(1) (5 points) State the $n$-th order Taylor theorem about $x$ and with remainder for $f(x + h)$.

(2) (5 points) Let $f$ be a function on $[a, b]$ and $x_0, \ldots, x_n$ distinct points of $[a, b]$. Then what does it mean for the polynomial $p(x)$ to interpolate $f$ at the points $x_0, \ldots, x_n$?

(3) (5 points) Let $f$ be $n + 1$ times differentiable on $[a, b]$ and let $p(x)$ be the polynomial of degree $\leq n$ that interpolates $f$ at the distinct points $x_0, x_1, \ldots, x_n \in [a, b]$. What is the formula for the error $f(x) - p(x)$?

(4) (10 points) Let $x_0, \ldots, x_n$ be distinct points of $\mathbb{R}$.
   (a) Define the cardinal functions $\ell_0, \ldots, \ell_n$ determined by these points.

   (b) If $n \geq 2$ explain why $\sum_{i=0}^{n} x_i^2 \ell_i(x) = x^2$. 
(5) (15 points) Construct Newton’s interpolating polynomial for the data (you do not have to simplify your answer)

\[
\begin{array}{l|cccc}
  x & -1 & 1 & 3 & 4 \\
  \hline
  y & -9 & 2 & -3 & -4 \\
\end{array}
\]

(6) (20 points) Complete the following table of divided differences.

\[
\begin{array}{c|c|c|c|c}
  x & f[\ ] & f[,,] & f[,,,] & f[,,,,] \\
  \hline
  1 & -1 & & & \\
  3 & 5 & & & \\
  5 & 11 & & & \\
  6 & 59 & & & \\
\end{array}
\]
(7) (20 points) A interpolating polynomial of degree 20 is used to approximate \( \sin(x) \) on the interval \([-1, 1]\) at 21 equally spaced nodes. How accurate will this be?

(8) (20 points) Determine the error term in the approximation

\[
f'(x) \approx \frac{1}{2h} [4f(x + h) - 3f(x) - f(x + 2h)]
\]