three of which are on the same line. $\square$

2. Let k and ℓ be parallel lines. Show if m is a line, $m \neq k$ and m intersects k, then m also intersects ℓ.

Solution: As $m \neq k$ and these two lines intersect we see that they only intersect in one point (by the first axiom of affine geometry). Let $P = k \cap m$. Towards a contradiction assume that m does not intersect ℓ. Then m is parallel to ℓ (as $\ell \cap m = \varnothing$) and k is parallel to ℓ (given). Thus the point P has two line through it that are parallel to ℓ which contradicts the second axiom of affine geometry which tells us that there is only one line through P parallel to ℓ.

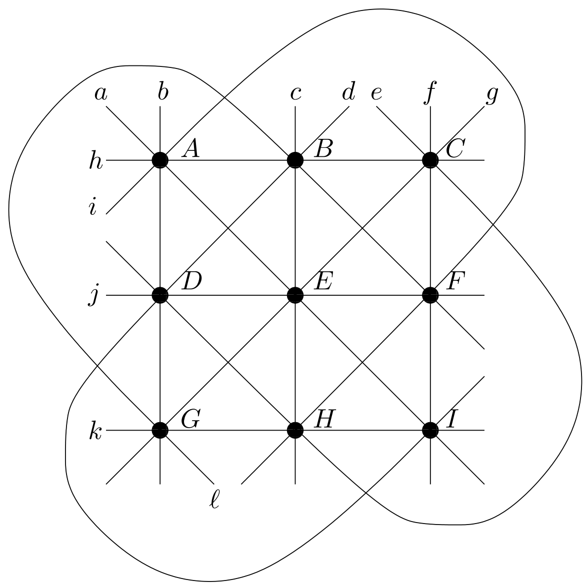


$\square$

3. In the figure below find the following

The line through F and parallel to g: $\underline{\quad i \quad}$

All lines parallel to d: $\underline{\quad d,\ g,\ i \quad}$



4. (a) Define the vector $\vec{a}$ and $\vec{b}$ are ***linearly independent***.

Solution: If α and β are scalars and

$$\alpha\vec{a} + \beta\vec{b} = \vec{0}$$

then $\alpha = \beta = 0$. ☐

(b) Show that the vector $\vec{a} = \langle 1, 0\rangle$ and $\vec{b} = \langle 1, 1\rangle$ are linearly independent.

Solution: Let α and β be scalars with

$$\alpha\vec{a} + \beta\vec{b} = \vec{0}$$

that is

$$\alpha\langle 1, 0\rangle + \beta\langle 1, 1\rangle = \langle 0, 0\rangle.$$

This is the same as the system of equations

$$\alpha + \beta = 0, \qquad \beta = 0.$$

Thus $\beta = 0$ and using this in $\alpha + \beta = 0$ gives $\alpha = 0$. Which is just what we needed to show that $\vec{a}$ and $\vec{b}$ are linearly independent. ☐

Alternate Solution: Let α and β be scalars with

$$\alpha\vec{a} + \beta\vec{b} = \vec{0}$$

that is

$$\alpha\langle 1, 0\rangle + \beta\langle 1, 1\rangle = \langle 0, 0\rangle.$$

This is the same as the system of equations

$$\begin{aligned} \alpha + \beta &= 0 \\ \beta &= 0. \end{aligned}$$

The determinant of this system is

$$\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 1 \cdot 0 = 1 \neq 0.$$

Therefore by Cramer's rule the only solution is $\alpha = \beta = 0$. And this is just what we needed to show that $\vec{a}$ and $\vec{b}$ are linearly independent. $\square$

Yet anther Solution: We have a theorem that tells us that two vectors $\vec{a}$ and $\vec{b}$ are linearly independent if and only if $\det(\vec{a}, \vec{b}) \neq 0$. In our case

$$\det(\vec{a}, \vec{b}) = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \neq 0$$

so the vectors are linearly independent. $\square$

5. Let a, b, and c be real numbers with at least one of a or b nonzero. Call subsets of $\mathbb{R}^2$ of the form

$$L(a, b, c) = \{(x, y) \in \mathbb{R}^2 : ax + by + c = 0\}$$

lines.

(a) Find the line through the points $P = (4, 5)$ and $Q = (6, 3)$.

Solution: A vector parallel to the line will be

$$Q - P = \langle 6 - 4, 3 - 5 \rangle = \langle 2, -2 \rangle.$$

Therefore a vector perpendicular to the line will be

$$\vec{n} = \langle -(-2), 2 \rangle = \langle 2, 2 \rangle.$$

So the equation of the line is

$$2(x - 4) + 2(y - 5) = 0$$

which can be rewritten as

$$2x + 2y - 18 = 0.$$

Therefore the line is $\underline{L(2, 2, -18)}$

As a check (and checking answers is always a good idea) we note

$$2(4) + 2(5) - 18 = 0, \qquad 2(6) + 2(3) - 18 = 0$$

which shows the points P and Q are indeed on the line $L(2, 2, -18)$. $\square$

(b) Given a line $\ell = L(a, b, c)$ find the line parallel to ℓ and through the point $P = (p_1, p_2)$.

Solution: The lines parallel to $\ell = L(a, b, c)$ are of the form $L(a, b, c')$, so all we have to do is find c'. The point P will be on this line if and only if

$$ap_1 + bp_2 + c' = 0.$$

Solve for c' to get

$$c' = -ap_1 - bp_2.$$

Thus the line is $\underline{L(a, b, -ap_1 - bp_2)}$ $\square$

(c) If $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ are points on the line $L(a, b, c)$. Define the ***midpoint*** between P and Q to be

$$M = \frac{1}{2}P + \frac{1}{2}Q.$$

Prove M is also on the line $L(a, b, c)$.

Solution: As the point P and Q are one the line we have

$$ap_1 + bp_2 + c = 0 \tag{1}$$
$$aq_1 + bq_2 + c = 0. \tag{2}$$

The point $M = (m_1, m_2)$ is

$$M = (m_1, m_2) = \frac{1}{2}P + \frac{1}{2}Q = \left(\frac{p_1}{2}, \frac{p_2}{2}\right) + \left(\frac{q_1}{2}, \frac{q_2}{2}\right) = \left(\frac{p_1 + q_1}{2}, \frac{p_2 + q_2}{2}\right)$$

Now check if the point M is on the line $L(a, b, c)$.

$$\begin{aligned} am_1 + bm_2 + c &= a\frac{p_1 + q_1}{2} + b\frac{p_2 + q_2}{2} + c \\ &= \frac{1}{2}(ap_1 + bp_2) + \frac{1}{2}(aq_1 + bq_2) + c \\ &= \frac{1}{2}(-c) + \frac{1}{2}(-c) + c && \text{(From Equations (1) and (2))} \\ &= 0 \end{aligned}$$

□

6. (a) State ***Cramer's rule*** about solutions of the system of equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f. \end{aligned}$$

Solution: If the determinant of the system is nonzero, that is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$$

then the system has a unique solution. This solution is given by

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

□

(b) If $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$, define $\det(\vec{a}, \vec{b})$.

Solution:

$$\det(\vec{a}, \vec{b}) = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

□

(c) Show that if $\det(\vec{a}, \vec{b}) \neq 0$, then for any vector $\vec{v}$ there are scalars α and β such that

$$\alpha\vec{a} + \beta\vec{b} = \vec{v}.$$

Solution: Let $\vec{a} = \langle a_1, a_2 \rangle$, $\vec{b} = \langle b_1, b_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$. Then we wish to solve $\alpha\vec{a} + \beta\vec{b} = \vec{v}$ for α and β. That is we want to solve

$$\alpha\langle a_1, a_2 \rangle + \beta\langle b_1, b_2 \rangle = \langle v_1, v_2 \rangle$$

which is equivalent to the system of two scalar equations:

$$\begin{aligned} a_1\alpha + b_1\beta &= v_1 \\ a_2\alpha + b_2\beta &= v_2. \end{aligned}$$

where α and β are the unknowns. The determinant of this system is

$$a_1 b_2 - a_2 b_1 = \det(\vec{a}, \vec{b}) \neq 0.$$

So by Cramer's rule this has a unique solution. $\square$

7. (a) Define the vectors $\vec{a}$ and $\vec{b}$ are ***linearly dependent***.

Solution: There exist scalars α and β not both zero such that

$$\alpha\vec{a} + \beta\vec{b} = \vec{0}.$$

$\square$

(b) Show for any vector $\vec{c}$, the vectors $\vec{a} = 3\vec{c}$ and $\vec{b} = -2\vec{c}$ are linearly dependent.

Solution: Let $\alpha = 2$ and $\beta = 3$. These are not zero and

$$\alpha\vec{a} + \beta\vec{b} = 2(3)\vec{c} + 3(-2)\vec{c} = \vec{0}.$$

Thus they are linearly dependent. $\square$