Math/Stat 511 Test #3 Name: Answer Key

Show your work! Answers that do not have a justification will receive no credit.

Grades on the Third Exam.

Here is the information on the third test. 16 people took the exam. The high score was a 100. Two people got a 53 which was the low score. The mean was 80.63 with a standard deviation of 14.71. The median was 83.5 The break down in the grades is in the table.

Grade	Range	Number	Percent
A	90–100	5	31.25%
B	80–89	4	25.00%
C	70–79	4	25.00%
D	60–69	1	6.25%
F	0–59	2	12.50%

1. (35 points) Let X be a random variable with the given function $M(t)$ as moment generating function. Then fill in the required information about X.

   Remark: Most of these are done by the same method. From the form of the we know what the distribution of X. We when this to fill in the rest of the information. The exception is part (d) where the definition of the moment generating function is used directly.

   (a) $M(t) = (.4 + .6e^t)^4$

   (i) What is the distribution of X?

   Solution: The moment generating function of a binomial $B(n, p)$ is $(q + pe^t)^n$ so we see that X is binomial $B(4, .6)$.

   (ii) What is the pdf of X?

   Solution: The pdf for a binomial $B(4, .6)$ random variable is

   $$f(x) = \binom{4}{x}(.6)^x(.4)^{4-x}, \qquad x = 0, 1, 2, 3, 4.$$

   Remark: Forgetting to put that $x = 0, 1, 2, 3, 4$ lost one point.

   (iii) What is $E(X)$?

   Solution: The mean is $E(X) = np = 4(.6) = 2.4$.

   (iv) What is $P(x \geq 4)$?

   Solution: As the range of X is $\{0, 1, 2, 3, 4\}$ this reduces to

   $$P(x \geq 4) = P(X = 4) = (.6)^4 = .1296.$$

   (b) $M(t) = e^{2(e^t-1)}$

   (i) What is the distribution of X?

   Solution: The moment generating function for a Poisson random variable with mean λ is $M(t) = e^{\lambda(e^t-1)}$. So X is a Poisson random variable with

   (ii) What is the pdf of X?

   Solution: The pdf of a Poisson random variable with mean $\lambda = 2$ is

   $$f(x) = \frac{2^x e^{-2}}{x!}, \qquad x = 0, 1, 2, \ldots$$

   Remark: Forgetting to put $x = 0, 1, 2, \ldots$ lost a point.

(iii) What is the expect value of X?
Solution: The expect value of a Poisson random variable with mean $\lambda = 2$ is

$$\mu = E(X) = \lambda = 2$$

(iv) What is the variance of X?
Solution: The variance of a Poisson random variable with mean $\lambda = 2$ is

$$\sigma^2 = \lambda = 2.$$

(c) $M(t) = e^{-7t+8t^2}$
(i) What is the distribution of X?
Solution: The moment generating function for a normal random variable with mean μ and variance σ^2 is $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$. So X is a normal random variable with $\mu = -7$ and $\frac{1}{2}\sigma^2 = 8$. Therefore $\sigma^2 = 16$. That is X is normal $N(-7, 16)$.
(ii) What is the pdf of X?
Solution: The pdf of a normal $N(-7, 16)$ random variable is

$$f(x) = \frac{1}{4\sqrt{2\pi}} e^{-\frac{(x+7)^2}{32}}.$$

(iii) What is the expect value of X?
Solution: $E(X) = \mu - 7$.
(iv) What is the variance of X?
Solution: $\mathrm{Var}(X) = \sigma^2 = 16$.
(d) $M(t) = .2e^{2t} + .5e^{4t} + .3e^{6t}$.
(i) What is the pdf of X?
Solution: The definition of the moment generating function of a discrete random variable is $M(t) = \sum_{x \in R} f(x)e^{xt}$ where R is the range of X. Therefore if we let X be a random variable with pdf

$$f(2) = .2, \qquad f(4) = .5, \qquad f(6) = .3$$

we have

$$M(t) = f(2)e^{2t} + f(4)e^{4t} + f(6)e^{6t} = .2e^{2t} + .5e^{4t} + .3e^{6t}.$$

Therefore this f is the pdf of X.
(ii) What is $P(X = 6)$?
Solution: $P(X = 6) = f(6) = .3$.

2. (10 points) The probably that a person has a side effect from a certain type of pain relief pull is .01. If 1000 people use this drug, then what is the probability that at most 8 people have the side effect?
Solution: If X is the number of people out of the 1000 that have the side effect, then X has a binomial distribution $B(1000, .01)$. Then $n = 1000$ is large and the mean $\mu = np = 1000(.01) = 10$ is small so the Poisson applies. Let Y be the Poisson with the same mean as X, that is $E(Y) = \lambda = 10$. Then we can use the table for the Poisson distribution to get

$$P(X \leq 8) \approx P(Y \leq 8) = .333$$

Remark: Using a computer I found the value of $P(X \leq 8) = .3317$ to four decimal places. Therefore the error in using the Poisson approximation is less than .001 in this case.

3. (15 points) Let X be a random variable of continuous type with pdf

$$f(x) = \begin{cases} c(1+x) & -1 \le x \le 0 \\ 0 & \text{elsewhere} \end{cases}$$

(a) Find the value of c
Solution: Use that a pdf must have integral one.

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = c\int_{-1}^{0} (1+x)\,dx = c\frac{(1+x)^2}{2}\bigg|_{x=-1}^{0} = \frac{c}{2}.$$

Therefore $c = 2$.

(b) What is the expect value of X?
Solution: From the definition of $E(X)$

$$E(X) = \int_{-\infty}^{\infty} xf(x)\,dx = 2\int_{-1}^{0} x(1+x)\,dx = 2\left(\frac{x^2}{x} + \frac{x^3}{3}\right)\bigg|_{x=-1}^{0} = -\frac{1}{3}$$

(c) What is the distribution function $F(x)$ of $f(x)$?
Solution: The definition of $F(x)$ is $F(x) = P(X \le x)$ and as the range of X is $-1 \le x \le 0$ we have $F(x) = 0$ for $x \le 0$ and $F(x) = 1$ for $x \ge 0$. For $-1 < x < 0$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = 2\int_{-1}^{x} (t+1)\,dt = (t+1)^2\bigg|_{-1}^{x} = (x+1)^2.$$

Summarizing:

$$F(x) = \begin{cases} 0, & x \le -1 \\ (1+x)^2, & -1 \le x \le 0 \\ 1, & 0 \le x. \end{cases}$$

4. (10 points) Cars arrive at a toll booth at a mean rate of two a minute according to a Poisson distribution. What is the probability that the toll collector has to wait longer that 5 minutes to collect 12 tolls? **You can leave your answer as an integral.**
Solution: Let W be the time required to collect 12 tolls. Then this is the waiting time for 12 occurances of a Poisson process with mean $\lambda = 2$. Therefore W has a Gamma distirbution with parmeters $\alpha = 12$ and $\theta = 1/\lambda = 1/2 = .5$. The pdf for such a Gamma distribution is

$$f(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)\theta^{\alpha}}e^{-x/\theta} = \frac{x^{11}}{\Gamma(12)(.5)^{12}}e^{-2x} = \frac{x^{11}}{11!(.5)^{12}}e^{-2x}, \qquad x \ge 0$$

Therefore the probability we want is

$$P(W > 5) = \int_{5}^{\infty} = \frac{x^{11}}{11!(.5)^{12}}e^{-2x}\,dx.$$

Solution: If you want to pratice using your calculator for intergals the value I get for this is $P(W > 5) = .696771461$

5. (15 points) Let X have a normal distribution with mean $\mu = 5$ and variance $\sigma^2 = 9$. Then find the following probabilities. **Remark:** In all of this we will be using that

$$Z = \frac{X-\mu}{\sigma} = \frac{X-5}{3}$$

is a standard normal so that we can use the table of values for the standard normal.

(a) $P(X \geq 5)$
Solution:
$$P(X \geq 5) = P\left(Z = \frac{X-5}{3} \geq \frac{5-5}{3}\right) = .5000$$

(b) $P(X \leq 7.5)$
Solution:
$$P(X \leq 7.5) = p\left(Z = \frac{X-5}{3} \leq \frac{7.5-5}{3} = .833\right) = .7967$$

(c) $P(2 \leq X \leq 7)$
Solution:
$$P(2 \leq X \leq 7) = P\left(\frac{2-5}{3} \leq Z = \frac{X-5}{3} \leq \frac{7-5}{3}\right) = P(-1 \leq Z \leq .67) = .5894$$

Remark: Rounding off 2/3 as .66 rather than .67 lead to the answer .5867 which also got full credit.

6. (10 points) If X has the chi-square distribution $\chi^2(23)$ then find a and b so that $P(a < X < b) = 0.95$ and $P(X < a) = 0.025$.
Solution: This is a stright off the homework and is just a matter of reading the answer off the tables.
$$a = 11.69, \qquad b = 38.08$$

7. (5 points) Let X be the value of a number chosen at random from the interval $3 \leq x \leq 12$. What is the probability that X is between 5 and 9.
Solution: In this setting the pharse "at random" is just a buzz word for "uniform distiution" $U(3, 12)$. This has the pdf
$$f(x) = \frac{1}{12-3} = \frac{1}{9}, \qquad 3 \leq x \leq 12.$$
Therefore
$$P(5 < X < 9) = \int_5^9 f(x)\,dx = \int_5^9 \frac{1}{9}\,dx = \frac{4}{9}.$$