Axioms and Basic Results in Plane Geometry

Undefined Terms

Point, line, incidence, between, congruent.

Incidence Axioms

IA 1: For any pair of distinct points P and Q there is a unique line through P and Q.
IA 2: Any line has at least two points on it.
IA 3: There are three points A, B, C not all on the same line.

Betweenness Axioms

BA 1: If $A * B * C$, then A, B, and C are three distinct points all lying on the same line, and $C * B * A$.

BA 2: Given any two points B and D, there exist points A, C, and E on $\overleftrightarrow{BD}$ so that $A*B*D$, $B * C * D$, and $B * D * E$.

BA 3: If A, B, C are distinct points on the same line, then exactly one of them is between the other two.

Definition: If A and B are distinct points, then the **segment** $\overline{AB}$ is the set of points X so that $X = A$, $X = B$, or $A * X * B$.

Definition: If A and B are distinct points, then the **ray** $\overrightarrow{AB}$ is the segment $\overline{AB}$ together with the set of points X so that $A * B * X$.

Definition: If ℓ is a line and A, B are points not on ℓ, then A and B are on the **same side** of ℓ iff $A = B$ or the segment $\overline{AB}$ does not meet ℓ. They are on opposite sides of ℓ if $A \neq B$ and $\overline{AB}$ intersects ℓ.

BA 4 (Plane Separation): For any line ℓ and points A, B, and C not on ℓ:
(i) if A and B are one the same side of ℓ and B and C are on the same side of ℓ, then A and C are on the same side of ℓ
(ii) if A and B are on opposite sides of ℓ and B and C are on opposite sides of ℓ, then A and C are on the same side of ℓ.

Congruence Axioms

CA 1: If A and B are distinct points and if A' is any point, then for each ray $\overrightarrow{r}$ starting at A' there is a unique point B' on $\overrightarrow{r}$ so that $B' \neq A'$ and $\overline{AB} \cong \overline{A'B'}$.

CA 2: If $\overline{AB} \cong \overline{CD}$ and $\overline{AB} \cong \overline{EF}$, then $\overline{CD} \cong \overline{EF}$. If $\overline{AB} \cong \overline{CD}$ then $\overline{CD} \cong \overline{AB}$. Also any segment is congruent to itself.

CA 3: If $A * B * C$, $A' * B' * C'$, $\overline{AB} \cong \overline{A'B'}$, and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$

CA 4: If Given any angle $\sphericalangle BAC$ (where, by definition of angle, $\overrightarrow{AB}$ is not opposite $\overrightarrow{AC}$), and given any ray $\overrightarrow{A'B'}$, there is a unique ray $\overrightarrow{A'C'}$ on a given side of the line $\overleftrightarrow{A'B'}$ so that $\sphericalangle B'A'C' \cong \sphericalangle ABC$.

CA 5: If $\sphericalangle A \cong \sphericalangle B$ and $\sphericalangle A \cong \sphericalangle C$, then $\sphericalangle A \cong B$. If $\sphericalangle A \cong \sphericalangle B$, then $\sphericalangle B \cong \sphericalangle A$. Also any angle is congruent to itself.

Definition: Two triangles $\triangle ABC$ and $\triangle DEF$ are **congruent** (written $\triangle ABC \cong \triangle DEF$) iff $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, $\overline{AC} \cong \overline{DF}$, $\sphericalangle ABC \cong \sphericalangle DEF$, $\sphericalangle BCA \cong \sphericalangle EFD$, and $\sphericalangle CBA \cong \sphericalangle FDE$.

CA 6 (SAS): If two sides and the included angle of a triangle are congruent respectively to two sides and the included angle of anther triangle, then the two triangles are congruent.

Axioms of Continuity

Circular Continuity Principle: If a circle γ has on point inside and one point outside of anther circle γ', then the two circles interest in two points.

Elementary Continuity Principle: If one endpoint of a circle is inside of a circle and the other is outside, then the segment interests the circle.

Archimedes' Axiom: If $\overline{AB}$ and $\overline{CD}$ are any segments, then there is number n so that if segment $\overrightarrow{CD}$ is laid off n times on the ray $\overrightarrow{AB}$, then a point E on $\overrightarrow{AB}$ is reached where $n \cdot CD \cong AE$ and $A * B * E$.

Aristotle's Axiom: Given and side of an acute angle and any segment $\overline{AB}$, there is a point Y on the given side of the angle such that if X is the foot of the perpendicular from Y to the other side of the angle, $\overline{XY} > \overline{AB}$.

Important Corollary to Aristotle's Axiom: Let $\overrightarrow{AB}$ be any ray, P an point not colinear with A and B, and $\sphericalangle XVY$ any acute angle. Then there exists a point R on the ray $\overrightarrow{AB}$ such that $\sphericalangle PRA < \sphericalangle XVY$.

Dedekind's Axiom: Suppose the set of all points of a line ℓ is the union $\Sigma_1 \cup \Sigma_2$ of two nonempty subsets such that no point of Σ_1 is between two points of Σ_2 and no point of Σ_2 is between two points of Σ_1. Then there is a unique point O on ℓ so that if $P_1 \in \Sigma_1$, $P_2 \in \Sigma_2$, with P_1, P_2, and O distinct, then $P_1 * O * P_2$.

Axiom of Parallelism

Hilbert's Parallel Axiom: For every line ℓ and every point P not on ℓ there is at most one line m through P and parallel to ℓ.

Basic Results About Incidence

Prop 2.1: If ℓ and m are distinct lines that are not parallel, then ℓ and m have exactly one point in common.

Definition: A set of lines $\mathcal{S}$ is **concurrent** if there is a point P so that every member of ℓ of $\mathcal{S}$ passes through P.

Prop 2.2: There exist three distinct lines that are not concurrent.

Prop 2.3: For every line there is a point not lying on it.

Prop 2.4: For every point there is at least one line not passing through it.

Prop 2.5: For every point P there exists at least two distinct lines that pass through P.

Basic Results About Betweenness

Prop 3.1: For every pair of distinct points A, B:

(i) $\overrightarrow{AB} \cap \overrightarrow{BA} = \overline{AB}$ (ii) $\overrightarrow{AB} \cup \overrightarrow{BA} = \overleftrightarrow{AB}$

Definition: Let ℓ be a line, and P a point not on ℓ. Then the **half plane** determined by ℓ and P is the set of points X that are on the same side of ℓ as P. The half plane determined by ℓ and P will also be called the **side** of ℓ of P. The line ℓ is said to **bound** any of its half planes.

Prop 3.2: Every line bounds exactly two half-planes and these half planes have no point in common.

Prop 3.3: If $A * B * C$ and $A * C * D$, then $B * C * D$ and $A * B * D$.

Corollary: If $A * B * C$ and $B * C * D$, then $A * B * D$ and $A * C * D$.

Prop 3.4 (Line separation property): If $C * A * B$ and ℓ is the line through A, B and C, then for every point P lying on ℓ, either P lies on the ray $\overrightarrow{AB}$ or the opposite ray $\overrightarrow{AC}$.

Pasch's Theorem: If $\triangle ABC$ is any triangle and ℓ is any line intersecting side $\overline{AB}$ in a point between A and B, then ℓ also intersects either side $\overline{AC}$ or side $\overline{BC}$. If C does not lie on ℓ, then ℓ does not intersect both $\overline{AC}$ and $\overline{BC}$.

Prop 3.5: Given $A * B * C$, then $\overline{AC} = \overline{AB} \cup \overline{BC}$ and B is the only point common to segments $\overline{AB}$ and $\overline{BC}$.

Prop 3.6: Given $A * B * C$. Then B is the only point common to the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$. Also $\overrightarrow{AB} = \overrightarrow{AC}$.

Definition: Given an angle $\sphericalangle CAB$, define a point D to be in the **interior** of $\sphericalangle CAB$ iff D is on the same side of $\overleftrightarrow{AC}$ as B and D is on the same side of $\overleftrightarrow{AB}$ as C.

Prop 3.7: Given an angle $\sphericalangle CAB$ and a point D lying on the line $\overleftrightarrow{BC}$. Then D is in the interior of $\sphericalangle CAB$ if and only if $B * D * C$.

Prop 3.8: If D is in the interior of $\sphericalangle CAB$, then;

(a) so is every point on $\overrightarrow{AD}$ except A,

(b) no point on the opposite ray to $\overrightarrow{AD}$ is in the interior of $\sphericalangle CAB$,

(c) if $C * A * E$, then B is in the interior of $\sphericalangle DAE$.

Definition: The ray $\overrightarrow{AD}$ is **between** rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ iff $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are not opposite rays and D is interior to angle $\sphericalangle CAB$.

Crossbar Theorem: If $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$, then $\overrightarrow{AD}$ intersects the segment $\overline{BC}$.

Definition: The **interior** of a triangle is the intersection of the interiors of its three angles. A point is in the **exterior** of the triangle iff it is not in the interior of the triangle and does not lie on any of the sides of the triangle.

Prop 3.9: (a) If a ray $\overrightarrow{r}$ emanating from an exterior point of $\triangle ABC$ intersects side $\overline{AB}$ at a point between A and B, then also $\overrightarrow{r}$ intersects side $\overline{AC}$ or $\overline{BC}$.

(b) If a ray emanates from an interior point of $\triangle ABC$, then if intersects one of the sides of $\triangle ABC$. If the ray does not go through a vertex of the triangle, then it only interests one of the sides of $\triangle ABE$.

Basic Results About Congruence

Corollary to SAS: Given $\triangle ABC$ and segment $\overline{DE} \cong \overline{AB}$, there is a unique point F on a given side of the line $\overleftrightarrow{DE}$ such that $\triangle ABC \cong \triangle DEF$.

Prop 3.10: If in triangle $\triangle ABC$ we have $\overline{AB} \cong \overline{AC}$, then $\sphericalangle B \cong \sphericalangle C$.

Prop 3.11 (Segment Subtraction): If $A * B * C$, $D * E * F$, $\overline{AB} \cong \overline{DE}$, and $\overline{AC} \cong \overline{DF}$, then $\overline{BC} \cong \overline{EF}$.

Prop 3.12: Given $\overline{AC} \cong \overline{DF}$, then for any point B between A and C, there is a unique point E between D and F so that $\overline{AB} \cong \overline{DE}$.

Definition: $\overline{AB} < \overline{CD}$ means there is a point E between C and D with $\overline{AB} \cong \overline{CE}$.

Prop 3.13:(Segment Ordering) (a) Exactly one of the following holds (trichotomy): $\overline{AB} < \overline{CD}$, $\overline{AB} \cong \overline{\{CD\}}$, or $\overline{AB} > \overline{CD}$.

(b) If $\overline{AB} < \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} < \overline{EF}$.

(c) If $\overline{AB} < \overline{CD}$ and $\overline{CD} < \overline{EF}$, then $\overline{AB} < \overline{EF}$.

(d) If $\overline{AB} < \overline{CD}$ and $\overline{CD} < \overline{EF}$, then $\overline{AB} < \overline{EF}$.

Definition: If two angles $\sphericalangle BAD$ and $\sphericalangle CAD$ have a common side $\overrightarrow{AD}$ and the two other sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$ form opposite rays, the angles are **supplements** of each other, or **supplementary angles**.

Prop 3.14: Supplements of congruent angles are congruent.

Definition: An angle $\sphericalangle BAD$ is a **right angle** if it has a supplementary angle to which it is congruent.

Definition: For the definition of **vertical angles** see page 24 of the text.

Prop 3.15: (a) Vertical angles are congruent to each other.

(b) An angle congruent to a right angle is a right angle.

Definition: Two lines are **perpendicular** if they intersect at right angles.

Prop 3.16: For every line ℓ and every point P there there is a line through P and perpendicular to ℓ.

Prop 3.17 (ASA Criterion for Congruence)**:** Given $\triangle ABC$ and $\triangle DEF$ with $\sphericalangle A \cong \sphericalangle D$, $\sphericalangle C \cong \sphericalangle F$, and $\overline{AC} \cong \overline{DF}$. Then $\triangle ABC \cong \triangle DEF$.
Prop 3.18 (Converse to prop 3.10)**:** If in the triangle $\triangle ABC$ we have $\sphericalangle B \cong \sphericalangle C$, then $\overline{AB}$ and $\overline{AC}$ and $\triangle ABC$ is isosceles.
Prop 3.19 (Angle Addition)**:** Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$, $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$, and $\sphericalangle GBA \cong \sphericalangle HED$. Then $\sphericalangle ABC \cong \sphericalangle DEF$.
Prop 3.20 (Angle Subtraction)**:** Given $\overrightarrow{BG}$ between $\overrightarrow{BA}$ and $\overrightarrow{BC}$, $\overrightarrow{EH}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$, $\sphericalangle CBG \cong \sphericalangle FEH$, and $\sphericalangle ABC \cong \sphericalangle DEF$. Then $\sphericalangle GBA \cong \sphericalangle HED$.
Definition: $\sphericalangle ABC < \sphericalangle DEF$ means there is a ray $\overrightarrow{EG}$ between $\overrightarrow{ED}$ and $\overrightarrow{EF}$ such that $\sphericalangle ABC \cong \sphericalangle GEF$.
Prop 3.21 (Ordering of Angles)**:** (a) Exactly one of the following holds (trichotomy): $\sphericalangle P < \sphericalangle Q$, $\sphericalangle P \cong \sphericalangle Q$, or $\sphericalangle P > \sphericalangle Q$.
(b) If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q \cong \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.
(c) If $\sphericalangle P \cong \sphericalangle Q$ and $\sphericalangle Q < \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.
(d) If $\sphericalangle P < \sphericalangle Q$ and $\sphericalangle Q < \sphericalangle R$, then $\sphericalangle P < \sphericalangle R$.
Prop 3.22 (SSS Criterion for Congruence)**:** Given $\triangle ABC$ and $\triangle DEF$. If $\overline{AB} \cong \overline{DE}$, $\overline{BC} \cong \overline{EF}$, and $\overline{AC} \cong \overline{DF}$, then $\triangle ABC \cong \triangle DEF$.
Prop 3.23 (Euclid's Fourth Postulate)**:** All right angles are congruent to each other.

Some Results in Neutral Geometry

Theorem 4.1 (Alternate Interior Angle Theorem)**:** If two lines cut by a transversal have a pair of congruent alternate interior angles, then the two lines are parallel.
Corollary 1: Two lines perpendicular to the same line are parallel. Hence, the perpendicular dropped from a point P not on the line ℓ to ℓ is unique (and the point at which the perpendicular intersects ℓ is called its **foot**.
Corollary 2: If ℓ is a line and P a point not on ℓ, then there is at least one line through P and parallel to ℓ.
Theorem 4.2 (Exterior Angle Theorem)**:** An exterior angle of a triangle is greater than either remote interior angle.
Proposition 4.1 (SAA Congruence Criterion)**:** Given triangles $\triangle ABC$ and $\triangle DEF$ with $\overline{AC} \cong \overline{DF}$, $\sphericalangle A \cong \sphericalangle D$ and $\sphericalangle B \cong \sphericalangle E$. Then $\triangle ABC \cong \triangle DEF$.
Proposition 4.2: Two right triangles are congruent if the hypotenuse and a leg of one are congruent respectively to the hypotenuse and a leg of the other.
Proposition 4.3 (Midpoints)**:** Every segment has a unique midpoint.
Proposition 4.4 (Bisectors)**:** (a) Every angle has a unique bisector.
(b) Every segment has a unique perpendicular bisector.
Proposition 4.5: In a triangle $\triangle ABC$, the greater angle lies opposite the greater side and the greater side lies lies opposite the greater angle. That is $\overline{AB} > \overline{BC}$ iff $\sphericalangle C > \sphericalangle A$
Theorem 4.3 : A. There is a unique degree measure assigned to each angle so that
(0) $(\sphericalangle A)^\circ$ is a real number between 0° and 180°.
(1) $(\sphericalangle A)^\circ = 90^\circ$ iff $\sphericalangle A$ is a right angle
(2) $(\sphericalangle A)^\circ = (\sphericalangle B)^\circ$ iff $\sphericalangle A \cong \sphericalangle B$
(3) If $\overrightarrow{AC}$ is interior to $\sphericalangle BAD$, then $(\sphericalangle BAD)^\circ = (\sphericalangle DAC)^\circ + (\sphericalangle CAB)^\circ$
(4) For every real number x between 0 and 180 there is an angle $\sphericalangle A$ with $(\sphericalangle A)^\circ = x^\circ$
(5) If $\sphericalangle A$ is supplementary to $\sphericalangle B$, then $(\sphericalangle A)^\circ + (\sphericalangle B)^\circ = 180^\circ$
(6) $\sphericalangle A < \sphericalangle B$ iff $(\sphericalangle A)^\circ < (\sphericalangle B)^\circ$.
B. Given a segment $\overline{OI}$ (called the *unit segment*) then there is a unique way to of assigning a length to $|\overline{AB}|$ to each segment $\overline{AB}$ so that

(7) $|\overline{AB}|$ is a positive real number and $|\overline{OI}| = 1$
(8) $|\overline{AB}| = |\overline{CD}|$ iff $\overline{AB} \cong \overline{CD}$
(9) $A * B * C$ iff $|\overline{AB}| + |\overline{BC}| = |\overline{AC}|$
(10) $\overline{AB} < \overline{CD}$ iff $|\overline{AB}| < |\overline{CD}|$
(11) For every positive real number x there is a segment $\overline{AB}$ with $|\overline{AB}| = x$.

Corollary 1: The sum of the angles of any *two* angles of a triangle is less than $180°$.
Corollary 2 Triangle Inequality**:** If A, B, and C are three non-colinear points, then $|\overline{AC}| + |\overline{AB}| \geq |\overline{BC}|$.
Theorem 4.4 Saccheri-Legendre**:** The sum of the angles of a triangle is $\leq 180°$.
Corollary 1: The sum of two angles of a triangles is less than or equal to the remote exterior angle.
Definition: The quadrilateral $\square ABCD$ is **convex** iff it has a pair of opposite sides, e.g., AB and CD, such that CD is contained in one of the half planes bounded by $\overleftrightarrow{AB}$ and $\overline{AB}$ is contained in on of the half-planes bounded by $\overleftrightarrow{CD}$.
Corollary 2: The sum of the measures of the angles in any convex quadrilateral is at most $360°$.
Euclid's Postulate V: If two lines are intersected by a transversal in such a way that the sum of two interior angles on one side of the transversal is less than $180°$, then the two lines meet on that side of the transversal.
Theorem 4.5: Euclid's fifth postulate $\iff$ Hilbert's parallel postulate.
Proposition 4.7: Hilbert's parallel postulate $\iff$ any line that intersects one of two parallel line intersects the other one.
Proposition 4.8: Hilbert's parallel postulate $\iff$ Converse to theorem 4.1 (which is the Alternate interiors angles theorem.)
Proposition 4.9: Hilbert's parallel postulate $\iff$ if t is transversal to both ℓ and m, $\ell \| m$, and $t \perp l$, then $t \perp m$.
Proposition 4.10: Hilbert's parallel postulate $\iff$ $k \| \ell$, $m \perp k$, and $n \perp \ell$, implies $m = n$ or $m \| n$.
Proposition 4.11: Hilbert's parallel postulate $\Longrightarrow$ the sum of every triangle is $180°$.
Definition: For any triangle $\triangle ABC$ the **defect** of $\triangle ABC$ is defined by $\delta(\triangle ABC) = 180° - ((\sphericalangle A)° + (\sphericalangle B)° + (\sphericalangle C)°)$
Theorem 4.6 Additivity of the defect**:** Let $\triangle ABC$ be any triangle and let D be a point between A and B. Then $\delta(\triangle ABC) = \delta(\triangle ACD) + \delta(\triangle BCD)$
Corollary: With the same hypothesis as the last theorem, the angel sum of $\triangle ABC$ is $180°$ iff the angle sum of both of $\triangle ACD$ and $\triangle BCD$ is $180°$.
Definition: A quadrilateral $\square ABCD$ is a **rectangle** iff all four of its angles are right angles.
Theorem 4.7: If there exists a triangle with angle sum $180°$, then a rectangle exists. It a rectangle exists, the every triangle ha angle sum equal to $180°$.
Corollary: If there exits a triangle with positive defect, then all triangles have positive defect.