Math/Stat 511 Test #1 Name: Solution Key.

Show your work! Answers that do not have a justification will receive no credit.

Remark: As we had been over these definitions before there was not much partial credit given for mistakes.

1. (20 Points)

(a) What is the mean $\overline{x}$ of $x_1, \ldots, x_n$? **Solution:**

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{or} \quad \overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

(b) What is the variance s^2 of $x_1, \ldots, x_n$? **Solution:** . Either of the formulas

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\overline{x} - x_i)^2 \qquad \text{or} \qquad s^1 = \frac{1}{n-1}\left(\sum_{i=1}^{n} x_i^2 - n\overline{x}^2\right)$$

is correct.

(c) Define what it means for events $A_1, \ldots, A_k$ to be mutually exclusive.
Solution: They are mutually exculsive iff $A_i \cap A_j = \varnothing$ for $i \neq j$. **Remark:** Giving $A_1 \cap A_2 \cap \ldots A_k = \varnothing$ only recived 1 out of 3 points.

(d) Define what if means for events A, B, and C to be independent.
Solution: The conditions are

$$P(A \cap B) = P(A)P(B), \quad P(A \cap C) = P(A)P(C),$$
$$P(B \cap C) = P(B)P(C), \quad P(A \cap B \cap C) = P(A)P(B)P(C).$$

Remark: The formula $P(A|B) = P(A)P(B)$ and the like got zero points (as it implies that if $P(B) \neq 0$ then $P(A) = 1$ or $P(A) = 1$) and only listing some of the four conditions got 1 out of 3 points.

(e) Let A and B be events with $P(B) \neq 0$. Then what is the definition of the conditional probability $P(A|B)$?
Solution: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

(f) Complete the following: **Probability** is a set function that assigns to each event A in the sample space S a number $P(A)$, called the probability of the event A, such that the following properties are satisfied:
Solution:
(i) $P(A) \geq 0$
(ii) $P(S) = 1$
(iii) If $A_1, A_2, \ldots$ are mutually exclusive then

$$P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$$

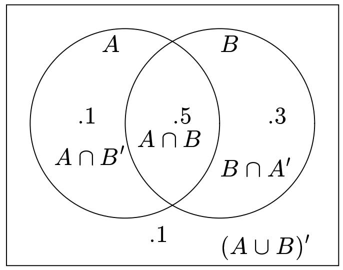
Remark: Not saying that $A_1, A_2, \ldots$ are mutually exclusive lost 2 out of five points.

2. (15 Points) Let A and B be events so that $P(A) = .6$, $P(B) = .8$, $P(A \cup B) = .9$. Then compute

(a) $P(A')$ **Solution:** $P(A') = 1 - P(A) = 1 - .6 = .4$.

(b) $P(A \cap B)$ **Solution:** $P(A \cap B) = P(A) + P(B) - P(A \cup B) = .6 + .8 - .9 = .5$.
Remark: Assuming A and B independent lost all 5 points.

(c) $P(A \cup B')$ **Solution:** The easiest method was to fill in the a Venn diagram:



where we see that $A \cup B' = A \cup ((A \cup B)')$ and A and $(A \cup B)'$ mutually exclusive so the probabilities add to give $P(A \cup B') = P(A) + P((A \cup B)') = .6 + .1 = .7$. **Remark:** Assuming that A and B were independent lost 4 out of 5 points. The formula $P(A \cup B') = P(A)P(B')$ lost all 5 points.

3. (10 Points) Alice and Bill play a chess tournament where the first person to win 10 games wins the tournament. How many different ways can Alice win the tournament in exactly 17 games?

Solution: If Alice wins the tournament on the 17-th game, then she won the 17-th game and also won exactly 9 of the first 16 games (so that her win on the 17-th game is her 10-th win). The number of ways to do this is the number of ways to choose the 9 wins out of the first 16 wins. That is

$$\binom{16}{9} = 11{,}440.$$

Remark: The most popular wrong answer was $\binom{17}{10} = 19{,}448$ which got 5 out of 10 points.

4. (5 Points) Assume the events A and B are independent and that $P(A) = .3$ and $P(B) = .6$. Then what is $P(A' \cup B)$?

Solution: . First note $P(A') = 1 - P(A) = 1 - .3 = .7$. Then A' and B are also independent so $P(A' \cap B) = P(A')P(B) = (.7)(.6) = .42$. And to finish:

$$P(A' \cup B) = P(A') + P(B) - P(A' \cap B) = .7 + .6 - .42 = .88\ .$$

Remark: Having the formula $P(A' \cup B) = P(A')P(B)$ lost 4 out of the five points as this is just about never true.

5. (15 Points) Let A_1 and A_2 be the events that that a person is left or right handed, respectively. Let B_1 and B_2 be the events that a person is left eye dominant or right eye dominant, respectively. A survey in one statistics class yielded the following data:

	B_1	B_2	Total
A_1	10	23	33
A_2	15	25	40
Total	25	48	73

Compute the following:

(a) $P(A_1)$ **Solution:** The total number of students is 73 and the number of students in the event A_1 is $\#(A_1) = 33$. Therefore

$$P(A_1) = \frac{33}{73} \approx .4520547\ldots$$

(b) $P(A_1|B_2)$ **Solution:** The conditional probability is computed by restriction the sample space down to the event B_2. This

$$P(A_1|B_2) = \frac{\#(A_1 \cap B_2)}{\#(B_2)} = \frac{23}{48} \approx .479166\ldots$$

(c) The probability that a member of the class is right eye dominant given that they are right handed. **Solution:** This is the probability $P(B_2|A_2)$ which we compute as

$$P(B_2|A_2) = \frac{\#(A_2 \cap B_2)}{\#(A_2)} = \frac{25}{40} = .625$$

Remark: The point of this part of the problem was translating the English into mathematics. Therefore computing $P(A_2|B_2)$ only received 1 out of 5 points.

6. (10 Points) Assuming that it is equally likely that a person be born on any of the 12 months of the year, then what is the probability that of 5 people chosen at random no two were born in the same month?

Solution: We consider as the sample space the total number of ordered samples with replacement out of the 12 months. Therefore the number elements in the sample space is 12^5. The number of these samples that have all the months distinct is the number of ordered ways to choose five elements with out of the 12 months with no replacement. That is ${}_{12}P_5$. Therefore the probability of no month being repeated is

$$P(\text{all 5 birth months distinct}) = \frac{{}_{12}P_5}{12^5} \approx .381944\ldots$$

This can also be done by use to the mutiplication rule for conditional probablities. This leads to

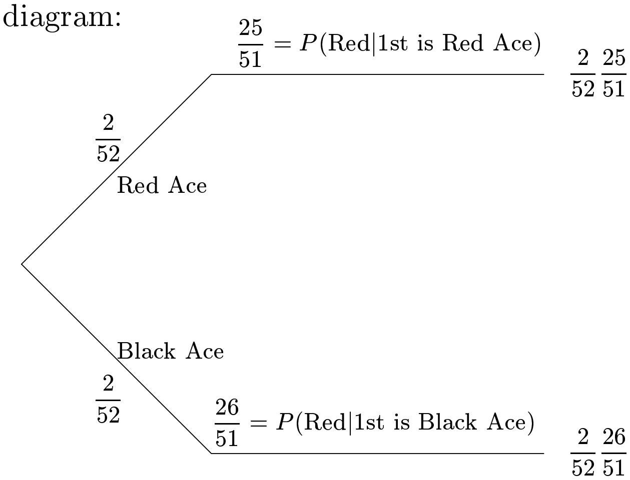
$$P(\text{all 5 birth months distinct}) = \frac{12}{12}\frac{11}{12}\frac{10}{12}\frac{9}{12}\frac{8}{12} \approx .381944\ldots$$

7. (10 Points) Two cards are drawn at random without replacement from a standard deck of 52 playing cards. Then compute

(a) The probability that both are clubs. **Solution:** This uses the mutiplication rule for conditional probablities

$$P(\text{Both clubs}) = P(\text{first is a club})P(\text{second is a club}|\text{first is a club}) = \frac{13}{52}\frac{12}{51} \approx .0588235\ldots$$

(b) The first one is an Ace and the second one is red. **Solution:** The easiest way to to this is make a tree diagram:

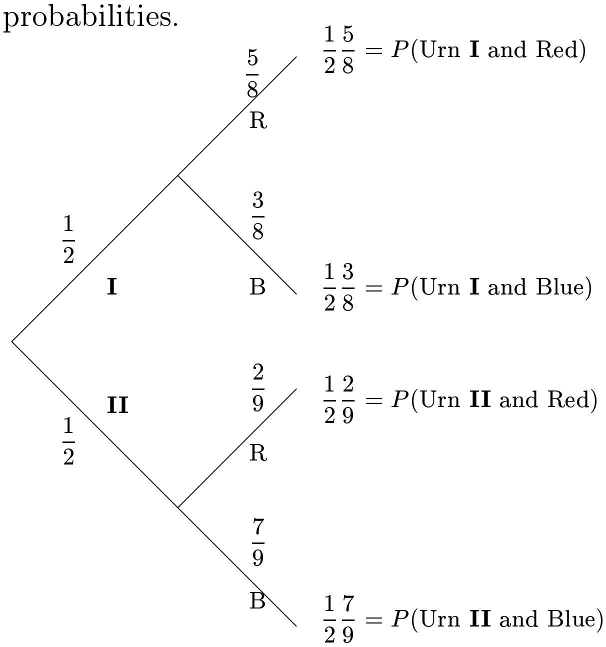


and so

$$P(\text{1-st is an Ace and 2-nd is red}) = \frac{2}{52}\frac{25}{51} + \frac{2}{52}\frac{26}{51} = \frac{1}{26} \approx .38461\ldots$$

8. (10 Points) An urn marked **I** contains 5 red and 3 blue balls. A second urn, marked **II**, contains 2 red and 7 blue balls. An experiment is done where one of the two urns is chosen at random and one ball is chosen from it.

Solution: Before doing any problems we make a tree diagram that summarizes all the outcomes and probabilities.



(a) Compute the probability that the ball is red. **Solution:** From the figure we see

$$P(\text{Red}) = P(\text{Urn } \mathbf{I} \text{ and Red}) + P(\text{Urn } \mathbf{II} \text{ and Red}) = \frac{1}{2}\frac{5}{8} + \frac{1}{2}\frac{2}{9} \approx .4236111\ldots$$

(b) Compute the conditional probability that the ball came from urn **I**, given that it is red. **Solution:**

$$P(\text{Urn } \mathbf{I}|\text{Red}) = \frac{P(\text{Urn } \mathbf{I} \text{ and Red})}{P(\text{Red})} = \frac{\frac{1}{2}\frac{5}{8}}{\frac{1}{2}\frac{5}{8} + \frac{1}{2}\frac{2}{9}} \approx .7377\ldots$$

9. (5 Points) Who is the standard deviate?
Solution: For my generation it was Hunter S. Thompson. In the opinion of this class I beat out Bill Clinton by one vote.