Chapter One

FUNCTIONS AND CHANGE

Functions are truly fundamental to mathematics. In everyday language we say, "The performance of the stock market is a function of consumer confidence" or "The patient's blood pressure is a function of the drugs prescribed." In each case, the word *function* expresses the idea that knowledge of one fact tells us another. In mathematics, the most important functions are those in which knowledge of one number tells us another number. If we know the length of the side of a square, its area is determined. If the circumference of a circle is known, its radius is determined.

Calculus starts with the study of functions. This chapter lays the foundation for calculus by surveying the behavior of some common functions. We also see ways of handling the graphs, tables, and formulas that represent these functions.

Calculus enables us to study change. In this chapter we see how to measure change and average rate of change.

1.1 WHAT IS A FUNCTION?

In mathematics, a *function* is used to represent the dependence of one quantity upon another.

Let's look at an example. In December 2000, the temperatures in Chicago were unusually low over winter vacation. The daily high temperatures for December 19–28 are given in Table 1.1.

Table 1.1 *Daily high temperature in Chicago, December 19–28, 2000*

Date (December 2000)	19	20	21	22	23	24	25	26	27	28
High temperature (°F)	20	17	19	7	20	11	17	19	17	20

Although you may not have thought of something so unpredictable as temperature as being a function, the temperature *is* a function of date, because each day gives rise to one and only one high temperature. There is no formula for temperature (otherwise we would not need the weather bureau), but nevertheless the temperature does satisfy the definition of a function: Each date, t, has a unique high temperature, H, associated with it.

We define a function as follows:

> A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

The input is called the *independent variable* and the output is called the *dependent variable*. In the temperature example, the set of dates $\{19, 20, 21, 22, 23, 24, 25, 26, 27, 28\}$ is the domain and the set of temperatures $\{7, 11, 17, 19, 20\}$ is the range. We call the function f and write $H = f(t)$. Notice that a function may have identical outputs for different inputs (December 20, 25, and 27, for example).

Some quantities, such as date, are *discrete*, meaning they take only certain isolated values (dates must be integers). Other quantities, such as time, are *continuous* as they can be any number. For a continuous variable, domains and ranges are often written using interval notation:

The set of numbers t such that $a \le t \le b$ is written $[a, b]$.

The set of numbers t such that $a < t < b$ is written (a, b).

Representation of Functions: Tables, Graphs, Formulas, and Words

Functions can be represented by tables, graphs, formulas, and descriptions in words. For example, the function giving the daily high temperatures in Chicago can be represented by the graph in Figure 1.1, as well as by Table 1.1.

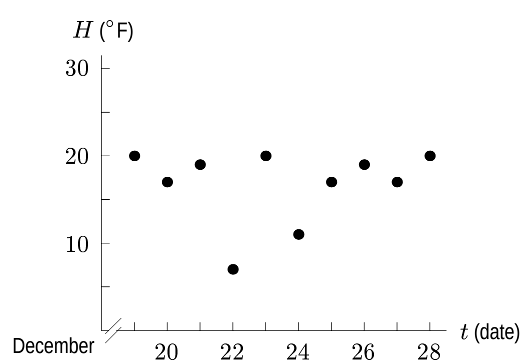


Figure 1.1: Chicago temperatures, December 2000

Other functions arise naturally as graphs. Figure 1.2 contains electrocardiogram (EKG) pictures showing the heartbeat patterns of two patients, one normal and one not. Although it is possible to construct a formula to approximate an EKG function, this is seldom done. The pattern of repetitions is what a doctor needs to know, and these are more easily seen from a graph than from a formula. However, each EKG gives electrical activity as a function of time.

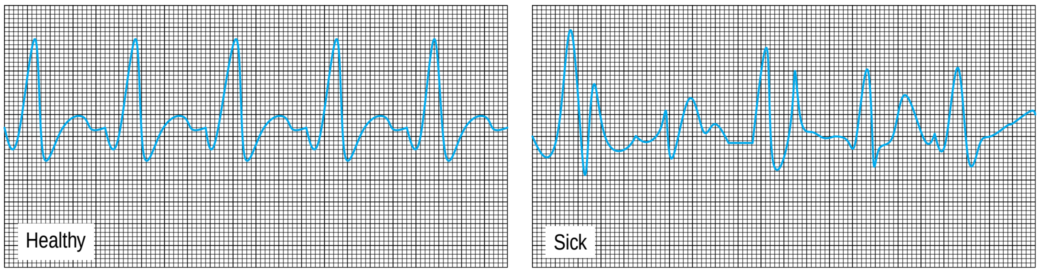


Figure 1.2: EKG readings on two patients

Consider the snow tree cricket. Surprisingly enough, all such crickets chirp at essentially the same rate if they are at the same temperature. That means that the chirp rate is a function of temperature. In other words, if we know the temperature, we can determine the chirp rate. Even more surprisingly, the chirp rate, C, in chirps per minute, increases steadily with the temperature, T, in degrees Fahrenheit, and can be computed, to a fair degree of accuracy, using the formula

$$C = f(T) = 4T - 160.$$

The graph of this function is in Figure 1.3.

Since $C = f(T)$ increases with T, we say that f is an *increasing function*.

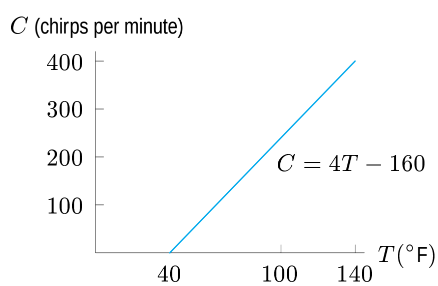


Figure 1.3: Cricket chirp rate as a function of temperature

Function Notation and Intercepts

We write $y = f(t)$ to express the fact that y is a function of t. The independent variable is t, the dependent variable is y, and f is the name of the function. The graph of a function has an *intercept* where it crosses the horizontal or vertical axis. Horizontal intercepts are also called the *zeros* of the function.

Example 1 The value of a car, V, is a function of the age of the car, a, so $V = f(a)$.

(a) Interpret the statement $f(5) = 9$ in terms of the value of the car if V is in thousands of dollars and a is in years.

(b) In the same units, the value of a Honda[1] is approximated by $f(a) = 13.25 - 0.9a$. Find and interpret the vertical and horizontal intercepts of the graph of this depreciation function f.

[1] From data obtained from the Kelley Blue Book, www.kbb.com.

Solution

(a) Since $V = f(a)$, the statement $f(5) = 9$ means $V = 9$ when $a = 5$. This tells us that the car is worth \$9000 when it is 5 years old.

(b) Since $V = f(a)$, a graph of the function f has the value of the car on the vertical axis and the age of the car on the horizontal axis. The vertical intercept is the value of V when $a = 0$. It is $V = f(0) = 13.25$, so the Honda was valued at \$13,250 when new. The horizontal intercept is the value of a such that $V(a) = 0$, so

$$13.25 - 0.9a = 0$$
$$a = \frac{13.25}{0.9} = 14.7.$$

At age 15 years, the Honda has no value.

Since $V = f(a)$ decreases with a, we say that f is a *decreasing function*.

Problems for Section 1.1

1. The population of a city, P, in millions, is a function of t, the number of years since 1950, so $P = f(t)$. Explain the meaning of the statement $f(35) = 12$ in terms of the population of this city.

2. Which graph in Figure 1.4 best matches each of the following stories?[2] Write a story for the remaining graph.

(a) I had just left home when I realized I had forgotten my books, and so I went back to pick them up.
(b) Things went fine until I had a flat tire.
(c) I started out calmly but sped up when I realized I was going to be late.

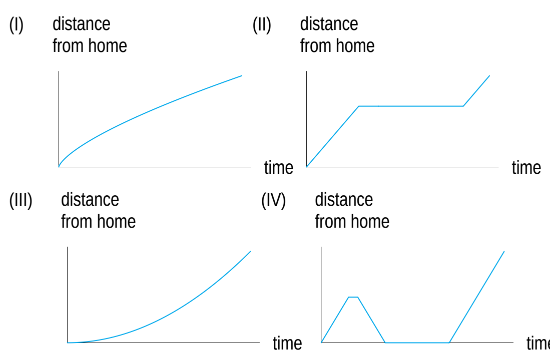


Figure 1.4

3. The number of sales per month, S, is a function of the amount, a, (in dollars) spent on advertising that month, so $S = f(a)$.

(a) Interpret the statement $f(1000) = 3500$.
(b) Which of the graphs in Figure 1.5 is more likely to represent this function?
(c) What does the vertical intercept of the graph of this function represent, in terms of sales and advertising?

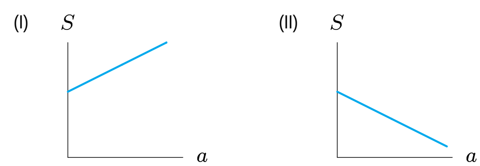


Figure 1.5

4. In the Andes mountains in Peru, the number, N, of species of bats is a function of the elevation, h, in feet above sea level, so $N = f(h)$.

(a) Interpret the statement $f(500) = 100$ in terms of bat species.
(b) What are the meanings of the vertical intercept, k, and horizontal intercept, c, in Figure 1.6?

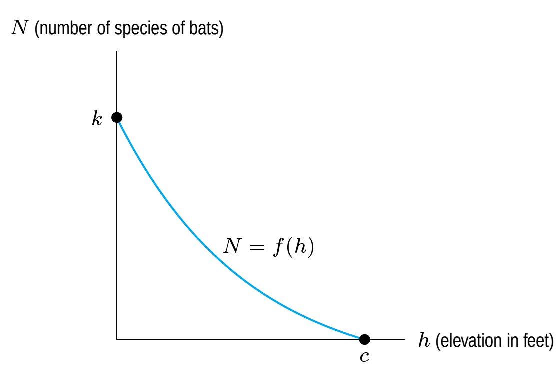


Figure 1.6

[2] Adapted from Jan Terwel, "Real Math in Cooperative Groups in Secondary Education." *Cooperative Learning in Mathematics*, ed. Neal Davidson, p. 234, (Reading: Addison Wesley, 1990).

5. An object is put outside on a cold day at time $t = 0$. Its temperature, $H = f(t)$, in $^\circ$C, is graphed in Figure 1.7.

   (a) What does the statement $f(30) = 10$ mean in terms of temperature? Include units for 30 and for 10 in your answer.

   (b) Explain what the vertical intercept, a, and the horizontal intercept, b, represent in terms of temperature of the object and time outside.

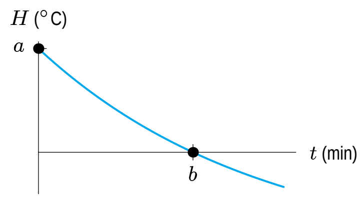


Figure 1.7

6. The population of Washington DC grew from 1900 to 1950, stayed approximately constant during the 1950s, and decreased from about 1960 to 2000. Graph the population as a function of years since 1900.

7. Financial investors know that, in general, the higher the expected rate of return on an investment, the higher the corresponding risk.

   (a) Graph this relationship, showing expected return as a function of risk.

   (b) On the figure from part (a), mark a point with high expected return and low risk. (Investors hope to find such opportunities.)

8. In tide pools on the New England coast, snails eat algae. Describe what Figure 1.8 tells you about the effect of snails on the diversity of algae.[3] Does the graph support the statement that diversity peaks at intermediate predation levels?

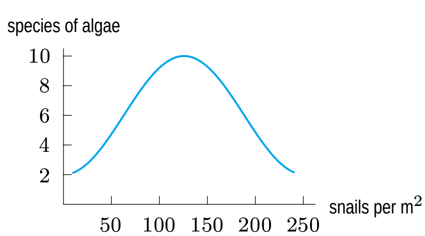


Figure 1.8

9. After an injection, the concentration of a drug in a patient's body increases rapidly to a peak and then slowly decreases. Graph the concentration of the drug in the body as a function of the time since the injection was given. Assume that the patient has none of the drug in the body before the injection. Label the peak concentration and the time it takes to reach that concentration.

10. A deposit is made into an interest-bearing account. Figure 1.9 shows the balance, B, in the account t years later.

    (a) What was the original deposit?

    (b) Estimate $f(10)$ and interpret it.

    (c) When does the balance reach \$5000?

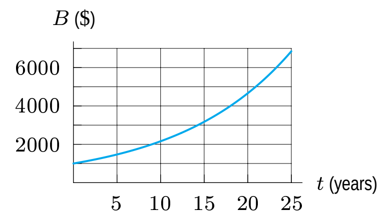


Figure 1.9

11. **(a)** A potato is put in an oven to bake at time $t = 0$. Which of the graphs in Figure 1.10 could represent the potato's temperature as a function of time?

    (b) What does the vertical intercept represent in terms of the potato's temperature?

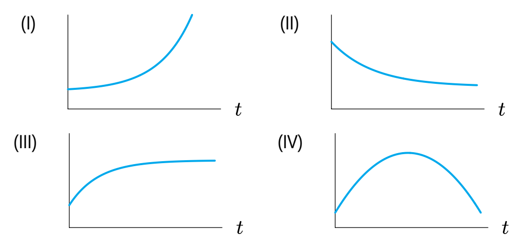


Figure 1.10

12. Figure 1.11 shows the amount of nicotine, $N = f(t)$, in mg, in a person's bloodstream as a function of the time, t, in hours, since the person finished smoking a cigarette.

    (a) Estimate $f(3)$ and interpret it in terms of nicotine.

    (b) About how many hours have passed before the nicotine level is down to 0.1 mg?

    (c) What is the vertical intercept? What does it represent in terms of nicotine?

    (d) If this function had a horizontal intercept, what would it represent?

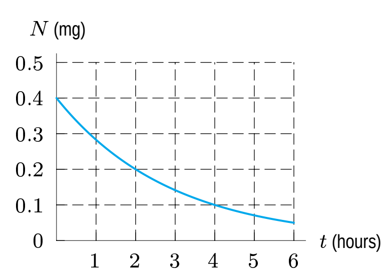


Figure 1.11

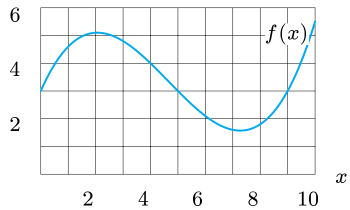
[3] Rosenzweig, M.L., *Species Diversity in Space and Time*, p. 343, (Cambridge: Cambridge University Press, 1995).

For the functions in Problems 13–17, find $f(5)$.

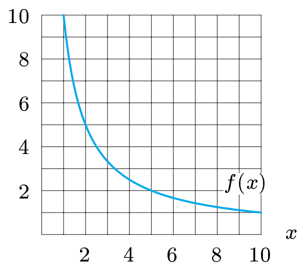
13. $f(x) = 2x + 3$

14. $f(x) = 10x - x^2$

15.



16.



17.

x	1	2	3	4	5	6	7	8
$f(x)$	2.3	2.8	3.2	3.7	4.1	5.0	5.6	6.2

18. It warmed up throughout the morning, and then suddenly got much cooler around noon, when a storm came through. After the storm, it warmed up before cooling off at sunset. Sketch temperature as a function of time.

19. When a patient with a rapid heart rate takes a drug, the heart rate plunges dramatically and then slowly rises again as the drug wears off. Sketch the heart rate against time from the moment the drug is administered.

20. Figure 1.12 shows fifty years of fertilizer use in the US, India, and the former Soviet Union.[4]

(a) Estimate fertilizer use in 1970 in the US, India, and the former Soviet Union.

(b) Write a sentence for each of the three graphs describing how fertilizer use has changed in each region over this 50-year period.

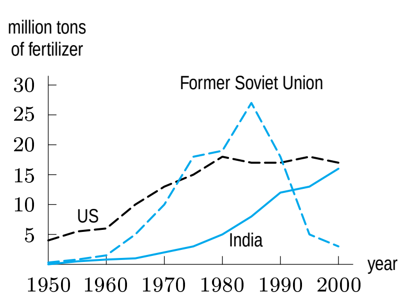


Figure 1.12

21. The gas mileage of a car (in miles per gallon) is highest when the car is going about 45 miles per hour and is lower when the car is going faster or slower than 45 mph. Graph gas mileage as a function of speed of the car.

22. A gas tank 6 meters underground springs a leak. Gas seeps out and contaminates the soil around it. Graph the amount of contamination as a function of the depth (in meters) below ground.

23. Describe what Figure 1.13 tells you about an assembly line whose productivity is represented as a function of the number of workers on the line.

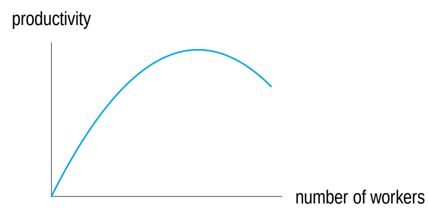


Figure 1.13

24. **(a)** The graph of $r = f(p)$ is in Figure 1.14. What is the value of r when p is 0? When p is 3?

(b) What is $f(2)$?

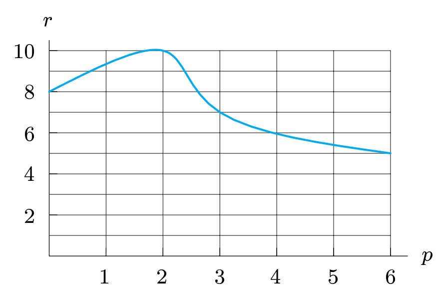


Figure 1.14

25. Let $y = f(x) = x^2 + 2$.

(a) Find the value of y when x is zero.

(b) What is $f(3)$?

(c) What values of x give y a value of 11?

(d) Are there any values of x that give y a value of 1?

[4]The Worldwatch Institute, *Vital Signs 2001*, p. 32, (New York: W.W. Norton, 2001).

1.2 LINEAR FUNCTIONS

Probably the most commonly used functions are the *linear functions*, whose graphs are straight lines. The chirp-rate and the Honda depreciation functions in the previous section are both linear. We now look at more examples of linear functions.

Olympic and World Records

During the early years of the Olympics, the height of the men's winning pole vault increased approximately 8 inches every four years. Table 1.2 shows that the height started at 130 inches in 1900, and increased by the equivalent of 2 inches a year between 1900 and 1912. So the height was a linear function of time.

Table 1.2 *Winning height (approximate) for Men's Olympic pole vault*

Year	1900	1904	1908	1912
Height (inches)	130	138	146	154

If y is the winning height in inches and t is the number of years since 1900, we can write

$$y = f(t) = 130 + 2t.$$

Since $y = f(t)$ increases with t, we see that f is an increasing function. The coefficient 2 tells us the rate, in inches per year, at which the height increases. This rate is the *slope* of the line in Figure 1.15. The slope is given by the ratio

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{146 - 138}{8 - 4} = \frac{8}{4} = 2 \text{ inches/year}.$$

Calculating the slope (rise/run) using any other two points on the line gives the same value.

What about the constant 130? This represents the initial height in 1900, when $t = 0$. Geometrically, 130 is the intercept on the vertical axis.

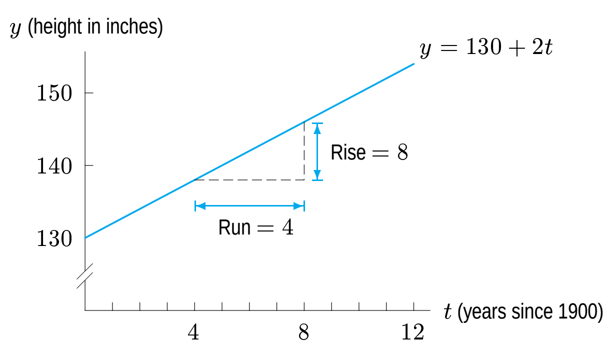


Figure 1.15: Olympic pole vault records

You may wonder whether the linear trend continues beyond 1912. Not surprisingly, it doesn't exactly. The formula $y = 130+2t$ predicts that the height in the 2000 Olympics would be 330 inches or 27 feet 6 inches, which is considerably higher than the actual value of 19 feet 4.27 inches. There is clearly a danger in *extrapolating* too far from the given data. You should also observe that the data in Table 1.2 is *discrete*, because it is given only at specific points (every four years). However, we have treated the variable t as though it were *continuous*, because the function $y = 130 + 2t$ makes sense for all values of t. The graph in Figure 1.15 is of the continuous function because it is a solid line, rather than four separate points representing the years in which the Olympics were held.

Example 1 If y is the world record time to run the mile, in seconds, and t is the number of years since 1900, then records show that, approximately,

$$y = g(t) = 260 - 0.4t.$$

Explain the meaning of the intercept, 260, and the slope, -0.4, in terms of the world record time to run the mile and sketch the graph.

Solution The intercept, 260, tells us that the world record was 260 seconds in 1900 (at $t = 0$). The slope, -0.4, tells us that the world record decreased at a rate of about 0.4 seconds per year. See Figure 1.16.

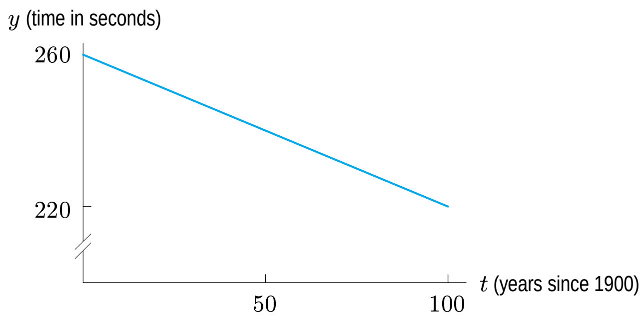


Figure 1.16: World record time to run the mile

Slope and Rate of Change

We use the symbol Δ (the Greek letter capital delta) to mean "change in," so Δx means change in x and Δy means change in y.

The slope of a linear function $y = f(x)$ can be calculated from values of the function at two points, given by x_1 and x_2, using the formula

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}} = \frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

The quantity $(f(x_2) - f(x_1))/(x_2 - x_1)$ is called a *difference quotient* because it is the quotient of two differences. (See Figure 1.17). Since slope $= \Delta y/\Delta x$, the slope represents the *rate of change* of y with respect to x. The units of the slope are y-units over x-units.

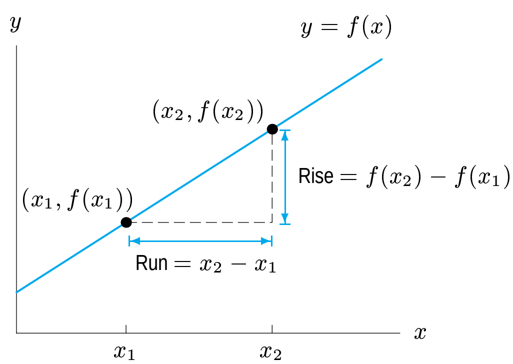


Figure 1.17: Difference quotient $= \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

Linear Functions in General

A **linear function** has the form

$$y = f(x) = b + mx.$$

Its graph is a line such that

- m is the **slope**, or rate of change of y with respect to x.
- b is the **vertical intercept**, or value of y when x is zero.

Notice that if the slope, m, is zero, we have $y = b$, a horizontal line. For a line of slope m through the point (x_0, y_0), we have

$$\text{Slope} = m = \frac{y - y_0}{x - x_0}.$$

Therefore we can write the equation of the line in the *point-slope form*:

The equation of a line of slope m through the point (x_0, y_0) is

$$y - y_0 = m(x - x_0).$$

Example 2 The solid waste generated each year in the cities of the US is increasing. The solid waste generated,[5] in millions of tons, was 205.2 in 1990 and 220.2 in 1998.

(a) Assuming that the amount of solid waste generated by US cities is a linear function of time, find a formula for this function by finding the equation of the line through these two points.

(b) Use this formula to predict the amount of solid waste generated in the year 2020.

Solution (a) We are looking at the amount of solid waste, W, as a function of year, t, and the two points are $(1990, 205.2)$ and $(1998, 220.2)$. The slope of the line is

$$m = \frac{\Delta W}{\Delta t} = \frac{220.2 - 205.2}{1998 - 1990} = \frac{15}{8} = 1.875 \text{ million tons/year.}$$

To find the equation of the line, we find the vertical intercept. We substitute the point $(1990, 205.2)$ and the slope $m = 1.875$ into the equation for W:

$$\begin{aligned} W &= b + mt \\ 205.2 &= b + (1.875)(1990) \\ 205.2 &= b + 3731.25 \\ -3526.05 &= b. \end{aligned}$$

The equation of the line is $W = -3526.05 + 1.875t$. Alternately, we could use the point-slope form of a line, $W - 205.2 = 1.875(t - 1990)$.

(b) To calculate solid waste predicted for the year 2020, we substitute $t = 2020$ into the equation of the line, $W = -3526.05 + 1.875t$, and calculate W:

$$W = -3526.05 + (1.875)(2020) = 261.45.$$

The formula predicts that in the year 2020, there will be 261.45 million tons of solid waste.

Recognizing Data from a Linear Function: Values of x and y in a table could come from a linear function $y = b + mx$ if differences in y-values are constant for equal differences in x.

[5] *Statistical Abstracts of the US*, 2000, Table 396.

Example 3 Which of the following tables of values could represent a linear function?

x	0	1	2	3
$f(x)$	25	30	35	40

x	0	2	4	6
$g(x)$	10	16	26	40

t	20	30	40	50
$h(t)$	2.4	2.2	2.0	1.8

Solution Since $f(x)$ increases by 5 for every increase of 1 in x, the values of $f(x)$ could be from a linear function with slope $= 5/1 = 5$.

Between $x = 0$ and $x = 2$, the value of $g(x)$ increases by 6 as x increases by 2. Between $x = 2$ and $x = 4$, the value of y increases by 10 as x increases by 2. Since the slope is not constant, $g(x)$ could not be a linear function.

Since $h(t)$ decreases by 0.2 for every increase of 10 in t, the values of $h(t)$ could be from a linear function with slope $= -0.2/10 = -0.02$.

Example 4 The data in the following table lie on a line. Find formulas for each of the following functions, and give units for the slope in each case:

(a) q as a function of p

(b) p as a function of q

p(dollars)	5	10	15	20
q(tons)	100	90	80	70

Solution (a) If we think of q as a linear function of p, then q is the dependent variable and p is the independent variable. We can use any two points to find the slope. The first two points give

$$\text{Slope} = m = \frac{\Delta q}{\Delta p} = \frac{90-100}{10-5} = \frac{-10}{5} = -2.$$

The units are the units of q over the units of p, or tons per dollar.

To write q as a linear function of p, we use the equation $q = b+mp$. We know that $m = -2$ and we can use any of the points in the table to find b. Substituting $p = 10$, $q = 90$ gives

$$\begin{aligned} q &= b + mp \\ 90 &= b + (-2)(10) \\ 90 &= b - 20 \\ 110 &= b. \end{aligned}$$

Thus, the equation of the line is

$$q = 110 - 2p.$$

(b) If we now consider p as a linear function of q, then p is the dependent variable and q is the independent variable. We have

$$\text{Slope} = m = \frac{\Delta p}{\Delta q} = \frac{10-5}{90-100} = \frac{5}{-10} = -0.5.$$

The units of the slope are dollars per ton.

Since p is a linear function of q, we have $p = b + mq$ and $m = -0.5$. To find b, we substitute any point from the table, such as $p = 10$, $q = 90$, into this equation:

$$\begin{aligned} p &= b + mq \\ 10 &= b + (-0.5)(90) \\ 10 &= b - 45 \\ 55 &= b. \end{aligned}$$

Thus, the equation of the line is

$$p = 55 - 0.5q.$$

Alternatively, we could take our answer to part (a), that is $q = 110 - 2p$, and solve for p.

Families of Linear Functions

Formulas such as $f(x) = b + mx$, in which the constants m and b can take on various values, represent a *family of functions*. All the functions in a family share certain properties—in this case, the graphs are lines. The constants m and b are called *parameters*. Figures 1.18 and 1.19 show graphs with several values of m and b. Notice the greater the magnitude of m, the steeper the line.

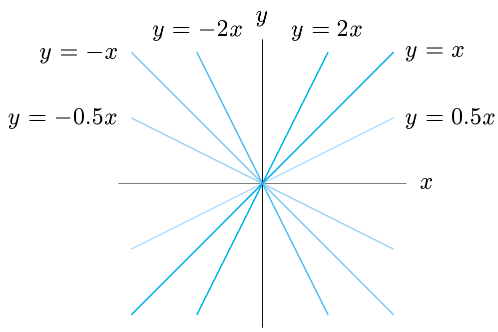


Figure 1.18: The family $y = mx$ (with $b = 0$)

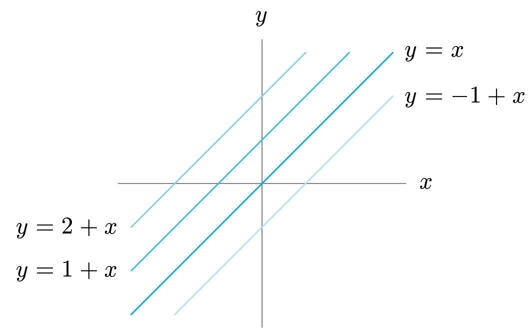


Figure 1.19: The family $y = b + x$ (with $m = 1$)

Problems for Section 1.2

For Problems 1–4, determine the slope and the y-intercept of the line whose equation is given.

1. $3x + 2y = 8$

2. $7y + 12x - 2 = 0$

3. $-4y + 2x + 8 = 0$

4. $12x = 6y + 4$

For Problems 5–8, find the equation of the line that passes through the given points.

5. $(0, 0)$ and $(1, 1)$

6. $(0, 2)$ and $(2, 3)$

7. $(4, 5)$ and $(2, -1)$

8. $(-2, 1)$ and $(2, 3)$

9. Match the graphs in Figure 1.20 with the following equations. (Note that the x and y scales may be unequal.)

(a) $y = x - 5$ **(b)** $-3x + 4 = y$
(c) $5 = y$ **(d)** $y = -4x - 5$
(e) $y = x + 6$ **(f)** $y = x/2$

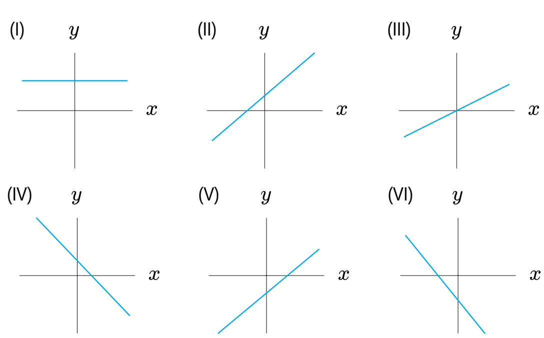


Figure 1.20

10. Figure 1.21 shows four lines given by equation $y = b + mx$. Match the lines to the conditions on the parameters m and b.

(a) $m > 0, b > 0$ **(b)** $m < 0, b > 0$
(c) $m > 0, b < 0$ **(d)** $m < 0, b < 0$

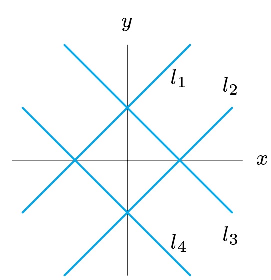


Figure 1.21

11. **(a)** Which two lines in Figure 1.22 have the same slope? Of these two lines, which has the larger y-intercept?
(b) Which two lines have the same y-intercept? Of these two lines, which has the larger slope?

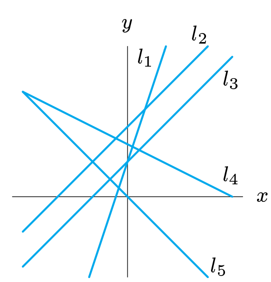


Figure 1.22

12. A cell phone company charges a monthly fee of $25 plus $0.05 per minute. Find a formula for the monthly charge, C, in dollars, as a function of the number of minutes, m, the phone is used during the month.

13. A city's population was 30,700 in the year 2000 and is growing by 850 people a year.

(a) Give a formula for the city's population, P, as a function of the number of years, t, since 2000.
(b) What is the population predicted to be in 2010?
(c) When is the population expected to reach 45,000?

14. A company rents cars at $40 a day and 15 cents a mile. Its competitor's cars are $50 a day and 10 cents a mile.

(a) For each company, give a formula for the cost of renting a car for a day as a function of the distance traveled.
(b) On the same axes, graph both functions.
(c) How should you decide which company is cheaper?

15. Which of the following tables could represent linear functions?

(a)

x	0	1	2	3
y	27	25	23	21

(b)

t	15	20	25	30
s	62	72	82	92

(c)

u	1	2	3	4
w	5	10	18	28

16. For each table in Problem 15 that could represent a linear function, find a formula for that function.

17. Find the linear equation used to generate the values in Table 1.3.

Table 1.3

x	5.2	5.3	5.4	5.5	5.6
y	27.8	29.2	30.6	32.0	33.4

18. A company's pricing schedule in Table 1.4 is designed to encourage large orders. (A gross is 12 dozen.) Find a formula for:

(a) q as a linear function of p.
(b) p as a linear function of q.

Table 1.4

q (order size, gross)	3	4	5	6
p (price/dozen)	15	12	9	6

19. World milk production rose at an approximately constant rate between 1960 and 1990.[6] See Figure 1.23.

(a) Estimate the vertical intercept and interpret it in terms of milk production.
(b) Estimate the slope and interpret it in terms of milk production.
(c) Give an approximate formula for milk production, M, as a function of t.

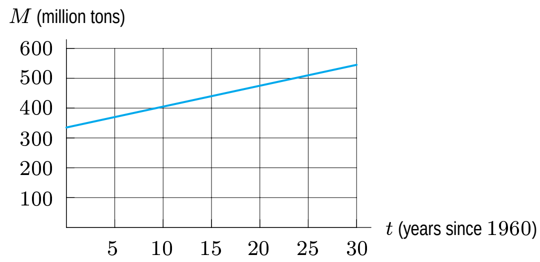


Figure 1.23

20. Figure 1.24 shows the distance from home, in miles, of a person on a 5-hour trip.

(a) Estimate the vertical intercept. Give units and interpret it in terms of distance from home.
(b) Estimate the slope of this linear function. Give units, and interpret it in terms of distance from home.
(c) Give a formula for distance, D, from home as a function of time, t in hours.

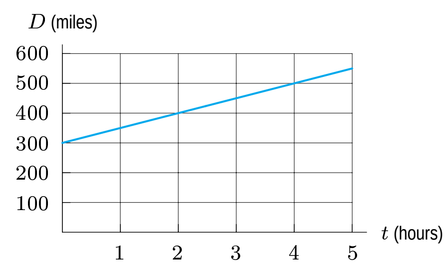


Figure 1.24

21. Values for the Canadian gold reserve, Q, in millions of fine troy ounces, are in Table 1.5.[7] Find a formula for the gold reserve as a linear function of time since 1986.

Table 1.5

Year	1986	1987	1988	1989	1990
Q (m. troy oz.)	19.72	18.48	17.24	16.00	14.76

22. Table 1.6 gives the average weight, w, in pounds, of American men in their sixties for various heights, h, in inches.[8]

(a) How do you know that the data in this table could represent a linear function?
(b) Find weight, w, as a linear function of height, h. What is the slope of the line? What are the units for the slope?
(c) Find height, h, as a linear function of weight, w. What is the slope of the line? What are the units for the slope?

Table 1.6

h (inches)	68	69	70	71	72	73	74	75
w (pounds)	166	171	176	181	186	191	196	201

[6] The Worldwatch Institute, *Vital Signs 2001*, p. 35, (New York: W.W. Norton, 2001).
[7] "Gold Reserves of Central Banks and Governments," *The World Almanac* (New Jersey: Funk and Wagnalls, 1992), p. 158.
[8] Adapted from "Average Weight of Americans by Height and Age", *The World Almanac* (New Jersey: Funk and Wagnalls, 1992), p. 956.

23. Search and rescue teams work to find lost hikers. Members of the search team separate and walk parallel to one another through the area to be searched. Table 1.7 shows the percent, P, of lost individuals found for various separation distances, d, of the searchers.[9]

Table 1.7

Separation distance d (ft)	20	40	60	80	100
Approximate percent found, P	90	80	70	60	50

(a) Explain how you know that the percent found, P, could be a linear function of separation distance, d.
(b) Find P as a linear function of d.
(c) What is the slope of the function? Give units and interpret the answer.
(d) What are the vertical and horizontal intercepts of the function? Give units and interpret the answers.

24. The monthly charge for a waste collection service is \$32 for 100 kg of waste and is \$48 for 180 kg of waste.

(a) Find a linear formula for the cost, C, of waste collection as a function of the number of kilograms of waste, w.
(b) What is the slope of the line found in part (a)? Give units and interpret your answer in terms of the cost of waste collection.
(c) What is the vertical intercept of the line found in part (a)? Give units with your answer and interpret it in terms of the cost of waste collection.

25. Sales of music compact disks (CDs) increased rapidly throughout the 1990s. Sales were 333.3 million in 1991 and 938.2 million in 1999.[10]

(a) Find a formula for sales, S, of music CDs, in millions of units, as a linear function of the number of years, t, since 1991.
(b) Give units for and interpret the slope and the vertical intercept of this function.
(c) Use the formula to predict music CD sales in 2005.

26. The number of species of coastal dune plants in Australia decreases as the latitude, in $^\circ$S, increases. There are 34 species at 11°S and 26 species at 44°S.[11]

(a) Find a formula for the number, N, of species of coastal dune plants in Australia as a linear function of the latitude, l, in $^\circ$S.
(b) Give units for and interpret the slope and the vertical intercept of this function.
(c) Graph this function between $l = 11^\circ$S and $l = 44^\circ$S. (Australia lies entirely within these latitudes.)

27. Residents of the town of Maple Grove who are connected to the municipal water supply are billed a fixed amount yearly plus a charge for each cubic foot of water used. A household using 1000 cubic feet was billed \$$90$, while one using 1600 cubic feet was billed \$$105$.

(a) What is the charge per cubic foot?
(b) Write an equation for the total cost of a resident's water as a function of cubic feet of water used.
(c) How many cubic feet of water used would lead to a bill of \$130?

28. A controversial 1992 Danish study[12] reported that the average male sperm count has decreased from 113 million per milliliter in 1940 to 66 million per milliliter in 1990.

(a) Express the average sperm count, S, as a linear function of the number of years, t, since 1940.
(b) A man's fertility is affected if his sperm count drops below about 20 million per milliliter. If the linear model found in part (a) is accurate, in what year will the average male sperm count fall below this level?

29. The graph of Fahrenheit temperature, $^\circ$F, as a function of Celsius temperature, $^\circ$C, is a line. You know that 212°F and 100°C both represent the temperature at which water boils. Similarly, 32°F and 0°C both represent water's freezing point.

(a) What is the slope of the graph?
(b) What is the equation of the line?
(c) Use the equation to find what Fahrenheit temperature corresponds to 20°C.
(d) What temperature is the same number of degrees in both Celsius and Fahrenheit?

30. You drive at a constant speed from Chicago to Detroit, a distance of 275 miles. About 120 miles from Chicago you pass through Kalamazoo, Michigan. Sketch a graph of your distance from Kalamazoo as a function of time.

1.3 RATES OF CHANGE

In the previous section, we saw that the height of the winning Olympic pole vault increased at an approximately constant rate of 2 inches/year between 1900 and 1912. Similarly, the world record for the mile decreased at an approximately constant rate of 0.4 seconds/year. We now see how to calculate rates of change when they are not constant.

[9] From *An Experimental Analysis of Grid Sweep Searching*, by J. Wartes (Explorer Search and Rescue, Western Region, 1974).
[10] From the *Recording Industry Association of America*, The World Almanac 2001, p. 313.
[11] Rosenzweig, M.L., *Species Diversity in Space and Time*, p. 292, (Cambridge: Cambridge University Press, 1995).
[12] "Investigating the Next Silent Spring", *US News and World Report*, p. 50-52, (March 11, 1996).

Example 1 Table 1.8 shows the height of the winning pole vault at the Olympics during the 1960s and 1980s. Find the rate of change of the winning height between 1960 and 1968, and between 1980 and 1988. In which of these two periods did the height increase faster than during the period 1900–1912?

Table 1.8 *Winning height in men's Olympic pole vault (approximate)*

Year	1960	1964	1968	...	1980	1984	1988
Height (inches)	185	201	212	...	227.5	226	237

Solution From 1900 to 1912, the height increased at 2 inches/year. To compare the 1960s and 1980s, we calculate

$$\text{Average rate of change of height 1960 to 1968} = \frac{\text{Change in height}}{\text{Change in time}} = \frac{212-185}{1968-1960} = 4.2 \text{ inches/year.}$$

$$\text{Average rate of change of height 1980 to 1988} = \frac{\text{Change in height}}{\text{Change in time}} = \frac{237-227.5}{1988-1980} = 2.375 \text{ inches/year.}$$

Thus, the height was increasing much more quickly during the 1960s than from 1900 to 1912. During the 1980s, the height was increasing slightly faster than from 1900 to 1912.

In Example 1, the function does not have a constant rate of change (it is not linear). However, we can compute an *average rate of change* over any interval. The word average is used because the rate of change may vary within the interval. We have the following general formula.

> If y is a function of t, so $y = f(t)$, then
>
> $$\textbf{Average rate of change of } y \text{ between } t=a \text{ and } t=b = \frac{\Delta y}{\Delta t} = \frac{f(b)-f(a)}{b-a}.$$
>
> The units of average rate of change of a function are units of y per unit of t.

The average rate of change of a linear function is the slope, and a function is linear if the rate of change is the same on all intervals.

Example 2 Using Figure 1.25, estimate the average rate of change of the number of farms[13] in the US between 1950 and 1970.

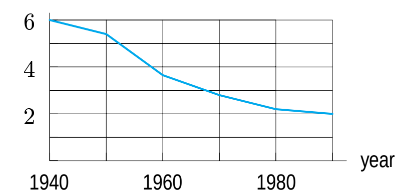


Figure 1.25: Number of farms in the US (in millions)

Solution Figure 1.25 shows that the number, N, of farms in the US was approximately 5.4 million in 1950 and approximately 2.8 million in 1970. If time, t, is in years, we have

$$\text{Average rate of change} = \frac{\Delta N}{\Delta t} = \frac{2.8-5.4}{1970-1950} = -0.13 \text{ million farms per year.}$$

The average rate of change is negative because the number of farms is decreasing. During this period, the number of farms decreased at an average rate of 0.13 million, or 130,000, farms per year.

[13] *The World Almanac* (New Jersey: Funk and Wagnalls, 1995), p. 135.

Increasing and Decreasing Functions

Since the rate of change of the height of the winning pole vault is positive, we know that the height is increasing. The rate of change of number of farms is negative, so the number of farms is decreasing. See Figure 1.26. In general:

A function f is **increasing** if the values of $f(x)$ increase as x increases.
A function f is **decreasing** if the values of $f(x)$ decrease as x increases.

The graph of an *increasing* function *climbs* as we move from left to right.
The graph of a *decreasing* function *descends* as we move from left to right.

Figure 1.26: Increasing and decreasing functions

We have looked at how an Olympic record and the number of farms change over time. In the next example, we look at a rate of change with respect to a quantity other than time.

Example 3 High levels of PCB (polychlorinated biphenyl, an industrial pollutant) in the environment affect pelicans' eggs. Table 1.9 shows that as the concentration of PCB in the eggshells increases, the thickness of the eggshell decreases, making the eggs more likely to break.[14]

Find the average rate of change in the thickness of the shell as the PCB concentration changes from 87 ppm to 452 ppm. Give units and explain why your answer is negative.

Table 1.9 *Thickness of pelican eggshells and PCB concentration in the eggshells*

Concentration, c, in parts per million (ppm)	87	147	204	289	356	452
Thickness, h, in millimeters (mm)	0.44	0.39	0.28	0.23	0.22	0.14

Solution Since we are looking for the average rate of change of thickness with respect to change in PCB concentration, we have

$$\text{Average rate of change of thickness} = \frac{\text{Change in the thickness}}{\text{Change in the PCB level}} = \frac{\Delta h}{\Delta c} = \frac{0.14 - 0.44}{452 - 87}$$
$$= -0.00082 \frac{\text{mm}}{\text{ppm}}.$$

The units are thickness units (mm) over PCB concentration units (ppm), or millimeters over parts per million. The average rate of change is negative because the thickness of the eggshell decreases as the PCB concentration increases. The thickness of pelican eggs decreases by an average of 0.00082 mm for every additional part per million of PCB in the eggshell.

[14] Risebrough, R. W., "Effects of environmental pollutants upon animals other than man." *Proceedings of the 6th Berkeley Symposium on Mathematics and Statistics, VI*, p. 443–463, (Berkeley: University of California Press, 1972).

Visualizing Rate of Change

For a function $y = f(x)$, the change in the value of the function between $x = a$ and $x = c$ is $\Delta y = f(c) - f(a)$. Since Δy is a difference of two y-values, it is represented by the vertical distance in Figure 1.27. The average rate of change of f between $x = a$ and $x = c$ is represented by the slope of the line joining the points A and C in Figure 1.28. This line is called the *secant line* between $x = a$ and $x = c$.

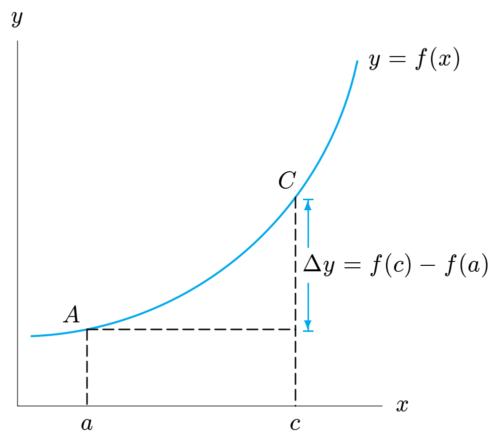


Figure 1.27: The change in a function is represented by a vertical distance

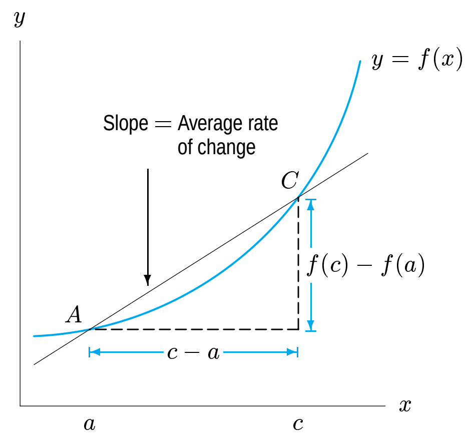


Figure 1.28: The average rate of change is represented by the slope of the line

Example 4

(a) Find the average rate of change of $y = f(x) = \sqrt{x}$ between $x = 1$ and $x = 4$.
(b) Graph $f(x)$ and represent this average rate of change as the slope of a line.
(c) Which is larger, the average rate of change of the function between $x = 1$ and $x = 4$ or the average rate of change between $x = 4$ and $x = 5$? What does this tell us about the graph of the function?

Solution

(a) Since $f(1) = \sqrt{1} = 1$ and $f(4) = \sqrt{4} = 2$, between $x = 1$ and $x = 4$, we have

$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{f(4) - f(1)}{4 - 1} = \frac{2 - 1}{3} = \frac{1}{3}.$$

(b) A graph of $f(x) = \sqrt{x}$ is given in Figure 1.29. The average rate of change of f between 1 and 4 is the slope of the secant line between $x = 1$ and $x = 4$.
(c) Since the secant line between $x = 1$ and $x = 4$ is steeper than the secant line between $x = 4$ and $x = 5$, the average rate of change between $x = 1$ and $x = 4$ is larger than it is between $x = 4$ and $x = 5$. The rate of change is decreasing. This tells us that the graph of this function is bending downward.

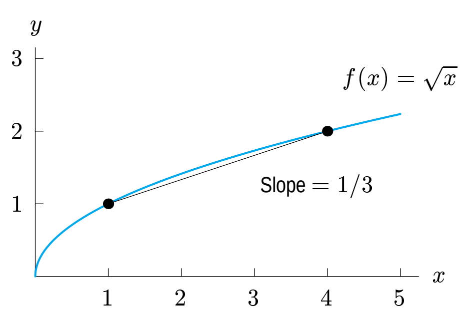


Figure 1.29: Average rate of change = Slope of secant line

Concavity

Figure 1.29 shows a graph that is bending downward because the rate of change is decreasing. The graph in Figure 1.27 bends upward because the rate of change of the function is increasing. We make the following definitions.

> The graph of a function is **concave up** if it bends upward as we move left to right; the graph is **concave down** if it bends downward. (See Figure 1.30.) A line is neither concave up nor concave down.

Figure 1.30: Concavity of a graph

Example 5 Using Figure 1.31, estimate the intervals over which:

(a) The function is increasing; decreasing. (b) The graph is concave up; concave down.

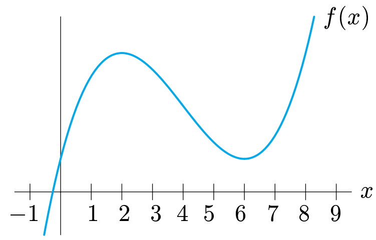


Figure 1.31

Solution

(a) The graph suggests that the function is increasing for $x < 2$ and for $x > 6$. It appears to be decreasing for $2 < x < 6$.

(b) The graph is concave down on the left and concave up on the right. It is difficult to tell exactly where the graph changes concavity, although it appears to be about $x = 4$. Approximately, the graph is concave down for $x < 4$ and concave up for $x > 4$.

Example 6 From the following values of $f(t)$, does f appear to be increasing or decreasing? Do you think its graph is concave up or concave down?

t	0	5	10	15	20	25	30
$f(t)$	12.6	13.1	14.1	16.2	20.0	29.6	42.7

Solution

Since the given values of $f(t)$ increase as t increases, f appears to be increasing. As we read from left to right, the change in $f(t)$ starts small and gets larger (for constant change in t), so the graph is climbing faster. Thus, the graph appears to be concave up. Alternatively, plot the points and notice that a curve through these points bends up.

Distance, Velocity, and Speed

A grapefruit is thrown up in the air. The height of the grapefruit above the ground first increases and then decreases. See Table 1.10.

Table 1.10 *Height, y, of the grapefruit above the ground t seconds after it is thrown*

t (sec)	0	1	2	3	4	5	6
y (feet)	6	90	142	162	150	106	30

Example 7 Find the change and average rate of change of the height of the grapefruit during the first 3 seconds. Give units and interpret your answers.

Solution The change in height during the first 3 seconds is $\Delta y = 162 - 6 = 156$ ft. This means that the grapefruit goes up a total of 156 feet during the first 3 seconds. The average rate of change during this 3 second interval is $156/3 = 52$ ft/sec. During the first 3 seconds, the grapefruit is rising at an average rate of 52 ft/sec.

The average rate of change of height with respect to time is *velocity*. You may recognize the units (feet per second) as units of velocity.

$$\text{Average velocity} = \frac{\text{Change in distance}}{\text{Change in time}} = \begin{matrix}\text{Average rate of change of distance}\\ \text{with respect to time}\end{matrix}$$

There is a distinction between *velocity* and *speed*. Suppose an object moves along a line. If we pick one direction to be positive, the velocity is positive if the object is moving in that direction and negative if it is moving in the opposite direction. For the grapefruit, upward is positive and downward is negative. Speed is the magnitude of velocity, so it is always positive or zero.

Example 8 Find the average velocity of the grapefruit over the interval $t = 4$ to $t = 6$. Explain the sign of your answer.

Solution Since the height is $y = 150$ feet at $t = 4$ and $y = 30$ feet at $t = 6$, we have

$$\text{Average velocity} = \frac{\text{Change in distance}}{\text{Change in time}} = \frac{\Delta y}{\Delta t} = \frac{30 - 150}{6 - 4} = -60 \text{ ft/sec.}$$

The negative sign means the height is decreasing and the grapefruit is moving downward.

Example 9 A car travels away from home on a straight road. Its distance from home at time t is shown in Figure 1.32. Is the car's average velocity greater during the first hour or during the second hour?

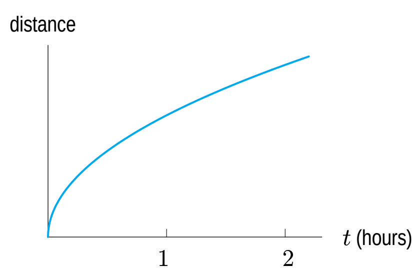


Figure 1.32: Distance of car from home

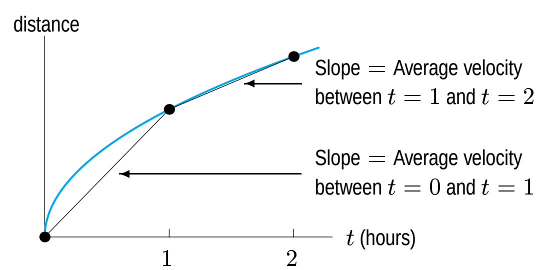


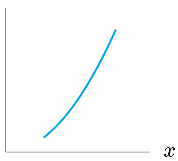
Figure 1.33: Average velocities of the car

Solution Average velocity is represented by the slope of a secant line. Figure 1.33 shows that the secant line between $t = 0$ and $t = 1$ is steeper than the secant line between $t = 1$ and $t = 2$. Thus, the average velocity is greater during the first hour.

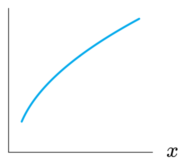
Problems for Section 1.3

In Problems 1–4, decide whether the graph is concave up, concave down, or neither.

1.



2.



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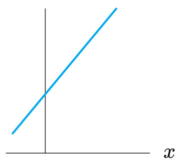

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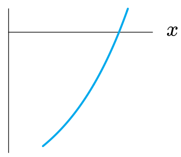


5. Graph a function $f(x)$ which is increasing everywhere and concave up for negative x and concave down for positive x.

6. Table 1.11 gives values of a function $w = f(t)$. Is this function increasing or decreasing? Is the graph of this function concave up or concave down?

Table 1.11

t	0	4	8	12	16	20	24
w	100	58	32	24	20	18	17

7. Identify the x-intervals on which the function graphed in Figure 1.34 is:

(a) Increasing and concave up
(b) Increasing and concave down
(c) Decreasing and concave up
(d) Decreasing and concave down

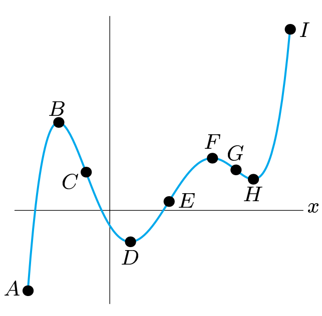


Figure 1.34

8. The total world marine catch[15] of fish, in tons, was 17 million in 1950 and 91 million in 1995. What was the average rate of change in the marine catch during this period? Give units and interpret your answer.

9. Figure 1.35 shows the total value of world exports (internationally traded goods), in billions of dollars.[16]

(a) Was the value of the exports higher in 1990 or in 1960? Approximately how much higher?
(b) Estimate the average rate of change between 1960 and 1990. Give units and interpret your answer in terms of the value of world exports.

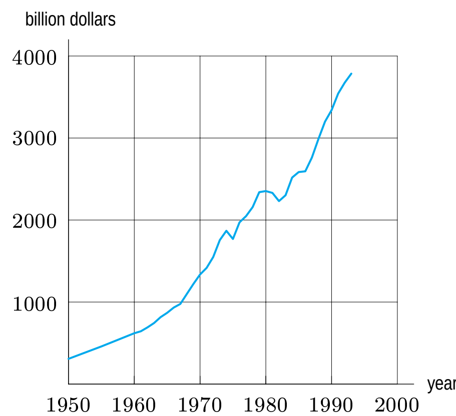


Figure 1.35

10. Table 1.12 shows world bicycle production.[17]

(a) Find the change in bicycle production between 1950 and 1990. Give units.
(b) Find the average rate of change in bicycle production between 1950 and 1990. Give units and interpret your answer in terms of bicycle production.

Table 1.12 *World bicycle production, in millions*

Year	1950	1960	1970	1980	1990	1993
Bicycles	11	20	36	62	90	108

11. Do you expect the average rate of change (in units per year) of each of the following to be positive or negative? Explain your reasoning.

(a) Number of acres of rain forest in the world.
(b) Population of the world.
(c) Number of polio cases each year in the US, since 1950.
(d) Height of a sand dune that is being eroded.
(e) Cost of living in the US.

[15] *Time magazine*, 11 August 1997, p. 67.
[16] Lester R. Brown, et al., *Vital Signs 1994*, p. 77, (New York: W. W. Norton, 1994).
[17] Lester R. Brown, et al., *Vital Signs 1994*, p. 87, (New York: W. W. Norton, 1994).

12. Table 1.13 shows the total amount spent on tobacco products in the US.

(a) What is the average rate of change in the amount spent on tobacco products between 1987 and 1993? Give units and interpret your answer in terms of money spent on tobacco products.

(b) During this six-year period, is there any interval during which the average rate of change was negative? If so, when?

Table 1.13 *Tobacco spending, in billions of dollars*

Year	1987	1988	1989	1990	1991	1992	1993
Spending	35.6	36.2	40.5	43.4	45.4	50.9	50.5

13. Find the average rate of change of $f(x) = 2x^2$ between $x = 1$ and $x = 3$.

14. Table 1.14 gives the net profit of The Gap, Inc, which operates nearly 2000 clothing stores.[18]

(a) Find the change in net profit between 1993 and 1996.

(b) Find the average rate of change in net profit between 1993 and 1996. Give units and interpret your answer.

(c) From 1990 to 1997, were there any one-year intervals during which the average rate of change was negative? If so, when?

Table 1.14 *Gap net profit, in millions of dollars*

Year	1990	1991	1992	1993	1994	1995	1996
Profit	144.5	229.9	210.7	258.4	350.2	354.0	452.9

15. Figure 1.11 on page 5 shows the amount of nicotine $N = f(t)$, in mg, in a person's bloodstream as a function of the time, t, in hours, since the last cigarette.

(a) Is the average rate of change in nicotine level positive or negative? Explain.

(b) Find the average rate of change in the nicotine level between $t = 0$ and $t = 3$. Give units and interpret your answer in terms of nicotine.

16. Find the average rate of change of $f(x) = 3x^2 + 4$ between $x = -2$ and $x = 1$. Illustrate your answer graphically.

17. When a deposit of $1000 is made into an account paying 8% interest, compounded annually, the balance, $\$B$, in the account after t years is given by $B = 1000(1.08)^t$. Find the average rate of change in the balance over the interval $t = 0$ to $t = 5$. Give units and interpret your answer in terms of the balance in the account.

18. Figure 1.36 shows the length, L, in cm, of a sturgeon (a type of fish) as a function of the time, t, in years.[19]

(a) Is the function increasing or decreasing? Is the graph concave up or concave down?

(b) Estimate the average rate of growth of the sturgeon between $t = 5$ and $t = 15$. Give units and interpret your answer in terms of the sturgeon.

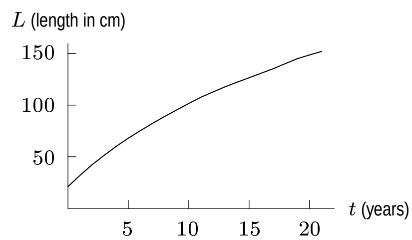


Figure 1.36

19. Table 1.15 shows the total US labor force, L. Find the average rate of change between 1930 and 1990; between 1930 and 1950; between 1950 and 1970. Give units and interpret your answers in terms of the labor force.[20]

Table 1.15 *US labor force, in thousands of workers*

Year	1930	1940	1950	1960	1970	1980	1990
L	29,424	32,376	45,222	54,234	70,920	90,564	103,905

20. The number of US households with cable television was 12,168,450 in 1977 and 65,929,420 in 1997. Estimate the average rate of change in the number of US households with cable television during this 20-year period. Give units and interpret your answer.

21. Table 1.16 gives the sales, S, of Intel Corporation, a leading manufacturer of integrated circuits.[21]

(a) Find the change in sales between 1991 and 1995.

(b) Find the average rate of change in sales between 1991 and 1995. Give units and interpret your answer.

(c) If the average rate of change continues at the same rate as between 1995 and 1997, in which year will the sales first reach 40,000 million dollars?

Table 1.16 *Intel sales, in millions of dollars*

Year	1990	1991	1992	1993	1994	1995	1996	1997
S	3921	4779	5844	8782	11,521	16,202	20,847	25,070

22. The volume of water in a pond over a period of 20 weeks is shown in Figure 1.37.

(a) Is the average rate of change of volume positive or negative over the following intervals?

(i) $t = 0$ and $t = 5$ (ii) $t = 0$ and $t = 10$
(iii) $t = 0$ and $t = 15$ (iv) $t = 0$ and $t = 20$

[18] From *Value Line Investment Survey*, November 21, 1997, (New York: Value Line Publishing, Inc.) p. 1700.
[19] Data from von Bertalanffy, L., *General System Theory*, p. 177, (New York: Braziller, 1968).
[20] *The World Almanac and Book of Facts 1995*, p. 154, (New Jersey: Funk & Wagnalls, 1994).
[21] From *Value Line Investment Survey*, January 23, 1998, (New York: Value Line Publishing, Inc.) p. 1060.

(b) During which of the following time intervals was the average rate of change larger?
(i) $0 \leq t \leq 5$ or $0 \leq t \leq 10$
(ii) $0 \leq t \leq 10$ or $0 \leq t \leq 20$

(c) Estimate the average rate of change between $t = 0$ and $t = 10$. Interpret your answer in terms of water.

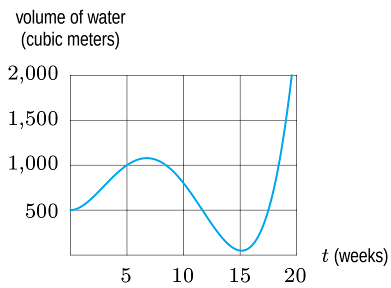


Figure 1.37

23. Table 1.17 shows the number of NCAA Division I men's basketball games.[22]

(a) Find the average rate of change in the number of games from 1983 to 1989. Give units.
(b) Find the annual increase in the number of games for each year from 1983 to 1989. (Your answer should be six numbers.)
(c) Show that the average rate of change found in part (a) is the average of the six yearly changes found in part (b).

Table 1.17 *NCAA Division I basketball*

Year	1983	1984	1985	1986	1987	1988	1989
Games	7957	8029	8269	8360	8580	8587	8677

24. A car starts slowly and then speeds up. Eventually the car slows down and stops. Graph the distance that the car has traveled against time.

25. Draw a graph of distance against time with the following properties: The average velocity is always positive and the average velocity for the first half of the trip is less than the average velocity for the second half of the trip.

26. When a new product is advertised, more and more people try it. However, the rate at which new people try it slows as time goes on.

(a) Graph the total number of people who have tried such a product against time.
(b) What do you know about the concavity of the graph?

27. Figure 1.38 shows the position of an object at time t.

(a) Draw a line on the graph whose slope represents the average velocity between $t = 2$ and $t = 8$.
(b) Is average velocity greater between $t = 0$ and $t = 3$ or between $t = 3$ and $t = 6$?
(c) Is average velocity positive or negative between $t = 6$ and $t = 9$?

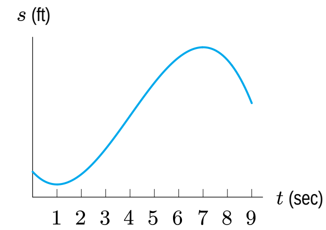


Figure 1.38

28. In an experiment, a lizard is encouraged to run as fast as possible. Figure 1.39 shows the distance run in meters as a function of the time in seconds.[23]

(a) If the lizard were running faster and faster, what would be the concavity of the graph? Does this match what you see?
(b) Estimate the average velocity of the lizard during this 0.8 second experiment.

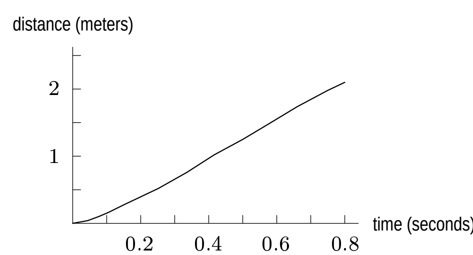


Figure 1.39

29. Sketch reasonable graphs for the following. Pay particular attention to the concavity of the graphs.

(a) The total revenue generated by a car rental business, plotted against the amount spent on advertising.
(b) The temperature of a cup of hot coffee standing in a room, plotted as a function of time.

30. Each of the functions g, h, k in Table 1.18 is increasing, but each increases in a different way. Which of the graphs in Figure 1.40 best fits each function?

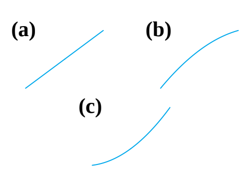


Figure 1.40

Table 1.18

t	$g(t)$	$h(t)$	$k(t)$
1	23	10	2.2
2	24	20	2.5
3	26	29	2.8
4	29	37	3.1
5	33	44	3.4
6	38	50	3.7

[22] *The World Almanac and Book of Facts 1992*, p. 863, (New York: Pharos Books, 1991).
[23] Data from Huey, R.B. and Hertz, P.E., "Effects of Body Size and Slope on the Acceleration of a Lizard", *J. Exp. Biol.*, Volume 110, 1984, p. 113-123.

1.4 APPLICATIONS OF FUNCTIONS TO ECONOMICS

In this section, we look at some of the functions of interest to decision-makers in a firm or industry.

The Cost Function

The **cost function**, $C(q)$, gives the total cost of producing a quantity q of some good.

What sort of function do you expect C to be? The more goods that are made, the higher the total cost, so C is an increasing function. Costs of production can be separated into two parts: the *fixed costs*, which are incurred even if nothing is produced, and the *variable cost*, which depends on how many units are produced. We consider linear functions in this section. Nonlinear cost functions are discussed in Section 2.5.

An Example: Manufacturing Costs

Let's consider a company that makes radios. The factory and machinery needed to begin production are fixed costs, which are incurred even if no radios are made. The costs of labor and raw materials are variable costs since these quantities depend on how many radios are made. The fixed costs for this company are \$24,000 and the variable costs are \$7 per radio. Then

$$\begin{aligned}\text{Total costs for the company} &= \text{Fixed cost} + \text{Variable cost} \\ &= 24{,}000 + 7 \cdot \text{Number of radios},\end{aligned}$$

so, if q is the number of radios produced,

$$C(q) = 24{,}000 + 7q.$$

This is the equation of a line with slope 7 and vertical intercept 24,000.

Example 1 Graph the cost function $C(q) = 24{,}000 + 7q$. Label the fixed costs and variable cost per unit.

Solution The graph of the cost function is the line in Figure 1.41. The fixed costs are represented by the vertical intercept of 24,000. The variable cost per unit is represented by the slope of 7, which is the change in cost corresponding to unit change in production.

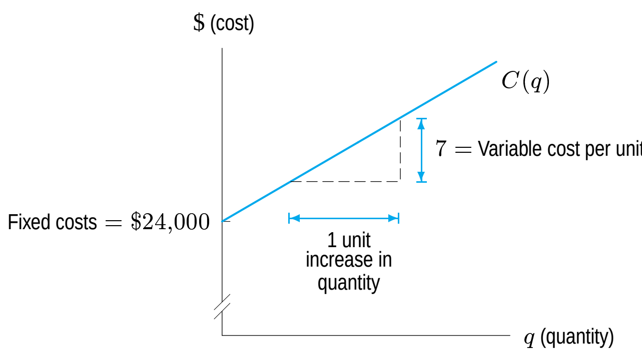


Figure 1.41: Cost function for the radio manufacturer

If $C(q)$ is a linear cost function,

- Fixed costs are represented by the vertical intercept.
- Variable costs per unit are represented by the slope.

Example 2 In each case, draw a graph of a linear cost function satisfying the given conditions:

(a) Fixed costs are large but variable cost per unit is small.
(b) There are no fixed costs but variable cost per unit is high.

Solution

(a) The graph is a line with a large vertical intercept and a small slope. See Figure 1.42.
(b) The graph is a line with a vertical intercept of zero (so the line goes through the origin) and a large positive slope. See Figure 1.43. Figures 1.42 and 1.43 have the same scales.

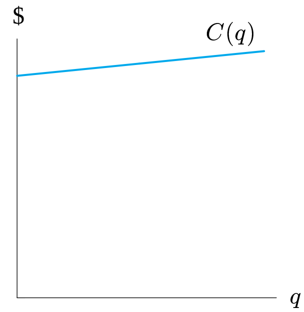


Figure 1.42: Large fixed costs, small variable cost per unit

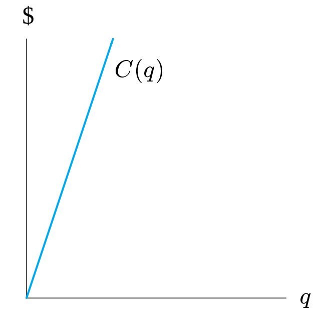


Figure 1.43: No fixed costs, high variable cost per unit

The Revenue Function

> The **revenue function**, $R(q)$, gives the total revenue received by a firm from selling a quantity, q, of some good.

If the good sells for a price of p per unit, and the quantity sold is q, then

$$\text{Revenue} = \text{Price} \cdot \text{Quantity}, \quad \text{so} \quad R = pq.$$

If the price does not depend on the quantity sold, so p is a constant, the graph of revenue as a function of q is a line through the origin, with slope equal to the price p.

Example 3 If radios sell for \$15 each, sketch the manufacturer's revenue function. Show the price of a radio on the graph.

Solution Since $R(q) = pq = 15q$, the revenue graph is a line through the origin with a slope of 15. See Figure 1.44. The price is the slope of the line.

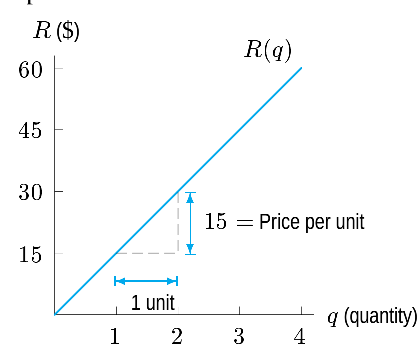


Figure 1.44: Revenue function for the radio manufacturer

Example 4 Sketch graphs of the cost function $C(q) = 24{,}000 + 7q$ and the revenue function $R(q) = 15q$ on the same axes. For what values of q does the company make money? Explain your answer graphically.

Solution The company makes money whenever revenues are greater than costs, so we want to find the values of q for which the graph of $R(q)$ lies above the graph of $C(q)$. See Figure 1.45.

We find the point at which the graphs of $R(q)$ and $C(q)$ cross:

$$\begin{aligned} \text{Revenue} &= \text{Cost} \\ 15q &= 24{,}000 + 7q \\ 8q &= 24{,}000 \\ q &= 3000. \end{aligned}$$

Thus, the company makes a profit if it produces and sells more than 3000 radios. The company loses money if it produces and sells fewer than 3000 radios.

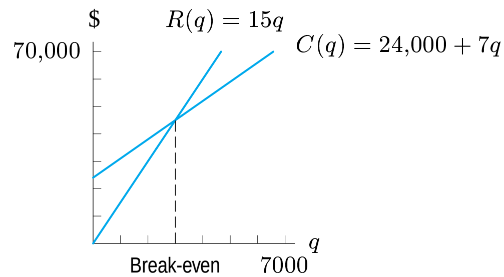


Figure 1.45: Cost and revenue functions for the radio manufacturer: What values of q generate a profit?

The Profit Function

Decisions are often made by considering the profit, usually written[24] as π to distinguish it from the price, p. We have

$$\text{Profit} = \text{Revenue} - \text{Cost} \quad \text{so} \quad \pi = R - C.$$

The *break-even point* for a company is the point where the profit is zero and revenue equals cost. A break-even point can refer either to a quantity q at which revenue equals cost, or to a point on a graph.

Example 5 Find a formula for the profit function of the radio manufacturer. Graph it, marking the break-even point.

Solution Since $R(q) = 15q$ and $C(q) = 24{,}000 + 7q$, we have

$$\pi(q) = 15q - (24{,}000 + 7q) = -24{,}000 + 8q.$$

Notice that the negative of the fixed costs is the vertical intercept and the break-even point is the horizontal intercept. See Figure 1.46.

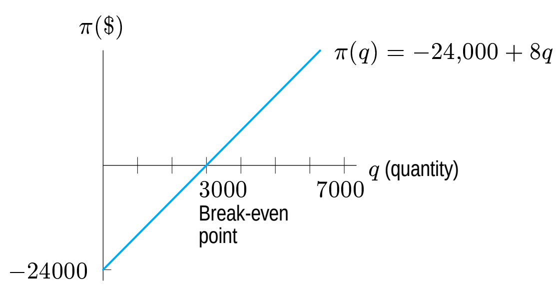


Figure 1.46: Profit for radio manufacturer

[24] This π has nothing to do with the area of a circle, and merely stands for the Greek equivalent of the letter "p."

Example 6

(a) Using Table 1.19, estimate the break-even point for this company.
(b) Find the company's profit if 1000 units are produced.
(c) What price do you think the company is charging for its product?

Table 1.19 *Company's estimates of cost and revenue for a product*

q	500	600	700	800	900	1000	1100
$C(q)$	5000	5500	6000	6500	7000	7500	8000
$R(q)$	4000	4800	5600	6400	7200	8000	8800

Solution

(a) The break-even point is the value of q for which revenue equals cost. Since revenue is below cost at $q = 800$ and revenue is greater than cost at $q = 900$, the break-even point is between 800 and 900. The values in the table suggest that the break-even point is closer to 800, as the cost and revenue are closer there. A reasonable estimate for the break-even point is $q = 830$.
(b) If the company produces 1000 units, the cost is \$7500 and the revenue is \$8000, so the profit is $8000 - 7500 = 500$ dollars.
(c) From the data given, it appears that $R(q) = 8q$. This indicates the company is selling the product for \$8 each.

The Marginal Cost, Marginal Revenue, and Marginal Profit

In economics and business, the terms marginal cost, marginal revenue, and marginal profit are used for the rate of change of cost, revenue, and profit, respectively. The term *marginal* is used to highlight the rate of change as an indicator of how the cost, revenue, or profit changes in response to a one unit (that is, marginal) change in quantity. For example, for the cost, revenue, and profit functions of the radio manufacturer, the marginal cost is 7 dollars/item (the additional cost of producing one more item is \$7), the marginal revenue is 15 dollars/item (the additional revenue from selling one more item is \$15), and the marginal profit is 8 dollars/item (the additional profit from selling one more item is \$8).

The Depreciation Function

Suppose that the radio manufacturer has a machine that costs \$20,000. The managers of the company plan to keep the machine for ten years and then sell it for \$3000. We say the value of their machine *depreciates* from \$20,000 today to a resale value of \$3000 in ten years. The depreciation formula gives the value, $V(t)$, of the machine as a function of the number of years, t, since the machine was purchased. We assume that the value of the machine depreciates linearly.

The value of the machine when it is new ($t = 0$) is \$20,000, so $V(0) = 20{,}000$. The resale value at time $t = 10$ is \$3000, so $V(10) = 3000$. We have

$$\text{Slope} = m = \frac{3000 - 20{,}000}{10 - 0} = \frac{-17{,}000}{10} = -1700.$$

This slope tells us that the value of the machine is decreasing at a rate of \$1700 per year. Since $V(0) = 20{,}000$, the vertical intercept is 20,000, so

$$V(t) = 20{,}000 - 1700t.$$

Supply and Demand Curves

The quantity, q, of an item that is manufactured and sold, depends on its price, p. We usually assume that as the price increases, manufacturers are willing to supply more of the product, whereas the quantity demanded by consumers falls. Since manufacturers and consumers react differently to changes in price, there are two curves relating p and q.

> The **supply curve**, for a given item, relates the quantity, q, of the item that manufacturers are willing to make per unit time to the price, p, for which the item can be sold.
> The **demand curve** relates the quantity, q, of an item demanded by consumers per unit time to the price, p, of the item.

Economists often think of the quantities supplied and demanded as functions of price. However, for historical reasons, the economists put price (the independent variable) on the vertical axis and quantity (the dependent variable) on the horizontal axis. (The reason for this state of affairs is that economists originally took price to be the dependent variable and put it on the vertical axis. Later, when the point of view changed, the axes did not.) Thus, typical supply and demand curves look like those shown in Figure 1.47.

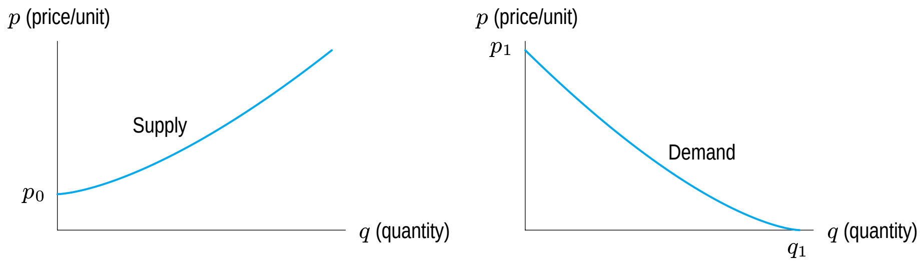


Figure 1.47: Supply and demand curves

Example 7 What is the economic meaning of the prices p_0 and p_1 and the quantity q_1 in Figure 1.47?

Solution The vertical axis corresponds to a quantity of zero. Since the price p_0 is the vertical intercept on the supply curve, p_0 is the price at which the quantity supplied is zero. In other words, unless the price is above p_0, the suppliers will not produce anything. The price p_1 is the vertical intercept on the demand curve, so it corresponds to the price at which the quantity demanded is zero. In other words, unless the price is below p_1, consumers won't buy any of the product.

The horizontal axis corresponds to a price of zero, so the quantity q_1 on the demand curve is the quantity that would be demanded if the price were zero—or the quantity that could be given away if the item were free.

Equilibrium Price and Quantity

If we plot the supply and demand curves on the same axes, as in Figure 1.48, the graphs cross at the *equilibrium point*. The values p^* and q^* at this point are called the *equilibrium price* and *equilibrium quantity*, respectively. It is assumed that the market naturally settles to this equilibrium point. (See Problem 23.)

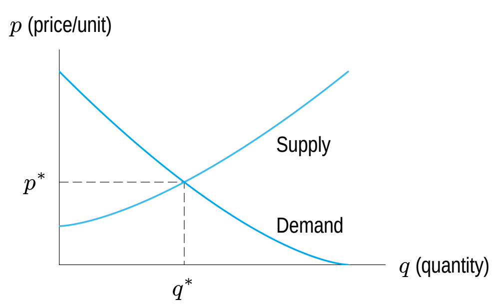


Figure 1.48: The equilibrium price and quantity

Example 8 Find the equilibrium price and quantity if

$$\text{Quantity supplied } = S(p) = 3p - 50 \quad \text{and} \quad \text{Quantity demanded } = D(p) = 100 - 2p.$$

Solution

To find the equilibrium price and quantity, we find the point at which

$$\begin{aligned} \text{Supply} &= \text{Demand} \\ 3p - 50 &= 100 - 2p \\ 5p &= 150 \\ p &= 30. \end{aligned}$$

The equilibrium price is \$30. To find the equilibrium quantity, we use either the demand curve or the supply curve. At a price of \$30, the quantity produced is $100 - 2(30) = 100 - 60 = 40$ items. The equilibrium quantity is 40 items. In Figure 1.49, the demand and supply curves intersect at $p^* = 30$ and $q^* = 40$.

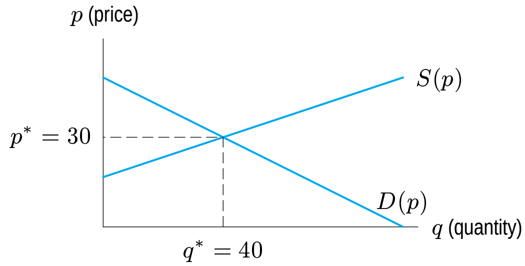


Figure 1.49: Equilibrium: $p^* = 30$, $q^* = 40$

The Effect of Taxes on Equilibrium

As in Example 8, suppose that the supply and demand curves for a product are

$$S(p) = 3p - 50 \quad \text{and} \quad D(p) = 100 - 2p.$$

What effect do taxes have on the equilibrium price and quantity for this product? And who (the producer or the consumer) ends up paying for the tax? We distinguish between two types of taxes.[25] A *specific tax* is a fixed amount per unit of a product sold regardless of the selling price. This is the case with such items as gasoline, alcohol, and cigarettes. A specific tax is usually imposed on the producer. A *sales tax* is a fixed percentage of the selling price. Many cities and states collect sales tax on a wide variety of items. A sales tax is usually imposed on the consumer. We consider a specific tax now; a sales tax is considered in Problems 27 and 28.

Suppose a specific tax of \$5 per unit is imposed upon suppliers. This means that a selling price of p dollars does not bring forth the same quantity supplied as before, since suppliers only receive $p - 5$ dollars. The amount supplied corresponds to $p - 5$, while the amount demanded still corresponds to p, the price the consumers pay. We have

$$\begin{aligned} \text{Quantity demanded} &= D(p) = 100 - 2p \\ \text{Quantity supplied} &= S(p - 5) = 3(p - 5) - 50 \\ &= 3p - 15 - 50 \\ &= 3p - 65. \end{aligned}$$

What are the equilibrium price and quantity in this situation? At the equilibrium price, we have

$$\begin{aligned} \text{Demand} &= \text{Supply} \\ 100 - 2p &= 3p - 65 \\ 165 &= 5p \\ p &= 33. \end{aligned}$$

The equilibrium price is now \$33. In Example 8, the equilibrium price is \$30, so the equilibrium price increases by \$3 as a result of the tax. Notice that this is less than the amount of the tax. The

[25] Adapted from Barry Bressler, *A Unified Approach to Mathematical Economics*, p. 81–88, (New York: Harper & Row, 1975).

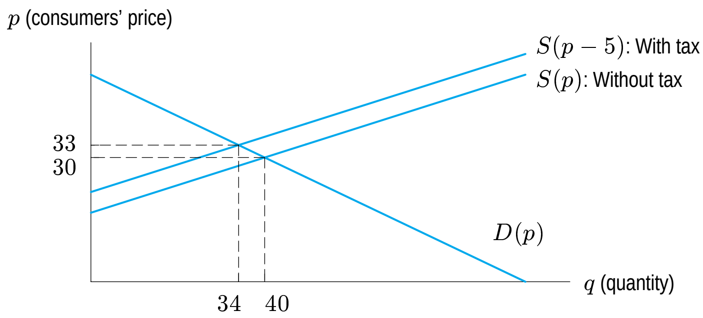


Figure 1.50: Specific tax shifts the supply curve, altering the equilibrium price and quantity

consumer ends up paying $\$3$ more than if the tax did not exist. However the government receives $\$5$ per item. Thus the producer pays the other $\$2$ of the tax, retaining $\$28$ of the price paid per item. Thus, although the tax was imposed on the producer, some of the tax is passed on to the consumer in terms of higher prices. The actual cost of the tax is split between the consumer and the producer.

The equilibrium quantity is now 34 units, since the quantity demanded is $D(33) = 34$. Not surprisingly, the tax has reduced the number of items sold. See Figure 1.50.

A Budget Constraint

An ongoing debate in the federal government concerns the allocation of money between defense and social programs. In general, the more that is spent on defense, the less that is available for social programs, and vice versa. Let's simplify the example to guns and butter. Assuming a constant budget, we show that the relationship between the number of guns and the quantity of butter is linear. Suppose that there is $\$12{,}000$ to be spent and that it is to be divided between guns, costing $\$400$ each, and butter, costing $\$2000$ a ton. Suppose the number of guns bought is g, and the number of tons of butter is b. Then the amount of money spent on guns is $\$400g$, and the amount spent on butter is $\$2000b$. Assuming all the money is spent,

$$\text{Amount spent on guns} + \text{Amount spent on butter} = \$12{,}000$$

or

$$400g + 2000b = 12{,}000.$$

Thus, dividing both sides by 400,

$$g + 5b = 30.$$

This equation is the budget constraint. It represents an *implicitly defined function*, because neither g nor b is given explicitly in terms of the other. If we solve for g, we get

$$g = 30 - 5b,$$

which is an explicit formula for g in terms of b. Similarly, solving for b leads to

$$b = \frac{30 - g}{5} \quad \text{or} \quad b = 6 - 0.2g,$$

which gives b as an *explicit function* of g. Since the explicit functions

$$g = 30 - 5b \quad \text{and} \quad b = 6 - 0.2g$$

are linear, the graph of the budget constraint is a line. See Figure 1.51.

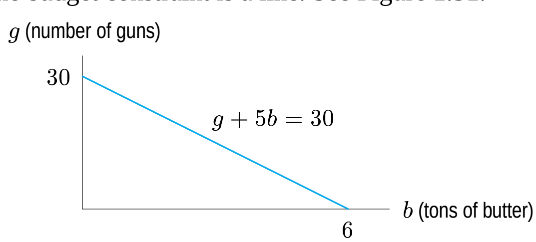


Figure 1.51: Budget constraint

Problems for Section 1.4

1. An amusement park charges an admission fee of \$7 per person as well as an additional \$1.50 for each ride.

(a) For one visitor, find the park's total revenue $R(n)$ as a function of the number of rides, n, taken.
(b) Find $R(2)$ and $R(8)$ and interpret your answers in terms of amusement park fees.

2. A company has cost function $C(q) = 4000 + 2q$ dollars and revenue function $R(q) = 10q$ dollars.

(a) What are the fixed costs for the company?
(b) What is the variable cost per unit?
(c) What price is the company charging for its product?
(d) Graph $C(q)$ and $R(q)$ on the same axes and label the break-even point, q_0. Explain how you know the company makes a profit if the quantity produced is greater than q_0.
(e) Find the break-even point q_0.

3. Values of a linear cost function are in Table 1.20. What are the fixed costs and the variable cost per unit (the marginal cost)? Find a formula for the cost function.

Table 1.20

q	0	5	10	15	20
$C(q)$	5000	5020	5040	5060	5080

4. **(a)** Estimate the fixed costs and the variable cost per unit for the cost function in Figure 1.52.
(b) Estimate $C(10)$ and interpret it in terms of cost.

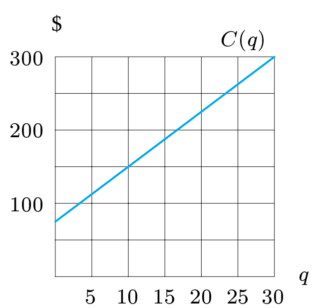


Figure 1.52

5. Figure 1.53 shows cost and revenue for a company.

(a) Approximately what quantity does this company have to produce to make a profit?
(b) Estimate the profit generated by 600 units.

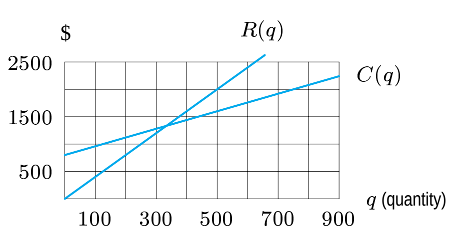


Figure 1.53

6. **(a)** What are the fixed costs and the variable cost per unit for the cost function in Figure 1.54?
(b) Explain what the fact that $C(100) = 2500$ tells you about costs.

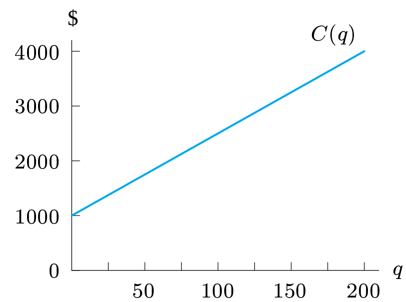


Figure 1.54

7. In Figure 1.55, which shows the cost and revenue functions for a product, label each of the following:

(a) Fixed costs **(b)** Break-even quantity

(c) Quantities at which the company:
(i) Makes a profit
(ii) Loses money

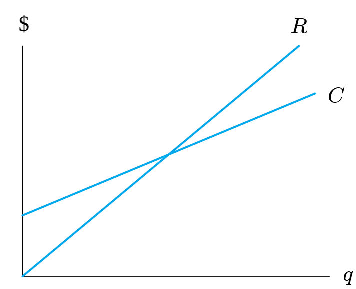


Figure 1.55

8. A company has cost and revenue functions, in dollars, given by $C(q) = 6000 + 10q$ and $R(q) = 12q$.

(a) Find the cost and revenue if the company produces 500 units. Does the company make a profit? What about 5000 units?
(b) Find the break-even point and illustrate it graphically.

9. A company producing jigsaw puzzles has fixed costs of \$6000 and variable costs of \$2 per puzzle. The company sells the puzzles for \$5 each.

(a) Find formulas for the cost function, the revenue function, and the profit function.
(b) Sketch a graph of $R(q)$ and $C(q)$ on the same axes. What is the break-even point, q_0, for the company?

10. A company that makes Adirondack chairs has fixed costs of \$5000 and variable costs of \$30 per chair. The company sells the chairs for \$50 each.

(a) Find formulas for the cost and revenue functions.
(b) Find the marginal cost and marginal revenue.
(c) Graph the cost and the revenue functions on the same axes.
(d) Find the break-even point.

11. Production costs for manufacturing running shoes consist of a fixed overhead of \$650,000 plus variable costs of \$20 per pair of shoes. Each pair of shoes sells for \$70.

(a) Find the total cost, $C(q)$, the total revenue, $R(q)$, and the total profit, $\pi(q)$, as a function of the number of pairs of shoes produced, q.
(b) Find the marginal cost, marginal revenue, and marginal profit.
(c) How many pairs of shoes must be produced and sold for the company to make a profit?

12. **(a)** Give an example of a possible company where the fixed costs are zero (or very small).
(b) Give an example of a possible company where the variable cost per unit is zero (or very small).

13. A photocopying company has two different price lists. The first price list is \$100 plus 3 cents per copy; the second price list is \$200 plus 2 cents per copy.

(a) For each price list, find the total cost as a function of the number of copies needed.
(b) Determine which price list is cheaper for 5000 copies.
(c) For what number of copies do both price lists charge the same amount?

14. A \$15,000 robot depreciates linearly to zero in 10 years.

(a) Find a formula for its value as a function of time.
(b) How much is the robot worth three years after it is purchased?

15. A \$50,000 tractor has a resale value of \$10,000 twenty years after it was purchased. Assume that the value of the tractor depreciates linearly from the time of purchase.

(a) Find a formula for the value of the tractor as a function of the time since it was purchased.
(b) Graph the value of the tractor against time.
(c) Find the horizontal and vertical intercepts, give units, and interpret them.

16. You have a budget of \$1000 for the year to cover your books and social outings. Books cost (on average) \$40 each and social outings cost (on average) \$10 each. Let b denote the number of books purchased per year and s denote the number of social outings in a year.

(a) What is the equation of your budget constraint?
(b) Graph the budget constraint. (It doesn't matter which variable you put on which axis.)
(c) Find the vertical and horizontal intercepts, and give a financial interpretation for each.

17. A company has a total budget of \$500,000 and spends this budget on raw materials and personnel. The company uses m units of raw materials, at a cost of \$100 per unit, and hires r employees, at a cost of \$25,000 each.

(a) What is the equation of the company's budget constraint?
(b) Solve for m as a function of r.
(c) Solve for r as a function of m.

18. One of the graphs in Figure 1.56 is a supply curve, and the other is a demand curve. Which is which? Explain how you made your decision using what you know about the effect of price on supply and demand.

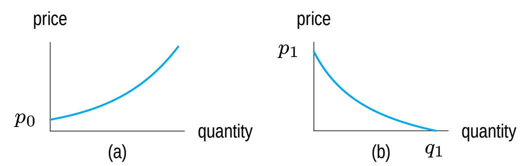


Figure 1.56

19. Table 1.21 gives data for the linear demand curve for a product, where p is the price of the product and q is the quantity sold every month at that price. Find formulas for the following functions. Interpret their slopes in terms of demand.

(a) q as a function of p. **(b)** p as a function of q.

Table 1.21

p (dollars)	16	18	20	22	24
q (tons)	500	460	420	380	340

20. A demand curve is given by $75p + 50q = 300$, where p is the price of the product, in dollars, and q is the quantity demanded at that price. Find p- and q-intercepts and interpret them in terms of consumer demand.

21. One of Tables 1.22 and 1.23 represents a supply curve; the other represents a demand curve.

(a) Which table represents which curve? Why?
(b) At a price of \$155, approximately how many items would consumers purchase?
(c) At a price of \$155, approximately how many items would manufacturers supply?
(d) Will the market push prices higher or lower than \$155?
(e) What would the price have to be if you wanted consumers to buy at least 20 items?
(f) What would the price have to be if you wanted manufacturers to supply at least 20 items?

Table 1.22

p (\$/unit)	182	167	153	143	133	125	118
q (quantity)	5	10	15	20	25	30	35

Table 1.23

p (\$/unit)	6	35	66	110	166	235	316
q (quantity)	5	10	15	20	25	30	35

22. The US production, Q, of copper in metric tons and the value, P, in thousands of dollars per metric ton are given[26] in Table 1.24. Plot the value as a function of production. Sketch a possible supply curve.

Table 1.24 *US copper production*

Year	1984	1985	1986	1987	1988	1989
Q	1103	1105	1144	1244	1417	1497
P	1473	1476	1456	1818	2656	2888

23. Figure 1.57 shows supply and demand for a product.

(a) What is the equilibrium price for this product? At this price, what quantity is produced?

(b) Choose a price above the equilibrium price—for example, $p = 12$. At this price, how many items are suppliers willing to produce? How many items do consumers want to buy? Use your answers to these questions to explain why, if prices are above the equilibrium price, the market tends to push prices lower (towards the equilibrium).

(c) Now choose a price below the equilibrium price—for example, $p = 8$. At this price, how many items are suppliers willing to produce? How many items do consumers want to buy? Use your answers to these questions to explain why, if prices are below the equilibrium price, the market tends to push prices higher (towards the equilibrium).

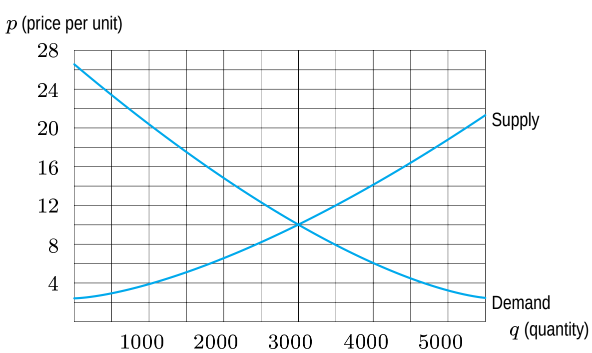


Figure 1.57

24. When the price, p, charged for a boat tour was \$25, the average number of passengers per week, N, was 500. When the price was reduced to \$20, the average number of passengers per week increased to 650. Find a formula for the demand curve, assuming that it is linear.

25. You have a budget of $\$k$ to spend on soda and suntan oil, which cost $\$p_1$ per liter and $\$p_2$ per liter respectively.

(a) Write an equation expressing the relationship between the number of liters of soda and the number of liters of suntan oil that you can buy if you exhaust your budget. This is your budget constraint.

(b) Graph the budget constraint, assuming that you can buy fractions of a liter. Label the intercepts.

(c) Suppose your budget is doubled. Graph the new budget constraint on the same axes.

(d) With a budget of $\$k$, the price of suntan oil doubles. Graph the new budget constraint on the same axes.

26. The demand and supply curves for a certain product are given in terms of price, p, by

$$D(p) = 2500 - 20p \quad \text{and} \quad S(p) = 10p - 500.$$

(a) Find the equilibrium price and quantity. Represent your answers on a graph.

(b) If a specific tax of \$6 per unit is imposed on suppliers, find the new equilibrium price and quantity. Represent your answers on the graph.

(c) How much of the \$6 tax is paid by consumers and how much by producers?

(d) What is the total tax revenue received by the government?

27. In Example 8, the demand and supply curves are given by $D(p) = 100 - 2p$ and $S(p) = 3p - 50$, the equilibrium price is \$30 and the equilibrium quantity is 40 units. Suppose that a sales tax of 5% is imposed on the consumer, so that the consumer pays $p + 0.05p$, while the supplier's price is p.

(a) Find the new equilibrium price and quantity.

(b) How much is paid in taxes on each unit? How much of this is paid by the consumer and how much by the producer?

28. Answer the questions in Problem 27, assuming that the sales tax is imposed on the supplier instead of the consumer, so that the supplier's price is $p - 0.05p$, while the consumer's price is p.

29. A corporate office provides the demand curve in Figure 1.58 to its ice cream shop franchises. At a price of \$1.00 per scoop, 240 scoops per day can be sold.

(a) Estimate how many scoops could be sold per day at a price of 50¢ per scoop. Explain.

(b) Estimate how many scoops per day could be sold at a price of \$1.50 per scoop. Explain.

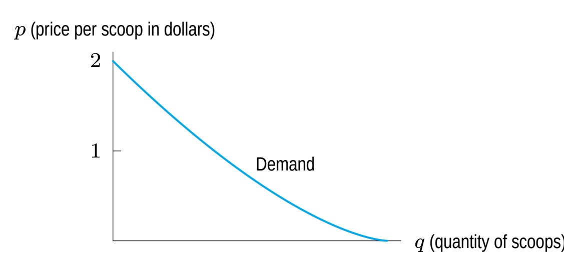


Figure 1.58

[26] "US Copper, Lead, and Zinc Production," *The World Almanac 1992*, p. 688.

30. Linear supply and demand curves are shown in Figure 1.59, with price on the vertical axis.

(a) Label the equilibrium price p_0 and the equilibrium quantity q_0 on the axes.

(b) Explain the effect on equilibrium price and quantity if the slope of the supply curve increases. Illustrate your answer graphically.

(c) Explain the effect on equilibrium price and quantity if the slope of the demand curve becomes more negative. Illustrate your answer graphically.

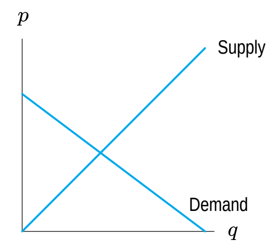


Figure 1.59

1.5 EXPONENTIAL FUNCTIONS

The function $f(x) = 2^x$, where the power is variable, is an *exponential function*. The number 2 is called the base. Exponential functions of the form $f(x) = a^x$, where a is a positive constant, are used to represent many phenomena in the natural and social sciences.

Population Growth

Table 1.25 contains data for the population of Mexico in the early 1980s. To see how the population is growing, we look at the yearly increases in population in the third column. If the population had been growing linearly, all the numbers in the third column would be the same. But populations usually grow faster as they get bigger, because there are more people to have babies, so it is not surprising that the numbers in the third column increase.

Table 1.25 *Population of Mexico (estimated)*

Year	Population (millions)	Increase in population (millions)
1980	67.38	
		1.75
1981	69.13	
		1.80
1982	70.93	
		1.84
1983	72.77	
		1.89
1984	74.66	
		1.94
1985	76.60	
		1.99
1986	78.59	

Suppose we divide each year's population by the previous year's population. We get, approximately,

$$\frac{\text{Population in 1981}}{\text{Population in 1980}} = \frac{69.13 \text{ million}}{67.38 \text{ million}} = 1.026$$
$$\frac{\text{Population in 1982}}{\text{Population in 1981}} = \frac{70.93 \text{ million}}{69.13 \text{ million}} = 1.026.$$

The fact that both calculations give 1.026 shows the population grew by about 2.6% between 1980 and 1981 *and* between 1981 and 1982. If we do similar calculations for other years, we find that the population grew by a factor of about 1.026, or 2.6%, every year. Whenever we have a constant growth factor (here 1.026), we have *exponential growth*. If t is the number of years since 1980,

$$\text{When } t = 0, \quad \text{population} = 67.38 = 67.38(1.026)^0.$$
$$\text{When } t = 1, \quad \text{population} = 69.13 = 67.38(1.026)^1.$$
$$\text{When } t = 2, \quad \text{population} = 70.93 = 69.13(1.026) = 67.38(1.026)^2.$$
$$\text{When } t = 3, \quad \text{population} = 72.77 = 70.93(1.026) = 67.38(1.026)^3.$$

So P, the population t years after 1980, is given by

$$P = 67.38(1.026)^t.$$

Since the variable t is in the exponent, this is an exponential function. The base, 1.026, represents the factor by which the population grows each year. Assuming that the formula holds for 50 years, the population graph has the shape in Figure 1.60. Since the population is growing, the function is increasing. Since the population grows faster as time passes, the graph is concave up. This behavior is typical of an exponential function. Even exponential functions that climb slowly at first, such as this one, climb extremely quickly eventually. That is why exponential population growth is considered by some to be such a threat to the world.

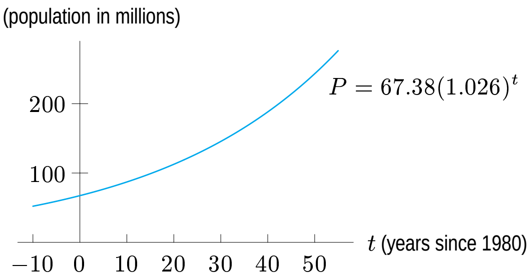


Figure 1.60: Population of Mexico (estimated): Exponential growth

Elimination of a Drug from the Body

Now we look at a quantity that is decreasing instead of increasing. When a patient is given medication, the drug enters the bloodstream. The rate at which the drug is metabolized and eliminated depends on the particular drug. For the antibiotic ampicillin, approximately 40% of the drug is eliminated every hour. A typical dose of ampicillin is 250 mg. Suppose $Q = f(t)$, where Q is the quantity of ampicillin, in mg, in the bloodstream at time t hours since the drug was given. At $t = 0$, we have $Q = 250$. Since the quantity remaining at the end of each hour is 60% of the quantity remaining the hour before, we have

$$\begin{aligned} f(0) &= 250 \\ f(1) &= 250(0.6) \\ f(2) &= 250(0.6)(0.6) = 250(0.6)^2 \\ f(3) &= 250(0.6)^2(0.6) = 250(0.6)^3. \end{aligned}$$

So, after t hours,

$$Q = f(t) = 250(0.6)^t.$$

This function is called an *exponential decay* function. As t increases, the function values get arbitrarily close to zero. The t-axis is a *horizontal asymptote* for this function.

Notice the way the values in Table 1.26 are decreasing. Each additional hour a smaller quantity of drug is removed than the previous hour (100 mg the first hour, 60 mg the second, and so on). This is because as time passes, there is less of the drug in the body to be removed. Thus, the graph in Figure 1.61 bends upward. Compare this to the exponential growth in Figure 1.60, where each step upward is larger than the previous one. Notice that both graphs are concave up.

Table 1.26 *Value of decay function*

t (hours)	Q (mg)
0	250
1	150
2	90
3	54
4	32.4
5	19.4

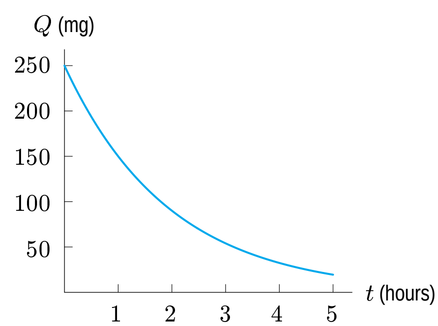


Figure 1.61: Drug elimination: Exponential decay

The General Exponential Function

Exponential growth is often described in terms of percent growth rates. The population of Mexico is growing at 2.6% per year, so it increases by a factor of $a = 1 + 0.026 = 1.026$ every year. Similarly, 40% of the ampicillin is removed every hour, so the quantity remaining decays by a factor of $a = 1 - 0.40 = 0.6$ each hour. We have the following general formulas.

> We say that P is an **exponential function** of t with base a if
>
> $$P = P_0 a^t,$$
>
> where P_0 is the initial quantity (when $t = 0$) and a is the factor by which P changes when t increases by 1. If $a > 1$, we have **exponential growth**; if $0 < a < 1$, we have **exponential decay**. The factor a is given by
>
> $$a = 1 + r$$
>
> where r is the decimal representation of the percent rate of change; r may be positive (for growth) or negative (for decay).

The largest possible domain for the exponential function is all real numbers,[27] provided $a > 0$.

Comparison Between Linear and Exponential Functions

Every exponential function changes at a constant percent, or *relative*, rate. For example, the population of Mexico increased at 2.6% per year. Every linear function changes at a constant (absolute) rate. For example, the Olympic pole vault record increased by 2 inches per year.

> A **linear** function has constant absolute rate of change.
> An **exponential** function has constant relative (or percent) rate of change.

Example 1 The amount of adrenaline in the body can change rapidly. Suppose the initial amount is 15 mg. Find a formula for A, the amount in mg, at a time t minutes later if A is:

(a) Increasing by 0.4 mg per minute.
(b) Decreasing by 0.4 mg per minute.
(c) Increasing by 3% per minute.
(d) Decreasing by 3% per minute.

Solution

(a) This is a linear function with initial quantity 15 and slope 0.4, so

$$A = 15 + 0.4t.$$

(b) This is a linear function with initial quantity 15 and slope -0.4, so

$$A = 15 - 0.4t.$$

(c) This is an exponential function with initial quantity 15 and base $1 + 0.03 = 1.03$, so

$$A = 15(1.03)^t.$$

(d) This is an exponential function with initial quantity 15 and base $1 - 0.03 = 0.97$, so

$$A = 15(0.97)^t.$$

[27] The reason we do not want $a \leq 0$ is that, for example, we cannot define $a^{1/2}$ if $a < 0$. Also, we do not usually have $a = 1$, since $P = P_0 a^t = P_0 1^t = P_0$ is then a constant function.

Example 2 Sales[28] at Borders Books and Music stores increased from \$78 million in 1991 to \$412 million in 1994. Assuming that sales have been increasing exponentially, find an equation of the form $P = P_0a^t$, where P is Borders sales in millions of dollars and t is the number of years since 1991. What is the percent growth rate?

Solution We know that $P = 78$ when $t = 0$, so $P_0 = 78$. To find a, we use the fact that $P = 412$ when $t = 3$. Substituting gives

$$P = P_0a^t$$
$$412 = 78a^3.$$

Dividing both sides by 78, we get

$$\frac{412}{78} = a^3$$
$$5.282 = a^3.$$

Taking the cube root of both sides gives

$$a = (5.282)^{1/3} = 1.74.$$

Since $a = 1.74$, the equation for Borders sales as a function of the number of years since 1991 is

$$P = 78(1.74)^t.$$

During this period, sales increased by 74% per year.

Recognizing Data from an Exponential Function: Values of t and P in a table could come from an exponential function $P = P_0a^t$ if ratios of P values are constant for equally spaced t values.

Example 3 Which of the following tables of values could correspond to an exponential function, a linear function, or neither? For those which could correspond to an exponential or linear function, find a formula for the function.

(a)

x	$f(x)$
0	16
1	24
2	36
3	54
4	81

(b)

x	$g(x)$
0	14
1	20
2	24
3	29
4	35

(c)

x	$h(x)$
0	5.3
1	6.5
2	7.7
3	8.9
4	10.1

Solution (a) We see that f cannot be a linear function, since $f(x)$ increases by different amounts ($24-16 = 8$ and $36 - 24 = 12$) as x increases by one. Could f be an exponential function? We look at the ratios of successive $f(x)$ values:

$$\frac{24}{16} = 1.5 \qquad \frac{36}{24} = 1.5 \qquad \frac{54}{36} = 1.5 \qquad \frac{81}{54} = 1.5.$$

[28] "How Borders Reads the Book Market," *US News & World Report*, October 30, 1995, p. 59–60.

Since the ratios are all equal to 1.5, this table of values could correspond to an exponential function with a base of 1.5. Since $f(0) = 16$, a formula for $f(x)$ is

$$f(x) = 16(1.5)^x.$$

Check by substituting $x = 0, 1, 2, 3, 4$ into this formula; you get the values given for $f(x)$.

(b) As x increases by one, $g(x)$ increases by 6 (from 14 to 20), then 4 (from 20 to 24), so g is not linear. We check to see if g could be exponential:

$$\frac{20}{14} = 1.43 \quad \text{and} \quad \frac{24}{20} = 1.2.$$

Since these ratios (1.43 and 1.2) are different, g is not exponential.

(c) For h, notice that as x increases by one, the value of $h(x)$ increases by 1.2 each time. So h could be a linear function with a slope of 1.2. Since $h(0) = 5.3$, a formula for $h(x)$ is

$$h(x) = 5.3 + 1.2x.$$

The Family of Exponential Functions and the Number e

The formula $P = P_0a^t$ gives a family of exponential functions with parameters P_0 (the initial quantity) and a (the base). The base tells us whether the function is increasing ($a > 1$) or decreasing ($0 < a < 1$). Since a is the factor by which P changes when t is increased by 1, large values of a mean fast growth; values of a near 0 mean fast decay. (See Figures 1.62 and 1.63.) All members of the family $P = P_0a^t$ are concave up if $P_0 > 0$.

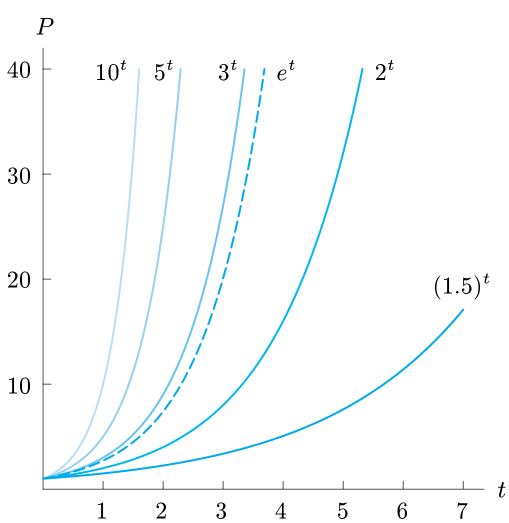


Figure 1.62: Exponential growth: $P = a^t$, for $a > 1$

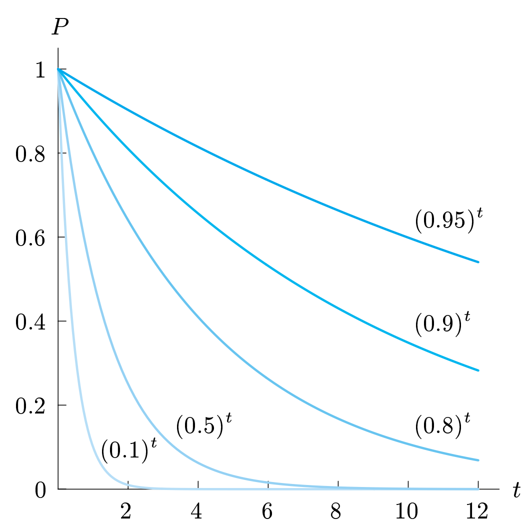


Figure 1.63: Exponential decay: $P = a^t$, for $0 < a < 1$

In practice the most commonly used base is the number $e = 2.71828\ldots$. The fact that most calculators have an e^x button is an indication of how important e is. Since e is between 2 and 3, the graph of $y = e^t$ in Figure 1.62 is between the graphs of $y = 2^t$ and $y = 3^t$.

The base e is used so often that it is called the natural base. At first glance, this is somewhat mysterious: What could be natural about using 2.71828... as a base? The full answer to this question must wait until Chapter 3, where you will see that many calculus formulas come out more neatly when e is used as the base.

Problems for Section 1.5

1. Each of the following functions gives the amount of a substance present at time t. In each case, give the amount present initially (at $t = 0$), state whether the function represents exponential growth or decay, and give the percent growth or decay rate.

(a) $A = 100(1.07)^t$ **(b)** $A = 5.3(1.054)^t$
(c) $A = 3500(0.93)^t$ **(d)** $A = 12(0.88)^t$

2. The following functions give the populations of four towns with time t in years.

(i) $P = 600(1.12)^t$ (ii) $P = 1{,}000(1.03)^t$
(iii) $P = 200(1.08)^t$ (iv) $P = 900(0.90)^t$

(a) Which town has the largest percent growth rate? What is the percent growth rate?
(b) Which town has the largest initial population? What is that initial population?
(c) Are any of the towns decreasing in size? If so, which one(s)?

3. A town has a population of 1000 people at time $t = 0$. In each of the following cases, write a formula for the population, P, of the town as a function of year t.

(a) The population increases by 50 people a year.
(b) The population increases by 5% a year.

4. An air-freshener starts with 30 grams and evaporates. In each of the following cases, write a formula for the quantity, Q grams, of air-freshener remaining t days after the start and sketch a graph of the function. The decrease is:

(a) 2 grams a day **(b)** 12% a day

5. A 50 mg dose of quinine is given to a patient to prevent malaria. Quinine leaves the body at a rate of 6% per hour.

(a) Find a formula for the amount, A (in mg), of quinine in the body t hours after the dose is given.
(b) How much quinine is in the body after 24 hours?
(c) Graph A as a function of t.
(d) Use the graph to estimate when 5 mg of quinine remains.

6. The world's economy has been expanding. In the year 2000, the gross world product (total output in goods and services) was 45 trillion dollars and was increasing at 4.7% a year.[29] Assume this growth rate continues.

(a) Find a formula for the gross world product, W (in trillions of dollars), as a function of t, the number of years since the year 2000.
(b) What is the predicted gross world product in 2010?
(c) Graph W as a function of t.
(d) Use the graph to estimate when the gross world product will pass 50 trillion dollars.

7. Figure 1.64 shows graphs of several cities' populations against time. Match each of the following descriptions to a graph and write a description to match each of the remaining graphs.

(a) The population increased at 5% per year.
(b) The population increased at 8% per year.
(c) The population increased by 5000 people per year.
(d) The population was stable.

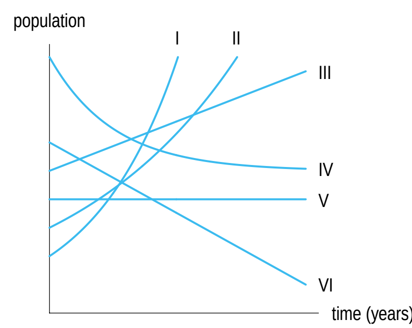


Figure 1.64

8. World population is approximately $P = 6.1(1.0126)^t$, with P in billions and t in years since 2000.

(a) What is the yearly percent rate of growth of the world population?
(b) What was the world population in 2000? What does this model predict for the world population in 2005?
(c) Use part (b) to find the average rate of change of the world population between 2000 and 2005.

9. The company that produces Cliffs Notes (abridged versions of classic literature) was started in 1958 with \$4000 and sold in 1998 for \$14,000,000. Find the annual percent increase in the value of this company, over the 40 years.

For Problems 10–11, find a possible formula for the function represented by the data.

10.

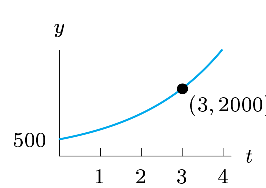
x	0	1	2	3
$f(x)$	4.30	6.02	8.43	11.80

11.

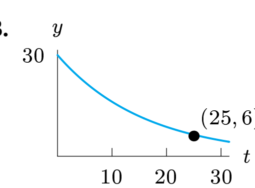
t	0	1	2	3
$g(t)$	5.50	4.40	3.52	2.82

Give a possible formula for the functions in Problems 12–13.

12.



13.



[29] The Worldwatch Institute, *Vital Signs* 2001, p. 57, (New York: W.W. Norton, 2001).

14. **(a)** Which (if any) of the functions in the following table could be linear? Find formulas for those functions.
(b) Which (if any) of these functions could be exponential? Find formulas for those functions.

x	$f(x)$	$g(x)$	$h(x)$
-2	12	16	37
-1	17	24	34
0	20	36	31
1	21	54	28
2	18	81	25

15. Determine whether each of the following tables of values could correspond to a linear function, an exponential function, or neither. For each table of values that could correspond to a linear or an exponential function, find a formula for the function.

(a)

x	$f(x)$
0	10.5
1	12.7
2	18.9
3	36.7

(b)

t	$s(t)$
-1	50.2
0	30.12
1	18.072
2	10.8432

(c)

u	$g(u)$
0	27
2	24
4	21
6	18

16. Find a formula for the number of zebra mussels in a bay as a function of the number of years since 1998, given that there were 2700 at the start of 1998 and 3186 at the start of 1999.

(a) Assume that the number of zebra mussels is growing linearly. Give units for the slope of the line and interpret it in terms of zebra mussels.
(b) Assume that the number of zebra mussels is growing exponentially. What is the percent rate of growth of the zebra mussel population?

17. During the 1980s, Costa Rica had the highest deforestation rate in the world, at 2.9% per year. (This is the rate at which land covered by forests is shrinking.) Of the land in Costa Rica covered by forests in 1980, what percent was still covered by forests in 1990?

18. The number of passengers using a railway fell from 190,205 to 174,989 during a 5-year period. Find the annual percentage decrease over this period.

19. Figure 1.65 shows world solar power output[30] during the 1990s.

(a) Explain why this graph could represent an exponential function.
(b) Give a possible formula for world solar power output, S (in megawatts), as a function of the number of years, t, since 1990.
(c) What is the annual percent change?
(d) Use the formula to predict world solar power output in 2005.

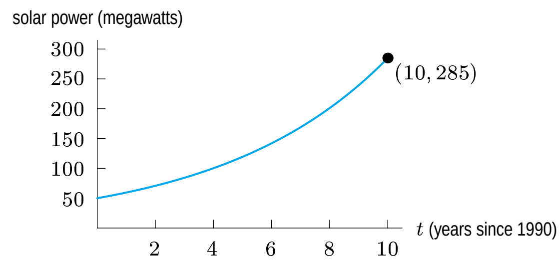


Figure 1.65

20. **(a)** Make a table of values for $y = e^x$ using $x = 0, 1, 2, 3$.
(b) Plot the points found in part (a). Does the graph look like an exponential growth or decay function?
(c) Make a table of values for $y = e^{-x}$ using $x = 0, 1, 2, 3$.
(d) Plot the points found in part (c). Does the graph look like an exponential growth or decay function?

21. Graph $y = 100e^{-0.4x}$. Describe what you see.

22. When the Olympic Games were held outside Mexico City in 1968, there was much discussion about the effect the high altitude (7340 feet) would have on the athletes. Assuming air pressure decays exponentially by 0.4% every 100 feet, by what percentage is air pressure reduced by moving from sea level to Mexico City?

23. Aircrafts require longer takeoff distances, called takeoff rolls, at high altitude airports because of diminished air density. The table shows how the takeoff roll for a certain light airplane depends on the airport elevation. (Takeoff rolls are also strongly influenced by air temperature; the data shown assume a temperature of 0° C.) Determine a formula for this particular aircraft that gives the takeoff roll as an exponential function of airport elevation.

Elevation (ft)	Sea level	1000	2000	3000	4000
Takeoff roll (ft)	670	734	805	882	967

24. The median price, P, of a home rose from \$50,000 in 1970 to \$100,000 in 1990. Let t be the number of years since 1970.

(a) Assume the increase in housing prices has been linear. Give an equation for the line representing price, P, in terms of t. Use this equation to complete column (a) of Table 1.27. Use units of \$1000.
(b) If instead the housing prices have been rising exponentially, find an equation of the form $P = P_0a^t$ to represent housing prices. Complete column (b) of Table 1.27.

[30]The Worldwatch Institute, *Vital Signs* 2001 , p. 47, (New York: W.W. Norton, 2001).

(c) On the same set of axes, sketch the functions represented in column (a) and column (b) of Table 1.27.

(d) Which model for the price growth do you think is more realistic?

Table 1.27

t	(a) Linear growth price in \$1000 units	(b) Exponential growth price in \$1000 units
0	50	50
10		
20	100	100
30		
40		

25. In 1994, the world's population was 5.6 billion and was growing at a rate of about 1.2% per year.

(a) Write a formula for the world's population as a function of time, t, in years since 1994.

(b) Find the predicted average rate of change in the world's population between 1994 and 2000. Give units with your answer.

(c) Find the predicted average rate of change in the world's population between 2010 and 2020. Give units with your answer.

26. Match the functions $h(s)$, $f(s)$, and $g(s)$, whose values are in Table 1.28, with the formulas

$$y = a(1.1)^s\ , \quad y = b(1.05)^s\ , \quad y = c(1.03)^s,$$

assuming a, b, and c are constants. Note that the function values have been rounded to two decimal places.

Table 1.28

s	$h(s)$	s	$f(s)$	s	$g(s)$
2	1.06	1	2.20	3	3.47
3	1.09	2	2.42	4	3.65
4	1.13	3	2.66	5	3.83
5	1.16	4	2.93	6	4.02
6	1.19	5	3.22	7	4.22

27. A photocopy machine can reduce copies to 80% of their original size. By copying an already reduced copy, further reductions can be made.

(a) If a page is reduced to 80%, what percent enlargement is needed to return it to its original size?

(b) Estimate the number of times in succession that a page must be copied to make the final copy less than 15% of the size of the original.

28. (a) Niki invested \$10,000 in the stock market. The investment was a loser, declining in value 10% per year each year for 10 years. How much was the investment worth after 10 years?

(b) After 10 years, the stock began to gain value at 10% per year. After how long will the investment regain its initial value (\$10,000)?

1.6 THE NATURAL LOGARITHM

In Section 1.5, we approximated the population of Mexico (in millions) by the function

$$P = f(t) = 67.38(1.026)^t,$$

where t is the number of years since 1980. Now suppose that instead of calculating the population at time t, we ask when the population will reach 200 million. We want to find the value of t for which

$$200 = f(t) = 67.38(1.026)^t.$$

We use logarithms to solve for a variable in an exponent.

Definition and Properties of the Natural Logarithm

We define the natural logarithm of x, written $\ln x$, as follows:

> The **natural logarithm** of x, written $\ln x$, is the power of e needed to get x. In other words,
>
> $$\ln x = c \quad \text{means} \quad e^c = x.$$
>
> The natural logarithm is sometimes written $\log_e x$.

For example, $\ln e^3 = 3$ since 3 is the power of e needed to give e^3. Similarly, $\ln(1/e) = \ln e^{-1} = -1$. A calculator gives $\ln 5 = 1.6094$, because $e^{1.6094} = 5$. However if we try to find $\ln(-7)$ on a calculator, we get an error message because e to any power is never negative or 0. In general

$\ln x$ is not defined if x is negative or 0.

To work with logarithms, we use the following properties:

Properties of the Natural Logarithm

1. $\ln(AB) = \ln A + \ln B$
2. $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$
3. $\ln(A^p) = p\ln A$
4. $\ln e^x = x$
5. $e^{\ln x} = x$

In addition, $\ln 1 = 0$ because $e^0 = 1$, and $\ln e = 1$ because $e^1 = e$.

Using the LN button on a calculator, we get the graph of $f(x) = \ln x$ in Figure 1.66. Observe that, for large x, the graph of $y = \ln x$ climbs very slowly as x increases. The x-intercept is $x = 1$, since $\ln 1 = 0$. For $x > 1$, the value of $\ln x$ is positive; for $0 < x < 1$, the value of $\ln x$ is negative.

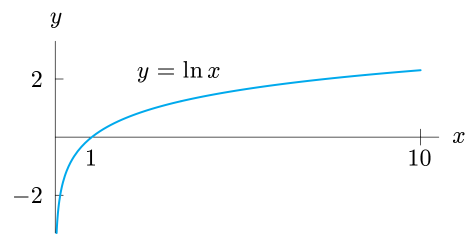


Figure 1.66: The natural logarithm function climbs very slowly

Solving Equations Using Logarithms

Natural logs can be used to solve for unknown exponents.

Example 1 Find t such that $3^t = 10$.

Solution First, notice that we expect t to be between 2 and 3, because $3^2 = 9$ and $3^3 = 27$. To find t exactly, we take the natural logarithm of both sides and solve for t:

$$\ln(3^t) = \ln 10.$$

The third property of logarithms tells us that $\ln(3^t) = t\ln 3$, so we have

$$t\ln 3 = \ln 10$$
$$t = \frac{\ln 10}{\ln 3}.$$

Using a calculator to find the natural logs gives

$$t = 2.096.$$

Example 2 We return to the question of when the population of Mexico reaches 200 million. To get an answer, we solve $200 = 67.38(1.026)^t$ for t, using logs.

Solution Dividing both sides of the equation by 67.38, we get

$$\frac{200}{67.38} = (1.026)^t.$$

Now take natural logs of both sides:

$$\ln\left(\frac{200}{67.38}\right) = \ln(1.026^t).$$

Using the fact that $\ln(1.026^t) = t\ln 1.026$, we get

$$\ln\left(\frac{200}{67.38}\right) = t\ln(1.026).$$

Solving this equation using a calculator to find the logs, we get

$$t = \frac{\ln(200/63.78)}{\ln(1.026)} = 42.4 \text{ years}.$$

Since $t = 0$ in 1980, this value of t corresponds to the year 2022.

Example 3 Find t such that $12 = 5e^{3t}$.

Solution It is easiest to begin by isolating the exponential, so we divide both sides of the equation by 5:

$$2.4 = e^{3t}.$$

Now take the natural logarithm of both sides:

$$\ln 2.4 = \ln(e^{3t}).$$

Since $\ln(e^x) = x$, we have

$$\ln 2.4 = 3t,$$

so, using a calculator, we get

$$t = \frac{\ln 2.4}{3} = 0.2918.$$

Exponential Functions with Base e

An exponential function with base a has formula

$$P = P_0a^t.$$

For any positive number a, we can write $a = e^k$ where $k = \ln a$. Thus, the exponential function can be rewritten as

$$P = P_0a^t = P_0(e^k)^t = P_0e^{kt}.$$

If $a > 1$, then k is positive, and if $0 < a < 1$, then k is negative. We conclude:

> Writing $a = e^k$, so $k = \ln a$, any exponential function can be written in two forms
>
> $$P = P_0a^t \quad \text{or} \quad P = P_0e^{kt}.$$
>
> - If $a > 1$, we have exponential growth; if $0 < a < 1$, we have exponential decay.
> - If $k > 0$, we have exponential growth; if $k < 0$, we have exponential decay.
> - k is called the *continuous* growth or decay rate.

The word continuous in continuous growth rate is used in the same way to describe continuous compounding of interest earned on money. See the Focus on Modeling Section on page 75.

Example 4 (a) Convert the function $P = 1000e^{0.05t}$ to the form $P = P_0a^t$.
(b) Convert the function $P = 500(1.06)^t$ to the form $P = P_0e^{kt}$.

Solution (a) Since $P = 1000e^{0.05t}$, we have $P_0 = 1000$. We want to find a so that

$$1000a^t = 1000e^{0.05t} = 1000(e^{0.05})^t.$$

We take $a = e^{0.05} = 1.0513$, so the following two functions give the same values:

$$P = 1000e^{0.05t} \quad \text{and} \quad P = 1000(1.0513)^t.$$

So a continuous growth rate of 5% is equivalent to a growth rate of 5.13% per unit time.

(b) We have $P_0 = 500$ and we want to find k with

$$500(1.06)^t = 500(e^k)^t,$$

so we take

$$1.06 = e^k$$
$$k = \ln(1.06) = 0.0583.$$

The following two functions give the same values:

$$P = 500(1.06)^t \quad \text{and} \quad P = 500e^{0.0583t}.$$

So a growth rate of 6% per unit time is equivalent to a continuous growth rate of 5.83%.

Example 5 Sketch graphs of $P = e^{0.5t}$, a continuous growth rate of 50%, and $Q = 5e^{-0.2t}$, a continuous decay rate of 20%.

Solution The graph of $P = e^{0.5t}$ is in Figure 1.67. Notice that the graph is the same shape as the previous exponential growth curves: increasing and concave up. The graph of $Q = 5e^{-0.2t}$ is in Figure 1.68; it has the same shape as other exponential decay functions.

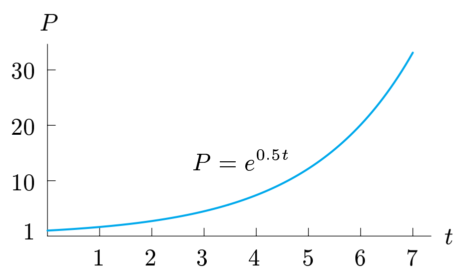


Figure 1.67: Continuous exponential growth function

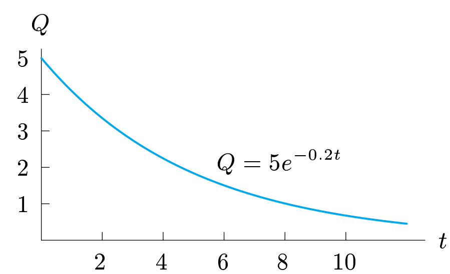


Figure 1.68: Continuous exponential decay function

Problems for Section 1.6

For Problems 1–16, solve for t using natural logarithms.

1. $5^t = 7$

2. $130 = 10^t$

3. $2 = (1.02)^t$

4. $10 = 2^t$

5. $100 = 25(1.5)^t$

6. $50 = 10 \cdot 3^t$

7. $a = b^t$

8. $10 = e^t$

9. $5 = 2e^t$

10. $e^{3t} = 100$

11. $10 = 6e^{0.5t}$

12. $40 = 100e^{-0.03t}$

13. $B = Pe^{rt}$

14. $2P = Pe^{0.3t}$

15. $7 \cdot 3^t = 5 \cdot 2^t$

16. $5e^{3t} = 8e^{2t}$

The functions in Problems 17–20 represent exponential growth or decay. What is the initial quantity? What is the growth rate? State if the growth rate is continuous.

17. $P = 5(1.07)^t$

18. $P = 7.7(0.92)^t$

19. $P = 3.2e^{0.03t}$

20. $P = 15e^{-0.06t}$

21. Write the exponential functions $P = e^{0.08t}$ and $Q = e^{-0.3t}$ in the form $P = a^t$ and $Q = b^t$.

22. A city's population is 1000 and growing at 5% a year.

(a) Find a formula for the population at time t years from now assuming that the 5% per year is an:

(i) Annual rate (ii) Continuous annual rate

(b) In each case in part (a), estimate the population of the city in 10 years.

23. The following formulas give the populations of four different towns, A, B, C, and D, with t in years from now.

$$P_A = 600e^{0.08t} \qquad P_B = 1000e^{-0.02t}$$
$$P_C = 1200e^{0.03t} \qquad P_D = 900e^{0.12t}$$

(a) Which town is growing fastest (that is, has the largest percentage growth rate)?
(b) Which town is the largest now?
(c) Are any of the towns decreasing in size? If so, which one(s)?

24. **(a)** A population, P, grows at a continuous rate of 2% a year and starts at 1 million. Write P in the form $P = P_0e^{kt}$, with P_0, k constants.
(b) Plot the population in part (a) against time.

25. **(a)** What is the continuous percent growth rate for the function $P = 10e^{0.15t}$?
(b) Write this function in the form $P = P_0a^t$.
(c) What is the annual (not continuous) percent growth rate for this function?
(d) Graph $P = 10e^{0.15t}$ and your answer to part (b) on the same axes. Explain what you see.

Write the functions in Problems 26–29 in the form $P = P_0a^t$. Which represent exponential growth and which represent exponential decay?

26. $P = 15e^{0.25t}$

27. $P = 2e^{-0.5t}$

28. $P = P_0e^{0.2t}$

29. $P = 7e^{-\pi t}$

In Problems 30–33, put the functions in the form $P = P_0e^{kt}$.

30. $P = 15(1.5)^t$

31. $P = 10(1.7)^t$

32. $P = 174(0.9)^t$

33. $P = 4(0.55)^t$

34. **(a)** What is the continuous percent growth rate for $P = 100e^{0.06t}$?
(b) Write this function in the form $P = P_0a^t$. What is the annual percent growth rate?

35. **(a)** What is the annual percent decay rate for $P = 25(0.88)^t$?
(b) Write this function in the form $P = P_0e^{kt}$. What is the continuous percent decay rate?

36. A fishery stocks a pond with 1000 young trout. The number of trout t years later is given by $P(t) = 1000e^{-0.5t}$.

(a) How many trout are left after six months? After 1 year?
(b) Find $P(3)$ and interpret it in terms of trout.
(c) At what time are there 100 trout left?
(d) Graph the number of trout against time, and describe how the population is changing. What might be causing this?

37. The population, P, in millions, of Nicaragua was 3.6 million in 1990 and growing at an annual rate of 3.4%. Let t be time in years since 1990.

(a) Express P as a function in the form $P = P_0a^t$.
(b) Express P as an exponential function using base e.
(c) Compare the annual and continuous growth rates.

38. The gross world product is $W = 45(1.047)^t$, where W is in trillions of dollars and t is years since 2000. Find a formula for gross world product using a continuous growth rate.

39. The population of the world can be represented by $P = 6.1(1.0126)^t$, where P is in billions of people and t is years since 2000. Find a formula for the population of the world using a continuous growth rate.

40. What annual percent growth rate is equivalent to a continuous percent growth rate of 8%?

41. What continuous percent growth rate is equivalent to an annual percent growth rate of 10%?

42. In 1980, there were about 170 million vehicles (cars and trucks) and about 227 million people in the United States. The number of vehicles has been growing at 4% a year, while the population has been growing at 1% a year. When was there, on average, one vehicle per person?

1.7 EXPONENTIAL GROWTH AND DECAY

Many quantities in nature change according to an exponential growth or decay function of the form $P = P_0e^{kt}$, where P_0 is the initial quantity and k is the continuous growth or decay rate.

Example 1 The Environmental Protection Agency (EPA) recently investigated a spill of radioactive iodine. The radiation level at the site was about 2.4 millirems/hour (four times the maximum acceptable limit of 0.6 millirems/hour), so the EPA ordered an evacuation of the surrounding area. The level of radiation from an iodine source decays at a continuous hourly rate of $k = -0.004$.

(a) What was the level of radiation 24 hours later?
(b) Find the number of hours until the level of radiation reached the maximum acceptable limit, and the inhabitants could return.

Solution (a) The level of radiation, R, in millirems/hour, at time t, in hours since the initial measurement, is given by

$$R = 2.4e^{-0.004t},$$

so the level of radiation 24 hours later was

$$R = 2.4e^{(-0.004)(24)} = 2.18 \text{ millirems per hour.}$$

(b) A graph of $R = 2.4e^{-0.004t}$ is in Figure 1.69. The maximum acceptable value of R is 0.6 millirems per hour, which occurs at approximately $t = 350$. Using logarithms, we have

$$\begin{aligned} 0.6 &= 2.4e^{-0.004t} \\ 0.25 &= e^{-0.004t} \\ \ln 0.25 &= -0.004t \\ t &= \frac{\ln 0.25}{-0.004} = 346.57. \end{aligned}$$

The inhabitants will not be able to return for 346.57 hours, or about 15 days.

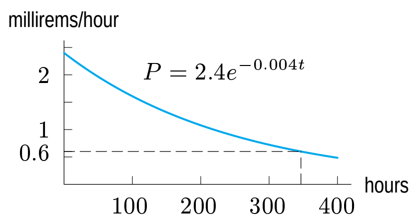


Figure 1.69: The level of radiation from radioactive iodine

Example 2 The population of Kenya was 19.5 million in 1984 and 21.2 million in 1986. Assuming the population increases exponentially, find a formula for the population of Kenya as a function of time.

Solution If we measure the population, P, in millions and time, t, in years since 1984, we can say

$$P = P_0e^{kt} = 19.5e^{kt},$$

where $P_0 = 19.5$ is the initial value of P. We find k by using the fact that $P = 21.2$ when $t = 2$, so

$$21.2 = 19.5e^{k\cdot 2}.$$

To find k, we divide both sides by 19.5, giving

$$\frac{21.2}{19.5} = e^{2k}.$$

Now take natural logs of both sides:

$$\ln\left(\frac{21.1}{19.5}\right) = \ln(e^{2k}).$$

Since $\ln(e^{2k}) = 2k$, this becomes

$$\ln\left(\frac{21.2}{19.5}\right) = 2k.$$

So, using a calculator, we get

$$k = \frac{1}{2}\ln\left(\frac{21.2}{19.5}\right) = 0.042,$$

and therefore

$$P = 19.5e^{0.042t}.$$

Since $k = 0.042 = 4.2\%$, the population of Kenya was growing at a continuous rate of 4.2% per year.

Doubling Time and Half-Life

Every exponential growth function has a fixed doubling time and every exponential decay function has a fixed half-life.

> The **doubling time** of an exponentially increasing quantity is the time required for the quantity to double.
> The **half-life** of an exponentially decaying quantity is the time required for the quantity to be reduced by a factor of one half.

Example 3 Show algebraically that every exponentially growing function has a fixed doubling time.

Solution Consider the exponential function $P = P_0 a^t$. For any base a with $a > 1$, there is a positive number d such that $a^d = 2$. We show that d is the doubling time. If the population is P at time t, then at time $t + d$, the population is

$$P_0 a^{t+d} = P_0 a^t a^d = (P_0 a^t)(2) = 2P.$$

So, no matter what the initial quantity and no matter what the initial time, the size of the population is doubled d time units later.

Example 4 The release of chlorofluorocarbons used in air conditioners and household sprays (hair spray, shaving cream, etc.) destroys the ozone in the upper atmosphere. The quantity of ozone, Q, is decaying exponentially at a continuous rate of 0.25% per year. What is the half-life of ozone? In other words, at this rate, how long will it take for half the ozone to disappear?

Solution If Q_0 is the initial quantity of ozone and t is in years, then

$$Q = Q_0 e^{-0.0025t}.$$

We want to find the value of t making $Q = Q_0/2$, so

$$\frac{Q_0}{2} = Q_0 e^{-0.0025t}.$$

Dividing both sides by Q_0 and taking natural logs gives

$$\ln\left(\frac{1}{2}\right) = -0.0025t,$$

so

$$t = \frac{\ln(1/2)}{-0.0025} = 277 \text{ years}.$$

Half the present atmospheric ozone will be gone in 277 years.

Financial Applications: Compound Interest

We deposit \$100 in a bank paying interest at a rate of 8% per year. How much is in the account at the end of the year? This depends on how often the interest is compounded. If the interest is paid into the account *annually*, that is only at the end of the year, then the balance in the account after one year is \$108. However, if the interest is paid twice a year, then 4% is paid at the end of the first six months and 4% at the end of the year. Slightly more money is earned this way, since the interest paid early in the year will earn interest during the rest of the year. This effect is called *compounding*.

In general, the more often interest is compounded, the more money is earned (although the increase may not be large). What happens if interest is compounded more frequently, such as every minute or every second? The benefit of increasing the frequency of compounding becomes negligible beyond a certain point. When that point is reached, we find the balance using the number e and we say that the interest per year is *compounded continuously*. If we have deposited \$100 in an account paying 8% interest per year compounded continuously, the balance after one year is $100e^{0.08} = \$108.33$. Compounding is discussed further in the Focus on Modeling section on page 75. In general:

> An amount P_0 is deposited in an account paying interest at a rate of r per year. Let P be the balance in the account after t years.
> - If interest is compounded annually, then $P = P_0(1+r)^t$.
> - If interest is compounded continuously, then $P = P_0e^{rt}$, where $e = 2.71828\ldots$.

We write P_0 for the initial deposit because it is the value of P when $t = 0$. Note that for a 7% interest rate, $r = 0.07$. If a rate is continuous, we will say so explicitly.

Example 5 A bank advertises an interest rate of 8% per year. If you deposit $5000, how much is in the account 3 years later if the interest is compounded (a) Annually? (b) Continuously?

Solution

(a) For annual compounding, $P = P_0(1+r)^t = 5000(1.08)^3 = \$6298.56$.

(b) For continuous compounding, $P = P_0e^{rt} = 5000e^{0.08\cdot 3} = \$6356.25$. As expected, the amount in the account 3 years later is larger if the interest is compounded continuously ($6356.25) than if the interest is compounded annually ($6298.56).

Example 6 If $10,000 is deposited in an account paying interest at a rate of 5% per year, compounded continuously, how long does it take for the balance in the account to reach $15,000?

Solution

Since interest is compounded continuously, we use $P = P_0e^{rt}$ with $r = 0.05$ and $P_0 = 10{,}000$. We want to find the value of t for which $P = 15{,}000$. The equation is

$$15{,}000 = 10{,}000e^{0.05t}.$$

Now divide both sides by $10{,}000$, then take logarithms and solve for t:

$$\begin{aligned} 1.5 &= e^{0.05t} \\ \ln(1.5) &= \ln(e^{0.05t}) \\ \ln(1.5) &= 0.05t \\ t &= \frac{\ln(1.5)}{0.05} = 8.1093. \end{aligned}$$

It takes about 8.1 years for the balance in the account to reach $15,000.

Example 7

(a) Calculate the doubling time, D, for interest rates of 2%, 3%, 4%, and 5% per year, compounded annually.

(b) Use your answers to part (a) to check that an interest rate of $i\%$ gives a doubling time approximated for small values of i by

$$D \approx \frac{70}{i} \text{ years.}$$

This is the "Rule of 70" used by bankers: To compute the approximate doubling time of an investment, divide 70 by the percent annual interest rate.

Solution

(a) We find the doubling time for an interest rate of 2% per year using the formula $P = P_0(1.02)^t$ with t in years. To find the value of t for which $P = 2P_0$, we solve

$$\begin{aligned} 2P_0 &= P_0(1.02)^t \\ 2 &= (1.02)^t \\ \ln 2 &= \ln(1.02)^t \\ \ln 2 &= t\ln(1.02) \qquad \text{(using the third property of logarithms)} \\ t &= \frac{\ln 2}{\ln 1.02} = 35.003 \text{ years.} \end{aligned}$$

With an annual interest rate of 2%, it takes about 35 years for an investment to double in value. Similarly, we find the doubling times for 3%, 4%, and 5% in Table 1.29.

Table 1.29 *Doubling time as a function of interest rate*

i (% annual growth rate)	2	3	4	5
D (doubling time in years)	35.003	23.450	17.673	14.207

(b) We compute $(70/i)$ for $i = 2, 3, 4, 5$. The results are shown in Table 1.30.

Table 1.30 *Approximate doubling time as a function of interest rate: Rule of* 70

i (% annual growth rate)	2	3	4	5
$(70/i)$ (Approximate doubling time in years)	35.000	23.333	17.500	14.000

Comparing Tables 1.29 and Table 1.30, we see that the quantity $(70/i)$ gives a reasonably accurate approximation to the doubling time, D, for the small interest rates we considered.

Present and Future Value

Many business deals involve payments in the future. For example, when a car is bought on credit, payments are made over a period of time. Being paid \$100 in the future is clearly worse than being paid \$100 today for many reasons. If we are given the money today, we can do something else with it—for example, put it in the bank, invest it somewhere, or spend it. Thus, even without considering inflation, if we are to accept payment in the future, we would expect to be paid more to compensate for this loss of potential earnings.[31] The question we consider now is, how much more?

To simplify matters, we consider only what we would lose by not earning interest; we do not consider the effect of inflation. Let's look at some specific numbers. Suppose we deposit \$100 in an account that earns 7% interest per year compounded annually, so that in a year's time we have \$107. Thus, \$100 today is worth \$107 a year from now. We say that the \$107 is the *future value* of the \$100, and that the \$100 is the *present value* of the \$107. In general, we say the following:

- The **future value**, B, of a payment, P, is the amount to which the P would have grown if deposited today in an interest-bearing bank account.
- The **present value**, P, of a future payment, B, is the amount that would have to be deposited in a bank account today to produce exactly B in the account at the relevant time in the future.

Due to the interest earned, the future value is larger than the present value. The relation between the present and future values depends on the interest rate, as follows.

Suppose B is the *future value* of P and P is the *present value* of B.
If interest is compounded annually at a rate r for t years, then

$$B = P(1+r)^t, \quad \text{or equivalently,} \quad P = \frac{B}{(1+r)^t}.$$

If interest is compounded continuously at a rate r for t years, then

$$B = Pe^{rt}, \quad \text{or equivalently,} \quad P = \frac{B}{e^{rt}} = Be^{-rt}.$$

The rate, r, is sometimes called the *discount rate*. The present value is often denoted by PV, and the future value by FV.

[31]This is referred to as the time value of money.

Example 8 You win the lottery and are offered the choice between \$1 million in four yearly installments of \$250,000 each, starting now, and a lump-sum payment of \$920,000 now. Assuming a 6% interest rate, compounded continuously, and ignoring taxes, which should you choose?

Solution We assume that you pick the option with the largest present value. The first of the four $\$250{,}000$ payments is made now, so

$$\text{Present value of first payment} = \$250{,}000.$$

The second payment is made one year from now and so

$$\text{Present value of second payment} = \$250{,}000e^{-0.06(1)}.$$

Calculating the present value of the third and fourth payments similarly, we find:

$$\begin{aligned}\text{Total present value} &= \$250{,}000 + \$250{,}000e^{-0.06(1)} + \$250{,}000e^{-0.06(2)} + \$250{,}000e^{-0.06(3)}\\ &= \$250{,}000 + \$235{,}441 + \$221{,}730 + \$208{,}818\\ &= \$915{,}989.\end{aligned}$$

Since the present value of the four payments is less than \$920,000, you are better off taking the \$920,000 now.

Alternatively, we can compare the future values of the two pay schemes. We calculate the future value of both schemes three years from now, on the date of the last \$250,000 payment. At that time,

$$\text{Future value of the lump-sum payment} = \$920{,}000e^{0.06(3)} = \$1{,}101{,}440.$$

The future value of the first \$250,000 payment is $\$250{,}000e^{0.06(3)}$. Calculating the future value of the other payments similarly, we find:

$$\begin{aligned}\text{Total future value} &= \$250{,}000e^{0.06(3)} + \$250{,}000e^{0.06(2)} + \$250{,}000e^{0.06(1)} + \$250{,}000\\ &= \$299{,}304 + \$281{,}874 + \$265{,}459 + \$250{,}000\\ &= \$1{,}096{,}637.\end{aligned}$$

As we expect, the future value of the $\$920{,}000$ payment is greater, so you are better off taking the $\$920{,}000$ now.

(Note: If you read the fine print, you will find that many lotteries do not make their payments right away, but often spread them out, sometimes far into the future. This is to reduce the present value of the payments made, so that the value of the prizes is less than it might first appear!)

Problems for Section 1.7

1. Find the doubling time of a quantity that is increasing by 7% per year.

2. If the quantity of a substance decreases by 4% in 10 hours, find its half-life.

3. The half-life of a radioactive substance is 12 days. There are 10.32 grams initially.
   - **(a)** Write an equation for the amount, A, of the substance as a function of time.
   - **(b)** When is the substance reduced to 1 gram?

4. The half-life of nicotine in the blood is 2 hours. A person absorbs 0.4 mg of nicotine by smoking a cigarette. Fill in the following table with the amount of nicotine remaining in the blood after t hours. Estimate the length of time until the amount of nicotine is reduced to 0.04 mg.

t (hours)	0	2	4	6	8	10
Nicotine (mg)	0.4					

5. If you deposit \$10,000 in an account earning interest at an 8% annual rate compounded continuously, how much money is in the account after five years?

6. You invest \$5000 in an account which pays interest compounded continuously.

(a) How much money is in the account after 8 years, if the annual interest rate is 4%?
(b) If you want the account to contain \$8000 after 8 years, what yearly interest rate is needed?

7. Suppose \$1000 is invested in an account paying interest at a rate of 5.5% per year. How much is in the account after 8 years if the interest is compounded

(a) Annually? **(b)** Continuously?

8. Each curve in Figure 1.70 represents the balance in a bank account into which a single deposit was made at time zero. Assuming continuously compounded interest, find:

(a) The curve representing the largest initial deposit.
(b) The curve representing the largest interest rate.
(c) Two curves representing the same initial deposit.
(d) Two curves representing the same interest rate.

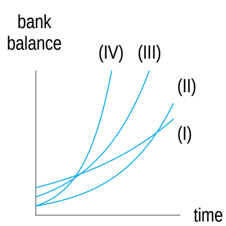


Figure 1.70

9. You need \$10,000 in your account 3 years from now and the interest rate is 8% per year, compounded continuously. How much should you deposit now?

10. Air pressure, P, decreases exponentially with the height, h, in meters above sea level:

$$P = P_0 e^{-0.00012h}$$

where P_0 is the air pressure at sea level.

(a) At the top of Mount McKinley, height 6198 meters (about 20,330 feet), what is the air pressure, as a percent of the pressure at sea level?
(b) The maximum cruising altitude of an ordinary commercial jet is around 12,000 meters (about 39,000 feet). At that height, what is the air pressure, as a percent of the sea level value?

11. An exponentially growing animal population numbers 500 at time $t = 0$; two years later, it is 1500. Find a formula for the size of the population in t years and find the size of the population at $t = 5$.

12. **(a)** Figure 1.71 shows exponential growth. Starting at $t = 0$, estimate the time for the population to double.
(b) Repeat part (a), but this time start at $t = 3$.
(c) Pick any other value of t for the starting point, and notice that the doubling time is the same no matter where you start.

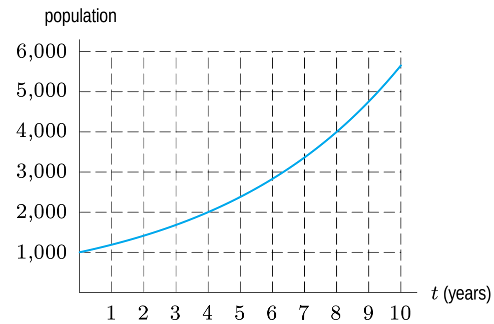


Figure 1.71

13. A population, currently 200, is growing at 5% per year.

(a) Write a formula for the population, P, as a function of time, t, years in the future.
(b) Graph P against t.
(c) Estimate the population 10 years from now.
(d) Use the graph to estimate the doubling time of the population.

14. The antidepressant fluoxetine (or Prozac) has a half-life of about 3 days. What percentage of a dose remains in the body after one day? After one week?

15. Figure 1.72 shows the balances in two bank accounts. Both accounts pay the same interest rate, but one compounds continuously and the other compounds annually. Which curve corresponds to which compounding method? What is the initial deposit in each case?

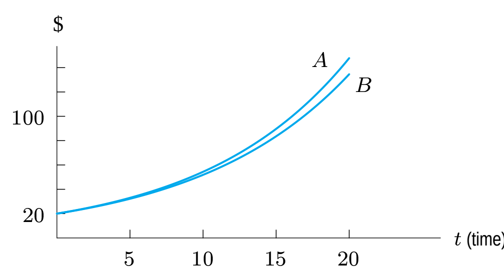


Figure 1.72

16. A cup of coffee contains 100 mg of caffeine, which leaves the body at a continuous rate of 17% per hour.

(a) Write a formula for the amount, A mg, of caffeine in the body t hours after drinking a cup of coffee.
(b) Graph the function from part (a). Use the graph to estimate the half-life of caffeine.
(c) Use logarithms to find the half-life of caffeine.

17. One of the main contaminants of a nuclear accident, such as that at Chernobyl, is strontium-90, which decays exponentially at a rate of approximately 2.5% per year.

(a) Write the percent of strontium-90 remaining, P, as a function of years, t, since the nuclear accident. [Hint: 100% of the contaminant remains at $t = 0$.]
(b) Graph P against t.
(c) Estimate the half-life of strontium-90.
(d) After the Chernobyl disaster, it was predicted that the region would not be safe for human habitation for 100 years. Estimate the percent of original strontium-90 remaining at this time.

18. The table shows yearly per capita health expenditures in the US.[32]

(a) Check that the increase in health expenditures is approximately exponential.
(b) Estimate the doubling time of health expenditures.

Year	1970	1972	1974	1976	1978	1980	1982
Expenditures	349	428	521	665	822	1055	1348

19. If \$12,000 is deposited in an account paying 8% interest per year, compounded continuously, how long will it take for the balance to reach \$20,000?

20. In 1994, the world's population was 5.6 billion, and the population was projected to reach 8.5 billion by the year 2030. What annual growth rate is projected?

21. If the size of a bacteria colony doubles in 5 hours, how long will it take for the number of bacteria to triple?

22. You want to invest money for your child's education in a certificate of deposit (CD). You want it to be worth $\$12{,}000$ in 10 years. How much should you invest if the CD pays interest at a 9% annual rate compounded
(a) Annually? **(b)** Continuously?

23. When you rent an apartment, you are often required to give the landlord a security deposit which is returned if you leave the apartment undamaged. In Massachusetts the landlord is required to pay the tenant interest on the deposit once a year, at a 5% annual rate, compounded annually. The landlord, however, may invest the money at a higher (or lower) interest rate. Suppose the landlord invests a $\$1000$ deposit at a yearly rate of

(a) 6%, compounded continuously
(b) 4%, compounded continuously.

In each case, determine the net gain or loss by the landlord at the end of the first year. (Give your answer to the nearest cent.)

24. Cumulative HIV infections increased worldwide approximately exponentially from 2.5 million in 1985 to 58.0 million in 2000.[33] (HIV is the virus that causes AIDS.)

(a) Give a formula for HIV infections, H, (in millions) as a function of the number of years, t, since 1985. Use the form $H = H_0 e^{kt}$. Graph this function.
(b) What was the annual continuous percent change in HIV infections between 1985 and 2000?
(c) In the year 2000, the number of new HIV infections was less than in the previous year. Does it appear that the exponential function is a reasonable way to model this data after the year 2000?

25. The island of Manhattan was sold for \$24 in 1626. Suppose the money had been invested in an account which compounded interest continuously.

(a) How much money would be in the account in the year 2000 if the yearly interest rate was
(i) 5%? (ii) 7%?
(b) If the yearly interest rate was 6%, in what year would the account be worth one million dollars?

26. In 1923, eighteen koala bears were introduced on Kangaroo Island off the coast of Australia. In 1993, the population was about 5000. Assuming exponential growth, find the (continuous) rate of growth of the population during this period. Find a formula for the population as a function of the number of years since 1923, and estimate the population in the year 2010.

27. The total world marine catch in 1950 was 17 million tons and in 1995 was 91 million tons.[34] If the marine catch is increasing exponentially, find the (continuous) rate of increase. Use it to predict the total world marine catch in the year 2020.

28. **(a)** Use the Rule of 70 to predict the doubling time of an investment which is earning 8% interest per year.
(b) Find the doubling time exactly, and compare your answer to part (a).

29. A radioactive substance has a half-life of 8 years. If 200 grams are present initially, how much remains at the end of 12 years? How long until only 10% of the original amount remains?

30. The quantity, Q, of radioactive carbon-14 remaining t years after an organism dies is given by the formula

$$Q = Q_0 e^{-0.000121t},$$

where Q_0 is the initial quantity.

(a) A skull uncovered at an archeological dig has 15% of the original amount of carbon-14 present. Estimate its age.
(b) Calculate the half-life of carbon-14.

[32] *Statistical Abstracts of the US 1988*, p. 86, Table 129.
[33] The Worldwatch Institute, *Vital Signs* 2001, p. 79, (New York: W.W. Norton, 2001).
[34] J. Madeleine Nash, "The Fish Crisis," *Time*, August 11, 1997, p. 67.

31. Owing to an innovative rural public health program, infant mortality in Senegal, West Africa, is being reduced at a rate of 10% per year. How long will it take for infant mortality to be reduced by 50%?

32. A picture supposedly painted by Vermeer (1632–1675) contains 99.5% of its carbon-14 (half-life 5730 years). From this information decide whether the picture is a fake. Explain your reasoning.

33. Interest is compounded annually. Consider the following choices of payments to you:

Choice 1: $1500 now and $3000 one year from now

Choice 2: $1900 now and $2500 one year from now

(a) If the interest rate on savings were 5% per year, which would you prefer?
(b) Is there an interest rate that would lead you to make a different choice? Explain.

34. A person is to be paid $2000 for work done over a year. Three payment options are being considered. Option 1 is to pay the $2000 in full now. Option 2 is to pay $1000 now and $1000 in a year. Option 3 is to pay the full $2000 in a year. Assume an annual interest rate of 5% a year, compounded continuously.

(a) Without doing any calculations, which option is the best option financially for the worker? Explain.
(b) Find the future value, in one year's time, of all three options.
(c) Find the present value of all three options.

35. A business associate who owes you $3000 offers to pay you $2800 now, or else pay you three yearly installments of $1000 each, with the first installment paid now. If you use only financial reasons to make your decision, which option should you choose? Justify your answer, assuming a 6% market interest rate, compounded continuously.

36. Big Tree McGee is negotiating his rookie contract with a professional basketball team. They have agreed to a three-year deal which will pay Big Tree a fixed amount at the end of each of the three years, plus a signing bonus at the beginning of his first year. They are still haggling about the amounts and Big Tree must decide between a big signing bonus and fixed payments per year, or a smaller bonus with payments increasing each year. The two options are summarized in the table. All values are payments in millions of dollars.

	Signing bonus	Year 1	Year 2	Year 3
Option #1	6.0	2.0	2.0	2.0
Option #2	1.0	2.0	4.0	6.0

(a) Big Tree decides to invest all income in stock funds which he expects to grow at a rate of 10% per year, compounded continuously. He would like to choose the contract option which gives him the greater future value at the end of the three years when the last payment is made. Which option should he choose?
(b) Calculate the present value of each contract offer.

37. A company is considering whether to buy a new machine, which costs $97,000. The cash flows (adjusted for taxes and depreciation) that would be generated by the new machine are given in the following table:

Year	1	2	3	4
Cash flow	$50,000	$40,000	$25,000	$20,000

(a) Find the total present value of the cash flows. Treat each year's cash flow as a lump sum at the end of the year and use a interest rate of 7.5% per year, compounded annually.
(b) Based on a comparison of the cost of the machine and the present value of the cash flows, would you recommend purchasing the machine?

38. You are buying a car that comes with a one-year warranty and are considering whether to purchase an extended warranty for $375. The extended warranty covers the two years immediately after the one-year warranty expires. You estimate that the yearly expenses that would have been covered by the extended warranty are $150 at the end of the first year of the extended warranty and $250 at the end of the second year. The interest rate is 5% per year, compounded annually. Should you buy the extended warranty? Explain.

39. You have the option of renewing the service contract on your three-year old dishwasher. The new service contract is for three years at a price of $200. The interest rate is 7.25% per year, compounded annually, and you estimate that the costs of repairs if you do not buy the service contract will be $50 at the end of the first year, $100 at the end of the second year, and $150 at the end of the third year. Should you buy the service contract? Explain.

1.8 NEW FUNCTIONS FROM OLD

We have studied linear and exponential functions, and the logarithm function. In this section, we learn how to create new functions by composing, stretching, and shifting functions we already know.

Composite Functions

A drop of water falls onto a paper towel. The area, A of the circular damp spot is a function of r, its radius, which is a function of time, t. We know $A = f(r) = \pi r^2$; suppose $r = g(t) = t + 1$. By substitution, we express A as a function of t:

$$A = f(g(t)) = \pi(t+1)^2.$$

The function $f(g(t))$ is a "function of a function," or a *composite function*, in which there is an *inside function* and an *outside function*. To find $f(g(2))$, we first add one ($g(2) = 2 + 1 = 3$) and then square and multiply by π. We have

$$f(g(2)) = \pi(2+1)^2 \quad = \quad \pi 3^2 \quad = \quad 9\pi.$$

↖First calculation ↖Second calculation

The inside function is $t + 1$ and the outside function is squaring and multiplying by π. In general, the inside function represents the calculation that is done first and the outside function represents the calculation done second.

Example 1 If $f(t) = t^2$ and $g(t) = t + 2$, find

(a) $f(t+1)$ (b) $f(t) + 3$ (c) $f(t+h)$ (d) $f(g(t))$ (e) $g(f(t))$

Solution

(a) Since $t + 1$ is the inside function, $f(t+1) = (t+1)^2$.
(b) Here 3 is added to $f(t)$, so $f(t) + 3 = t^2 + 3$.
(c) Since $t + h$ is the inside function, $f(t+h) = (t+h)^2$.
(d) Since $g(t) = t + 2$, substituting $t + 2$ into f gives $f(g(t)) = f(t+2) = (t+2)^2$.
(e) Since $f(t) = t^2$, substituting t^2 into g gives $g(f(t)) = g(t^2) = t^2 + 2$.

Example 2 If $f(x) = e^x$ and $g(x) = 5x + 1$, find (a) $f(g(x))$ (b) $g(f(x))$

Solution

(a) Substituting $g(x) = 5x + 1$ into f gives $f(g(x)) = f(5x+1) = e^{5x+1}$.
(b) Substituting $f(x) = e^x$ into g gives $g(f(x)) = g(e^x) = 5e^x + 1$.

Example 3 Using the following table, find $g(f(0))$, $f(g(0))$, $f(g(1))$, and $g(f(1))$.

x	0	1	2	3
$f(x)$	3	1	−1	−3
$g(x)$	0	2	4	6

Solution

To find $g(f(0))$, we first find $f(0) = 3$ from the table. Then we have $g(f(0)) = g(3) = 6$.
For $f(g(0))$, we must find $g(0)$ first. Since $g(0) = 0$, we have $f(g(0)) = f(0) = 3$.
Similar reasoning leads to $f(g(1)) = f(2) = -1$ and $g(f(1)) = g(1) = 2$.

We can write a composite function using a new variable u to represent the value of the inside function. For example

$$y = (t+1)^4 \quad \text{is the same as} \quad y = u^4 \quad \text{with} \quad u = t + 1.$$

Other expressions for u, such as $u = (t+1)^2$, with $y = u^2$, are also possible.

Example 4 Use a new variable u for the inside function to express each of the following as a composite function:

(a) $y = \ln(3t)$ (b) $w = 5(2r+3)^2$ (c) $P = e^{-0.03t}$

Solution

(a) We take the inside function to be $3t$, so $y = \ln u$ with $u = 3t$.
(b) We take the inside function to be $2r + 3$, so $w = 5u^2$ with $u = 2r + 3$.
(c) We take the inside function to be $-0.03t$, so $P = e^u$ with $u = -0.03t$.

Stretches of Graphs

The graph of $y = f(x)$ is in Figure 1.73. What does the graph of $y = 3f(x)$ look like? The factor 3 in the function $y = 3f(x)$ stretches each $f(x)$ value by multiplying it by 3. What does the graph of $y = -2f(x)$ look like? The factor -2 in the function $y = -2f(x)$ stretches $f(x)$ by multiplying by 2 and reflecting it about the x-axis. See Figure 1.74.

Multiplying a function by a constant, c, stretches the graph vertically (if $c > 1$) or shrinks the graph vertically (if $0 < c < 1$). A negative sign (if $c < 0$) reflects the graph about the x-axis, in addition to shrinking or stretching.

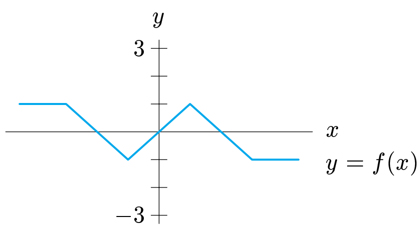


Figure 1.73: Graph of $f(x)$

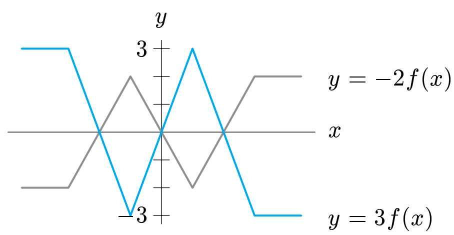


Figure 1.74: Multiples of the function $f(x)$

Shifted Graphs

Consider the function $y = x^2 + 4$. The y-coordinates for this function are exactly 4 units larger than the corresponding y-coordinates of the function $y = x^2$. So the graph of $y = x^2 + 4$ is obtained from the graph of $y = x^2$ by adding 4 to the y-coordinate of each point; that is, by moving the graph of $y = x^2$ up 4 units. (See Figure 1.75.)

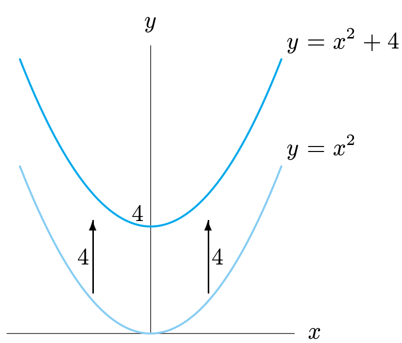


Figure 1.75: Vertical shift: Graphs of $y = x^2$ and $y = x^2 + 4$

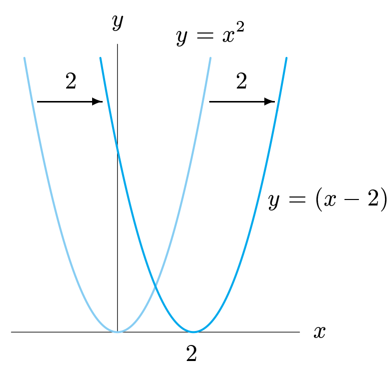


Figure 1.76: Horizontal shift: Graphs of $y = x^2$ and $y = (x - 2)^2$

A graph can also be shifted to the left or to the right. In Figure 1.76, we see that the graph of $y = (x - 2)^2$ is the graph of $y = x^2$ shifted to the right 2 units. In general,

- The graph of $y = f(x) + k$ is the graph of $y = f(x)$ moved up k units (down if k is negative).
- The graph of $y = f(x - k)$ is the graph of $y = f(x)$ moved to the right k units (to the left if k is negative).

Example 5 (a) A cost function, $C(q)$, for a company is shown in Figure 1.77. The fixed cost increases by \$1000. Sketch a graph of the new cost function.
(b) A supply curve, S, for a product is given in Figure 1.78. A new factory opens and produces 100 units of the product no matter what the price. Sketch a graph of the new supply curve.

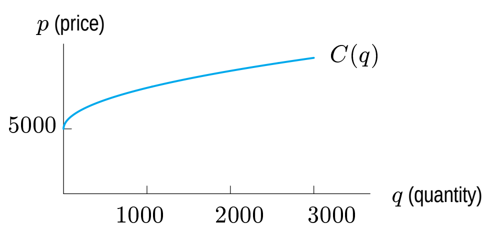


Figure 1.77: A cost function

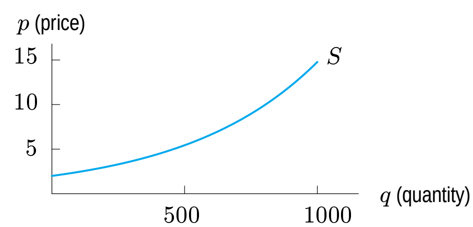


Figure 1.78: A supply function

Solution (a) For each quantity, the new cost is \$1000 more than the old cost. The new cost function is $C(q) + 1000$, whose graph is the graph of $C(q)$ shifted vertically up 1000 units. (See Figure 1.79.)
(b) To see the effect of the new factory, look at an example. At a price of 10 dollars, approximately 800 units are currently produced. With the new factory, this amount increases by 100 units, so the new amount produced is 900 units. At each price, the quantity produced increases by 100, so the new supply curve is S shifted horizontally to the right by 100 units. (See Figure 1.80.)

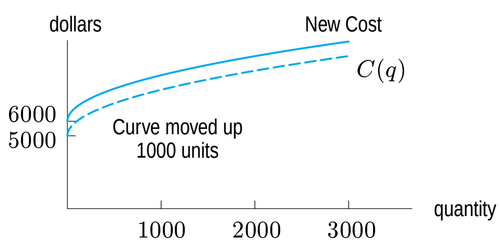


Figure 1.79: New cost function (original curve dashed)

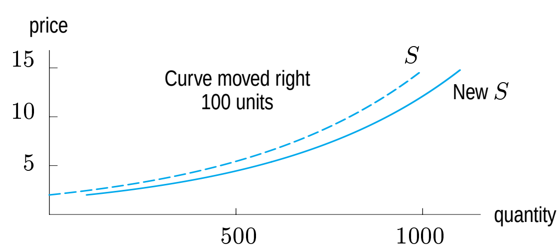


Figure 1.80: New supply curve (original curve dashed)

Problems for Section 1.8

1. For $g(x) = x^2 + 2x + 3$, find and simplify:

(a) $g(2+h)$ **(b)** $g(2)$
(c) $g(2+h) - g(2)$

2. If $f(x) = x^2 + 1$, find and simplify:

(a) $f(t+1)$ **(b)** $f(t^2+1)$
(c) $f(2)$ **(d)** $2f(t)$
(e) $[f(t)]^2 + 1$

3. Let $f(x) = 2x^2$ and $g(x) = x + 3$. Find the following:

(a) $f(g(x))$ **(b)** $g(f(x))$ **(c)** $f(f(x))$

4. Let $f(x) = x^2$ and $g(x) = 3x - 1$. Find the following:

(a) $f(2) + g(2)$ **(b)** $f(2) \cdot g(2)$
(c) $f(g(2))$ **(d)** $g(f(2))$

5. If $h(x) = x^3 + 1$ and $g(x) = \sqrt{x}$, find

(a) $g(h(x))$ **(b)** $h(g(x))$
(c) $h(h(x))$ **(d)** $g(x) + 1$
(e) $g(x+1)$

6. Let $f(x) = 2x + 3$ and $g(x) = \ln x$. Find formulas for each of the following functions.

(a) $g(f(x))$ **(b)** $f(g(x))$ **(c)** $f(f(x))$

7. Use the variable u for the inside function to express each of the following as a composite function:

(a) $y = (5t^2 - 2)^6$ **(b)** $P = 12e^{-0.6t}$
(c) $C = 12\ln(q^3 + 1)$

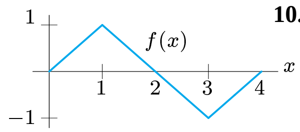
8. Use the variable u for the inside function to express each of the following as a composite function:

(a) $y = 2^{3x-1}$ **(b)** $P = \sqrt{5t^2 + 10}$
(c) $w = 2\ln(3r + 4)$

For the functions $f(x)$ in Problems 9–12, graph:

(a) $y = f(x) + 2$ **(b)** $y = f(x-1)$
(c) $y = 3f(x)$ **(d)** $y = -f(x)$

9.



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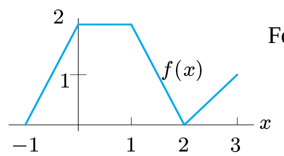

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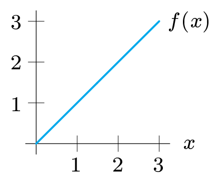

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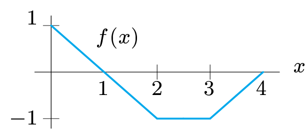


Graph the functions in Problems 13–18 using Figure 1.81.

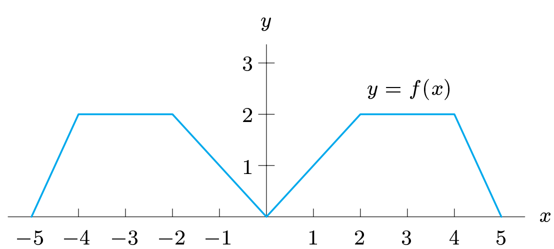


Figure 1.81

13. $y = 2f(x)$

14. $y = f(x) + 2$

15. $y = -3f(x)$

16. $y = f(x-1)$

17. $y = 2 - f(x)$

18. $y = 2f(x) - 1$

19. Use Table 1.31 to find:

(a) $f(g(0))$ **(b)** $f(g(1))$ **(c)** $f(g(2))$
(d) $g(f(2))$ **(e)** $g(f(3))$

Table 1.31

x	0	1	2	3	4	5
$f(x)$	10	6	3	4	7	11
$g(x)$	2	3	5	8	12	15

20. Make a table of values for each of the following functions using Table 1.31.

(a) $f(x) + 3$ **(b)** $f(x-2)$ **(c)** $5g(x)$
(d) $-f(x) + 2$ **(e)** $g(x-3)$ **(f)** $f(x) + g(x)$

For Problems 21–23, use the graphs in Figure 1.82.

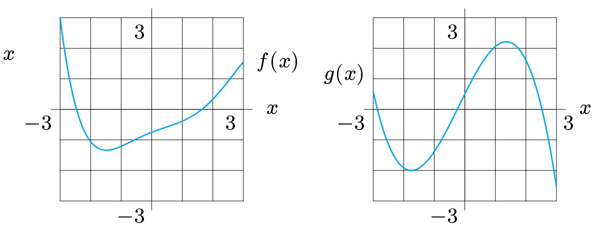


Figure 1.82

21. Estimate $f(g(1))$.

22. Estimate $g(f(2))$.

23. Estimate $f(f(1))$.

24. **(a)** Write an equation for a graph obtained by vertically stretching the graph of $y = x^2$ by a factor of 2, followed by a vertical upward shift of 1 unit. Sketch it.
(b) What is the equation if the order of the transformations (stretching and shifting) in part (a) is interchanged?
(c) Are the two graphs the same? Explain the effect of reversing the order of transformations.

25. For $f(n) = 3n^2 - 2$ and $g(n) = n + 1$, find and simplify:

(a) $f(n) + g(n)$
(b) $f(n)g(n)$
(c) The domain of $f(n)/g(n)$
(d) $f(g(n))$
(e) $g(f(n))$

Simplify the quantities in Problems 26–29 using $m(z) = z^2$.

26. $m(z+1) - m(z)$

27. $m(z+h) - m(z)$

28. $m(z) - m(z-h)$

29. $m(z+h) - m(z-h)$

For Problems 30–31, determine functions f and g such that $h(x) = f(g(x))$. [Note: There is more than one correct answer. Do not choose $f(x) = x$ or $g(x) = x$.]

30. $h(x) = (x+1)^3$

31. $h(x) = x^3 + 1$

32. Using Table 1.32, create a table of values for $f(g(x))$ and for $g(f(x))$.

Table 1.32

x	−3	−2	−1	0	1	2	3
$f(x)$	0	1	2	3	2	1	0
$g(x)$	3	2	2	0	−2	−2	−3

33. Using Figure 1.83, find $f(g(x))$ and $g(f(x))$ for $x = -3, -2, -1, 0, 1, 2, 3$. Then graph $f(g(x))$ and $g(f(x))$.

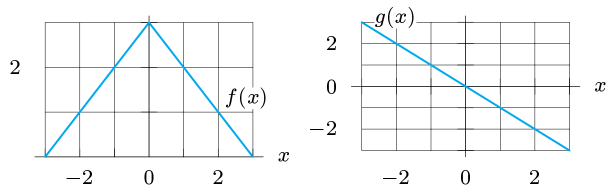


Figure 1.83

34. The Heaviside step function, H, is graphed in Figure 1.84. Graph the following functions.

(a) $2H(x)$ **(b)** $H(x)+1$ **(c)** $H(x+1)$
(d) $-H(x)$ **(e)** $H(-x)$

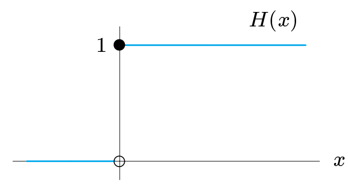


Figure 1.84

1.9 PROPORTIONALITY, POWER FUNCTIONS, AND POLYNOMIALS

Proportionality

A common functional relationship occurs when one quantity is *proportional* to another. For example, if apples are $1.40 a pound, we say the price you pay, p dollars, is proportional to the weight you buy, w pounds, because

$$p = f(w) = 1.40w.$$

As another example, the area, A, of a circle is proportional to the square of the radius, r:

$$A = f(r) = \pi r^2.$$

> We say y is (directly) **proportional** to x if there is a nonzero constant k such that
>
> $$y = kx.$$
>
> This k is called the constant of proportionality.

We also say that one quantity is *inversely proportional* to another if one is proportional to the reciprocal of the other. For example, the speed, v, at which you make a 50-mile trip is inversely proportional to the time, t, taken, because v is proportional to $1/t$:

$$v = 50\left(\frac{1}{t}\right) = \frac{50}{t}.$$

Notice that if y is directly proportional to x, then the magnitude of one variable increases (decreases) when the magnitude of the other increases (decreases). If, however, y is inversely proportional to x, then the magnitude of one variable increases when the magnitude of the other decreases.

Example 1 The heart mass of a mammal is proportional to its body mass.[35]

(a) Write a formula for heart mass, H, as a function of body mass, B.
(b) A human with a body mass of 70 kilograms has a heart mass of 0.42 kilograms. Use this information to find the constant of proportionality.
(c) Estimate the heart mass of a horse with a body mass of 650 kg.

[35] K. Schmidt-Nielson: *Scaling-Why is Animal Size So Important*? (Cambridge: CUP, 1984).

Solution

(a) Since H is proportional to B, for some constant k, we have

$$H = kB.$$

(b) We use the fact that $H = 0.42$ when $B = 70$ to solve for k:

$$H = kB$$
$$0.42 = k(70)$$
$$k = \frac{0.42}{70} = 0.006.$$

(c) Since $k = 0.006$, we have $H = 0.006B$, so the heart mass of the horse is given by

$$H = 0.006(650) = 3.9 \text{ kilograms}.$$

Example 2 The period of a pendulum, T, is the amount of time required for the pendulum to make one complete swing. For small swings, the period, T, is approximately proportional to the square root of l, the pendulum's length. So

$$T = k\sqrt{l} \quad \text{where } k \text{ is a constant.}$$

Notice that T is not directly proportional to l, but T is proportional to $\sqrt{l}$.

Example 3 An object's weight, w, is inversely proportional to the square of its distance, r, from the earth's center. So, for some constant k,

$$w = \frac{k}{r^2}.$$

Here w is not inversely proportional to r, but to r^2.

Power Functions

In each of the previous examples, one quantity is proportional to the power of another quantity. We make the following definition:

> We say that $Q(x)$ is a **power function** of x if $Q(x)$ is proportional to a constant power of x. If k is the constant of proportionality, and if p is the power, then
>
> $$Q(x) = k \cdot x^p.$$

For example, the function $H = 0.006B$ is a power function with $p = 1$. The function $T = k\sqrt{l} = kl^{1/2}$ is a power function with $p = 1/2$, and the function $w = k/r^2 = kr^{-2}$ is a power function with $p = -2$.

Example 4 Which of the following are power functions? For those which are, write the function in the form $y = kx^p$, and give the coefficient k and the exponent p.

(a) $y = \dfrac{5}{x^3}$ (b) $y = \dfrac{2}{3x}$ (c) $y = \dfrac{5x^2}{2}$

(d) $y = 5 \cdot 2^x$ (e) $y = 3\sqrt{x}$ (f) $y = (3x^2)^3$

Solution

(a) Since $y = 5x^{-3}$, this is a power function with $k = 5$ and $p = -3$.
(b) Since $y = (2/3)x^{-1}$, this is a power function with $k = 2/3$ and $p = -1$.
(c) Since $y = (5/2)x^2$, this is a power function with $k = 5/2 = 2.5$ and $p = 2$.
(d) This is not a power function. It is an exponential function.
(e) Since $y = 3x^{1/2}$, this is a power function with $k = 3$ and $p = 1/2$.
(f) Since $y = 3^3 \cdot (x^2)^3 = 27x^6$, this is a power function with $k = 27$ and $p = 6$.

Graphs of Power Functions

The graph of $y = x^2$ is shown in Figure 1.85. It is decreasing for negative x and increasing for positive x. Notice that it is bending upward, or concave up, for all x. The graph of $y = x^3$ is shown in Figure 1.86. Notice that it is bending downward, or concave down for negative x and bending upward, or concave up for positive x. The graph of $y = \sqrt{x} = x^{1/2}$ is shown in Figure 1.87. Notice that the graph is increasing and concave down.

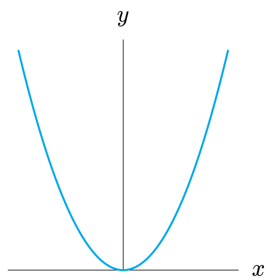


Figure 1.85: Graph of $y = x^2$

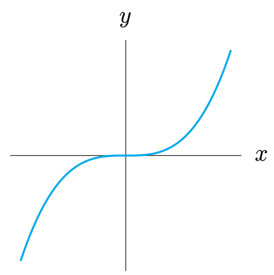


Figure 1.86: Graph of $y = x^3$

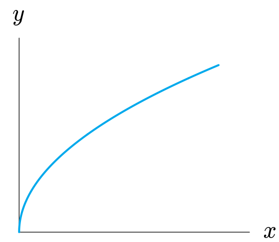


Figure 1.87: Graph of $y = x^{1/2}$

Example 5 If N is the average number of species found on an island and A is the area of the island, observations have shown[36] that N is approximately proportional to the cube root of A. Write a formula for N as a function of A and describe the shape of the graph of this function.

Solution For some positive constant k, we have

$$N = k\sqrt[3]{A} = kA^{1/3}.$$

It turns out that the value of k depends on the region of the world in which the island is found. The graph of N against A (for $A > 0$) has a shape similar to the graph in Figure 1.87. It is increasing and concave down. Thus, larger islands have more species on them (as we would expect), but the increase slows as the island gets larger.

The function $y = x^0 = 1$ has a graph that is a horizontal line. For negative powers, rewriting

$$y = x^{-1} = \frac{1}{x} \quad \text{and} \quad y = x^{-2} = \frac{1}{x^2}$$

makes it clear that as $x > 0$ increases, the denominators increase and the functions decrease. The graphs of $y = x^{-1}$ and $y = x^{-2}$ have both the x- and y-axes as asymptotes. (See Figure 1.88.)

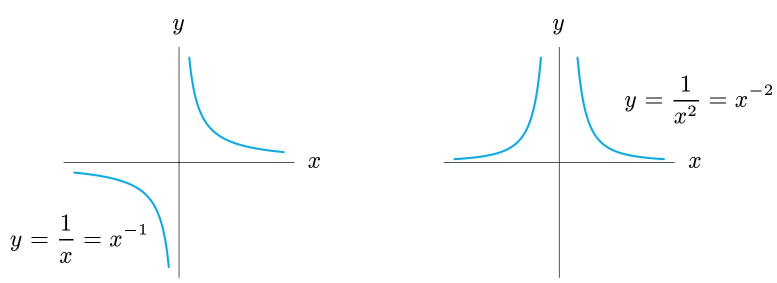


Figure 1.88: Graphs of negative powers of x

[36] *Scientific American*, p. 112, (September, 1989).

Polynomials

Sums of power functions with non-negative integer exponents are called *polynomials*, which are functions of the form

$$y = p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0.$$

Here, n is a nonnegative integer, called the *degree* of the polynomial, and a_n is a non-zero number called the *leading coefficient*. We call $a_n x^n$ the *leading term*. If $n = 2$, the polynomial is called quadratic; if $n = 3$, the polynomial is called cubic.

The shape of the graph of a polynomial depends on its degree. See Figure 1.89. The graph of a quadratic polynomial is a parabola. It opens up if the leading coefficient is positive (as in Figure 1.89) and opens down if the leading coefficient is negative. A cubic polynomial may have the shape of the graph of $y = x^3$, or the shape shown in Figure 1.89, or it may be a reflection of one of these about the x-axis.

Notice in Figure 1.89 that the graph of the quadratic "turns around" once, the cubic "turns around" twice, and the quartic (fourth degree) "turns around" three times. An n^{th} degree polynomial "turns around" at most $n - 1$ times (where n is a positive integer), but there may be fewer turns.

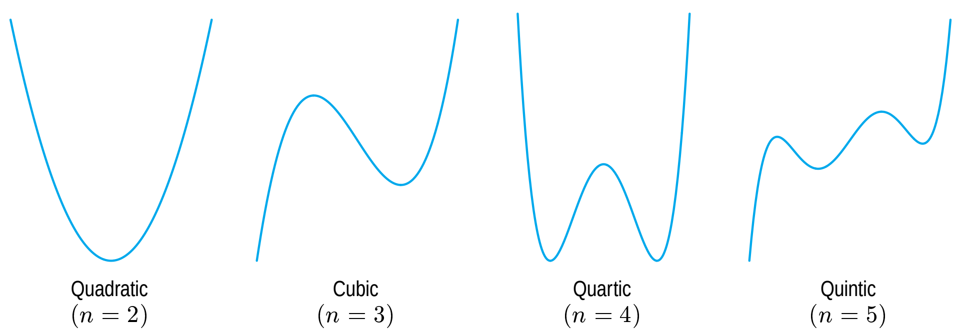


Figure 1.89: Graphs of typical polynomials of degree n

Example 6 A company finds that the average number of people attending a concert is 75 if the price is \$50 per person. At a price of \$35 per person, the average number of people in attendance is 120.

(a) Assume that the demand curve is a line. Write the demand, q, as a function of price, p.
(b) Use your answer to part (a) to write the revenue, R, as a function of price, p.
(c) Use a graph of the revenue function to determine what price should be charged to obtain the greatest revenue.

Solution (a) Two points on the line are $(p, q) = (50, 75)$ and $(p, q) = (35, 120)$. The slope of the line is

$$m = \frac{120 - 75}{35 - 50} = \frac{45}{-15} = -3 \text{ people/dollar.}$$

To find the vertical intercept of the line, we use the slope and one of the points:

$$\begin{aligned} 75 &= b + (-3)(50) \\ 225 &= b \end{aligned}$$

The demand function is $q = 225 - 3p$.

(b) Since $R = pq$ and $q = 225 - 3p$, we see that $R = p(225 - 3p) = 225p - 3p^2$.

(c) The revenue function is the quadratic polynomial graphed in Figure 1.90. The maximum revenue occurs at $p = 37.5$. Thus, the company maximizes revenue by charging \$37.50 per person.

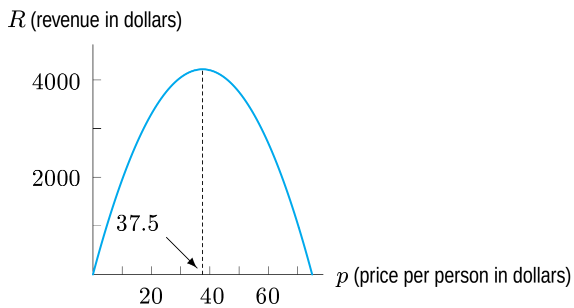


Figure 1.90: Revenue function for concert ticket sales

Example 7 Using a calculator or computer, sketch $y = x^4$ and $y = x^4 - 15x^2 - 15x$ for $-20 \le x \le 20$ and $0 \le y \le 200{,}000$. What do you observe?

Solution From Figure 1.91, we see that the two graphs look indistinguishable. The reason is that the leading term of each polynomial (the one with the highest power of x) is the same, namely x^4, and for the large values of x in this window, the leading term dominates the other terms.

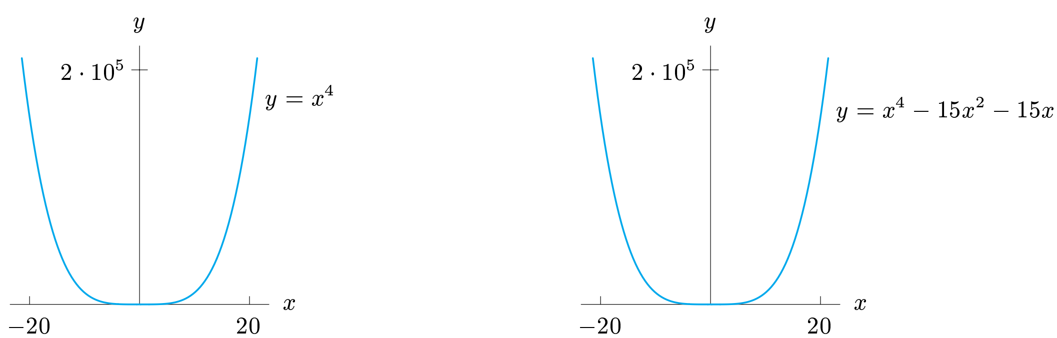


Figure 1.91: Graphs of $y = x^4$ and $y = x^4 - 15x^2 - 15x$ look almost indistinguishable in a large window

Problem 39 compares the graphs of these two functions in a smaller window with Figure 1.91.

We see in Example 7 that, from a distance, the polynomial $y = x^4 - 15x^2 - 15x$ looks like the power function $y = x^4$. In general, if the graph of a polynomial of degree n

$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

is viewed in a large enough window, it has approximately the same shape as the graph of the power function given by the leading term:

$$y = a_n x^n.$$

Problems for Section 1.9

In Problems 1–12, determine whether or not the function is a power function. If it is a power function, write it in the form $y = kx^p$ and give the values of k and p.

1. $y = \dfrac{3}{x^2}$

2. $y = 5\sqrt{x}$

3. $y = \dfrac{3}{8x}$

4. $y = 2^x$

5. $y = \dfrac{5}{2\sqrt{x}}$

6. $y = (3x^5)^2$

7. $y = \dfrac{2x^2}{10}$

8. $y = 3 \cdot 5^x$

9. $y = (5x)^3$

10. $y = \dfrac{8}{x}$

11. $y = \dfrac{x}{5}$

12. $y = 3x^2 + 4$

In Problems 13–16, write a formula representing the function.

13. The energy, E, expended by a swimming dolphin is proportional to the cube of the speed, v, of the dolphin.

14. The strength, S, of a beam is proportional to the square of its thickness, h.

15. The gravitational force, F, between two bodies is inversely proportional to the square of the distance d between them.

16. The average velocity, v, for a trip over a fixed distance, d, is inversely proportional to the time of travel, t.

17. The following table gives values for a function $p = f(t)$. Could p be proportional to t?

t	0	10	20	30	40	50
p	0	25	60	100	140	200

18. The surface area of a mammal, S, satisfies the equation $S = kM^{2/3}$, where M is the body mass, and the constant of proportionality k depends on the body shape of the mammal. A human of body mass 70 kilograms has surface area 18,600 cm^2. Find the constant of proportionality for humans. Find the surface area of a human with body mass 60 kilograms.

19. Biologists estimate that the number of animal species of a certain body length is inversely proportional to the square of the body length.[37] Write a formula for the number of animal species, N, of a certain body length as a function of the length, L. Are there more species at large lengths or at small lengths? Explain.

20. The circulation time of a mammal (that is, the average time it takes for all the blood in the body to circulate once and return to the heart) is proportional to the fourth root of the body mass of the mammal.

(a) Write a formula for the circulation time, T, in terms of the body mass B.

(b) If an elephant of body mass 5230 kilograms has a circulation time of 148 seconds, find the constant of proportionality.

(c) What is the circulation time of a human with body mass 70 kilograms?

For the functions in Problems 21–28:

(a) What is the degree of the polynomial? Is the leading coefficient positive or negative?

(b) What power function approximates $f(x)$ for large x? Without using a calculator or computer, sketch the graph of the function in a large window.

(c) Using a calculator or computer, sketch a graph of the function. How many turning points does the function have? How does the number of turning points compare to the degree of the polynomial?

21. $f(x) = x^2 + 10x - 5$

22. $f(x) = 5x^3 - 17x^2 + 9x + 50$

23. $f(x) = 8x - 3x^2$

24. $f(x) = 17 + 8x - 2x^3$

25. $f(x) = -9x^5 + 82x^3 + 12x^2$

26. $f(x) = 0.01x^4 + 2.3x^2 - 7$

27. $f(x) = 100 + 5x - 12x^2 + 3x^3 - x^4$

28. $f(x) = 0.2x^7 + 1.5x^4 - 3x^3 + 9x - 15$

29. Allometry is the study of the relative size of different parts of a body as a consequence of growth. In this problem, you will check the accuracy of an allometric equation: the weight of a fish is proportional to the cube of its length.[38] Table 1.33 relates the weight, y, in gm, of plaice (a type of fish) to its length, x, in cm. Does this data support the hypothesis that (approximately) $y = kx^3$? If so, estimate the constant of proportionality, k.

Table 1.33

x	y	x	y	x	y
33.5	332	37.5	455	41.5	623
34.5	363	38.5	500	42.5	674
35.5	391	39.5	538	43.5	724
36.5	419	40.5	574		

30. The DuBois formula relates a person's surface area, s, in m^2, to weight w, in kg, and height h, in cm, by

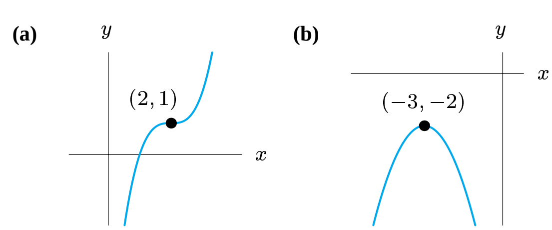
$$s = 0.01w^{0.25}h^{0.75}.$$

(a) What is the surface area of a person who weighs 65 kg and is 160 cm tall?

(b) What is the weight of a person whose height is 180 cm and who has a surface area of 1.5 m^2?

(c) For people of fixed weight 70 kg, solve for h as a function of s. Simplify your answer.

31. Use shifts of power functions to find a possible formula for each of the graphs.



[37] *US News & World Report*, August 18, 1997, p. 79.

[38] Adapted from "On the Dynamics of Exploited Fish Populations" by R. J. H. Beverton and S. J. Holt, *Fishery Investigations*, Series II, 19, 1957.

32. Find the average rate of change between $x = 0$ and $x = 10$ of each of the following functions: $y = x, y = x^2, y = x^3$, and $y = x^4$. Which has the largest average rate of change? Graph the four functions, and draw lines whose slopes represent these average rates of change.

33. The rate, R, at which a population in a confined space increases is proportional to the product of the current population, P, and the difference between the carrying capacity, L, and the current population. (The carrying capacity is the maximum population the environment can sustain.)

(a) Write R as a function of P.
(b) Sketch R as a function of P.

34. A standard tone of 20,000 dynes/cm^2 (about the loudness of a rock band) is assigned a value of 10. A subject listened to other sounds, such as a light whisper, normal conversation, thunder, a jet plane at takeoff, and so on. In each case, the subject was asked to judge the loudness and assign it a number relative to 10, the value of the standard tone. This is a "judgment of magnitude" experiment. The power law $J = al^{0.3}$ was found to model the situation well, where l is the actual loudness (measured in dynes/cm^2) and J is the judged loudness.

(a) What is the value of a?
(b) What is the judged loudness if the actual loudness is 0.2 dynes/cm^2 (normal conversation)?
(c) What is the actual loudness if judged loudness is 20?

35. Each of the graphs in Figure 1.92 is of a polynomial. The windows are large enough to show global behavior.

(a) What is the minimum possible degree of the polynomial?
(b) Is the leading coefficient of the polynomial positive or negative?

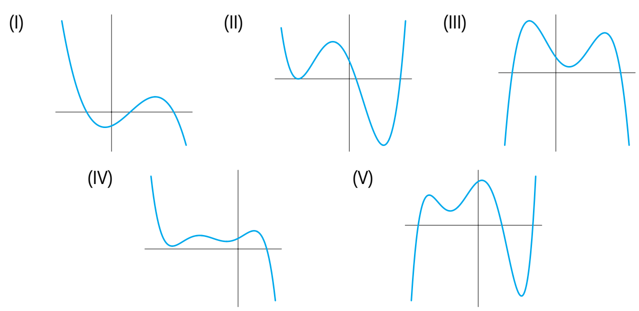


Figure 1.92

36. A sporting goods wholesaler finds that when the price of a product is \$25, the company sells 500 units per week. When the price is \$30, the number sold per week decreases to 460 units.

(a) Find the demand, q, as a function of price, p, assuming that the demand curve is linear.
(b) Use your answer to part (a) to write revenue as a function of price.
(c) Graph the revenue function in part (b). Find the price that maximizes revenue. What is the revenue at this price?

37. A health club has cost and revenue functions given by $C = 10{,}000 + 35q$ and $R = pq$, where q is the number of annual club members and p is the price of a one-year membership. The demand function for the club is $q = 3000 - 20p$.

(a) Use the demand function to write cost and revenue as functions of p.
(b) Graph cost and revenue as a function of p, on the same axes. (Note that price does not go above \$170 and that the annual costs of running the club reach \$120,000.)
(c) Explain why the graph of the revenue function has the shape it does.
(d) For what prices does the club make a profit?
(e) Estimate the annual membership fee that maximizes profit. Mark this point on your graph.

38. According to the National Association of Realtors,[39] the minimum annual gross income, m, in thousands of dollars, needed to obtain a 30-year home loan of A thousand dollars at 9% is given in Table 1.34. Note that the larger the loan, the greater the income needed.

Table 1.34

A	50	75	100	150	200
m	17.242	25.863	34.484	51.726	68.968

Table 1.35

r	8	9	10	11	12
m	31.447	34.484	37.611	40.814	44.084

Of course, not every mortgage is financed at 9%. In fact, excepting for slight variations, mortgage interest rates are generally determined not by individual banks but by the economy as a whole. The minimum annual gross income, m, in thousands of dollars, needed for a home loan of \$100,000 at various interest rates, r, is given in Table 1.35. Note that obtaining a loan at a time when interest rates are high requires a larger income.

(a) Is the size of the loan, A, proportional to the minimum annual gross income, m?
(b) Is the percentage rate, r, proportional to the minimum annual gross income, m?

39. Do the functions $y = x^4$ and $y = x^4 - 15x^2 - 15x$ look similar in the window $-4 \le x \le 4; -100 \le y \le 100$? Comment on the difference between your answer to this question and what you see in Figure 1.91.

40. Find a calculator window in which the graphs of $f(x) = x^3 + 1000x^2 + 1000$ and $g(x) = x^3 - 1000x^2 - 1000$ appear indistinguishable.

[39] "Income needed to get a Mortgage," *The World Almanac 1992*, p. 720.

1.10 PERIODIC FUNCTIONS

What Are Periodic Functions?

Many functions have graphs that oscillate, resembling a wave. Figure 1.93 shows the number of new housing construction starts in the US, 1977-1979, where t is time in quarter-years. Notice that few new homes begin construction during the first quarter of a year (January, February, and March), whereas many new homes are begun in the second quarter (April, May, and June). We expect this pattern of oscillations to continue into the future.

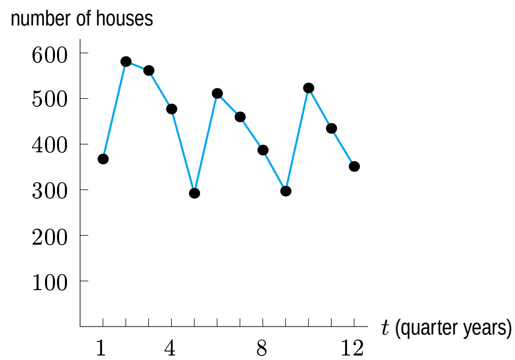


Figure 1.93: New housing construction starts, 1977–1979

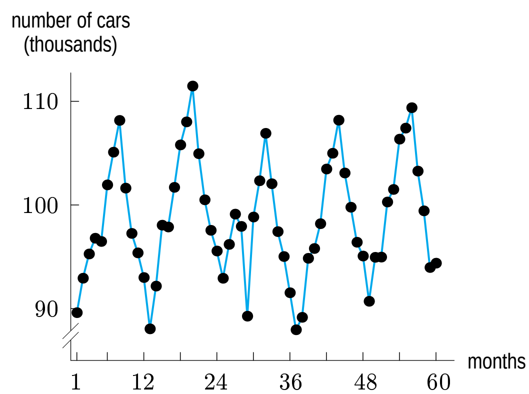


Figure 1.94: Traffic on the Golden Gate Bridge, 1976-80

Let's look at another example. Figure 1.94 is a graph of the number of cars (in thousands) traveling across the Golden Gate Bridge per month, from 1976-1980. Notice that traffic is at its minimum in January of each year (except 1978) and reaches its maximum in August of each year. Again, the graph looks like a wave.

Functions whose values repeat at regular intervals are called *periodic*. Many processes, such as the number of housing starts or the number of cars that cross the bridge, are approximately periodic. The water level in a tidal basin, the blood pressure in a heart, retail sales in the US, and the position of air molecules transmitting a musical note are also all periodic functions of time.

Amplitude and Period

Periodic functions repeat exactly the same cycle forever. If we know one cycle of the graph, we know the entire graph.

For any periodic function of time:
- The **amplitude** is half the difference between its maximum and minimum values.
- The **period** is the time for the function to execute one complete cycle.

Example 1 Estimate the amplitude and period of the new housing starts function shown in Figure 1.93.

Solution Figure 1.93 is not exactly periodic, since the maximum and minimum are not the same for each cycle. Nonetheless, the minimum is about 300, and the maximum is about 550. The difference between them is 250, so the amplitude is about $\frac{1}{2}(250) = 125$ houses.

The wave completes a cycle between $t = 1$ and $t = 5$, so the period is $t = 4$ quarter-years, or one year. The business cycle for new housing construction is one year.

Example 2 Figure 1.95 shows the temperature in an unopened freezer. Estimate the temperature in the freezer at 12:30 and at 2:45.

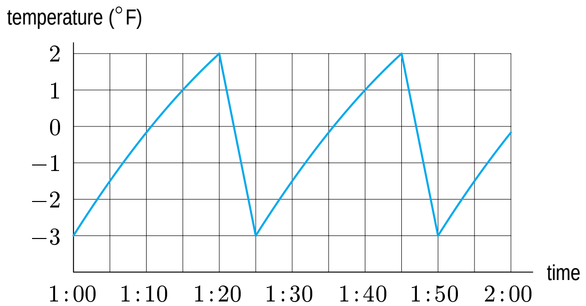


Figure 1.95: Oscillating freezer temperature. Estimate the temperature at 12:30 and 2:45

Solution The maximum and minimum values each occur every 25 minutes, so the period is 25 minutes. The temperature at 12:30 should be the same as at 12:55 and at 1:20, namely, 2°F. Similarly, the temperature at 2:45 should be the same as at 2:20 and 1:55, or about -1.5°F.

The Sine and Cosine

Many periodic functions are represented using the functions called *sine* and *cosine*. The keys for the sine and cosine on a calculator are usually labeled as [sin] and [cos].

Warning: Your calculator can be in either "degree" mode or "radian" mode. For this book, always use "radian" mode.

Graphs of the Sine and Cosine

The graphs of the sine and the cosine functions are periodic; see Figures 1.96 and 1.97. Notice that the graph of the cosine function is the graph of the sine function, shifted $\pi/2$ to the left.

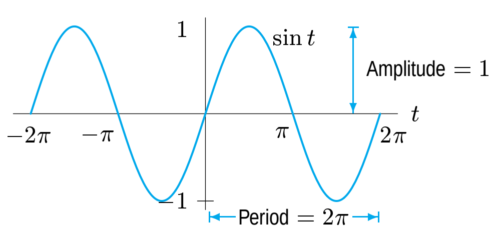


Figure 1.96: Graph of $\sin t$

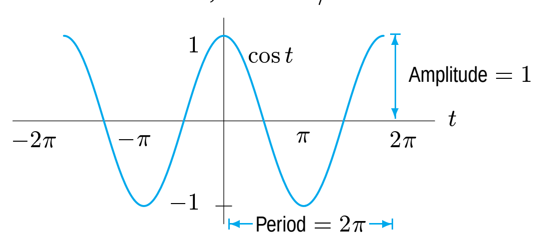


Figure 1.97: Graph of $\cos t$

The maximum and minimum values of $\sin t$ are $+1$ and -1, so the amplitude of the sine function is 1. The graph of $y = \sin t$ completes a cycle between $t = 0$ and $t = 2\pi$; the rest of the graph repeats this portion. The period of the sine function is 2π.

Example 3 Use a graph of $y = 3\sin 2t$ to estimate the amplitude and period of this function.

Solution In Figure 1.98, the waves have a maximum of $+3$ and a minimum of -3, so the amplitude is 3. The graph completes one complete cycle between $t = 0$ and $t = \pi$, so the period is π.

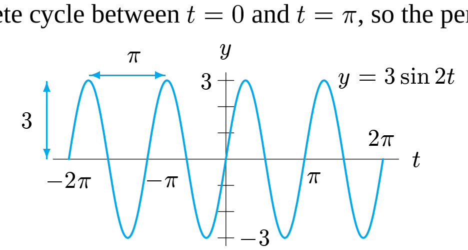


Figure 1.98: The amplitude is 3 and the period is π

Example 4 Explain how the graphs of each of the following functions differ from the graph of $y = \sin t$.

(a) $y = 6\sin t$ (b) $y = 5 + \sin t$ (c) $y = \sin(t + \frac{\pi}{2})$

Solution

(a) The graph of $y = 6\sin t$ is in Figure 1.99. The maximum and minimum values are $+6$ and -6, so the amplitude is 6. This is the graph of $y = \sin t$ stretched vertically by a factor of 6.

(b) The graph of $y = 5 + \sin t$ is in Figure 1.100. The maximum and minimum values of this function are 6 and 4, so the amplitude is $(6 - 4)/2 = 1$. The amplitude (or size of the wave) is the same as for $y = \sin t$, since this is a graph of $y = \sin t$ shifted up 5 units.

(c) The graph of $y = \sin(t + \frac{\pi}{2})$ is in Figure 1.101. This has the same amplitude, namely 1, and period, namely 2π, as the graph of $y = \sin t$. It is the graph of $y = \sin t$ shifted $\pi/2$ units to the left. (In fact, this is the graph of $y = \cos t$.)

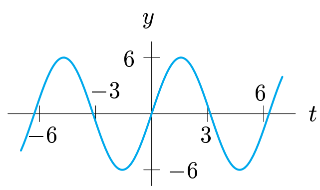


Figure 1.99: Graph of $y = 6\sin t$

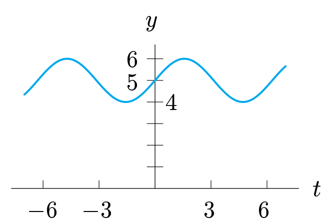


Figure 1.100: Graph of $y = 5 + \sin t$

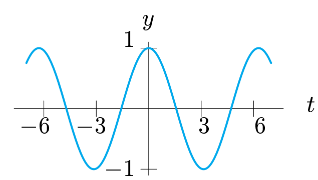


Figure 1.101: Graph of $y = \sin(t + \frac{\pi}{2})$

Families of Curves: The Graph of $y = A\sin(Bt)$

The constants A and B in the expression $y = A\sin(Bt)$ are called *parameters*. We can study families of curves by varying one parameter at a time and studying the result.

Example 5

(a) Graph $y = A\sin t$ for several positive values of A. Describe the effect of A on the graph.

(b) Graph $y = \sin(Bt)$ for several positive values of B. Describe the effect of B on the graph.

Solution

(a) From the graphs of $y = A\sin t$ for $A = 1, 2, 3$ in Figure 1.102, we see that A is the amplitude.

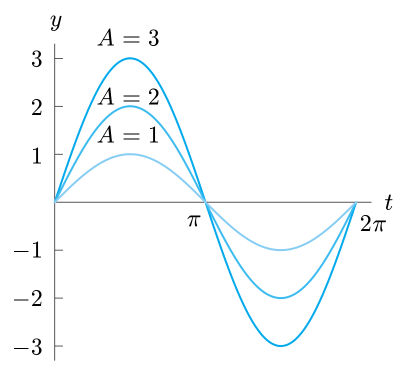


Figure 1.102: Graphs of $y = A\sin t$ with $A = 1,2,3$

(b) The graphs of $y = \sin(Bt)$ for $B = \frac{1}{2}, B = 1$, and $B = 2$ are shown in Figure 1.103. When $B = 1$, the period is 2π; when $B = 2$, the period is π; and when $B = \frac{1}{2}$, the period is 4π. The parameter B affects the period of the function. The graphs suggest that the larger B is, the shorter the period. In fact, the period is $2\pi/B$.

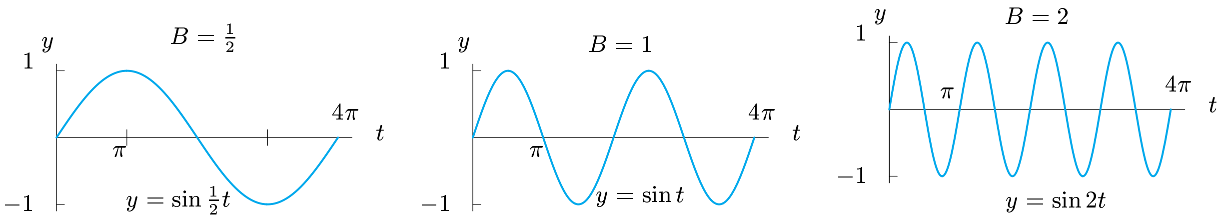
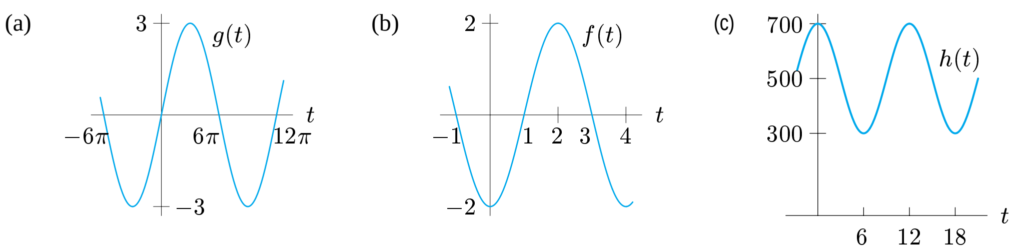


Figure 1.103: Graphs of $y = \sin(Bt)$ with $B = \frac{1}{2}, 1, 2$

In Example 5, the amplitude of $y = A\sin(Bt)$ was determined by the parameter A, and the period was determined by the parameter B. In general, we have

> The functions $y = A\sin(Bt) + C$ and $y = A\cos(Bt) + C$ are periodic with
>
> $$\text{Amplitude} = |A|, \qquad \text{Period} = \frac{2\pi}{|B|}, \qquad \text{Vertical shift} = C$$

Example 6 Find possible formulas for the following periodic functions.



Solution

(a) This function looks like a sine function of amplitude 3, so $g(t) = 3\sin(Bt)$. Since the function executes one full oscillation between $t = 0$ and $t = 12\pi$, when t changes by 12π, the quantity Bt changes by 2π. This means $B \cdot 12\pi = 2\pi$, so $B = 1/6$. Therefore, $g(t) = 3\sin(t/6)$ has the graph shown.

(b) This function looks like an upside down cosine function with amplitude 2, so $f(t) = -2\cos(Bt)$. The function completes one oscillation between $t = 0$ and $t = 4$. Thus, when t changes by 4, the quantity Bt changes by 2π, so $B \cdot 4 = 2\pi$, or $B = \pi/2$. Therefore, $f(t) = -2\cos(\pi t/2)$ has the graph shown.

(c) This function looks like a cosine function. The maximum is 700 and the minimum is 300, so the amplitude is $\frac{1}{2}(700 - 300) = 200$. The height halfway between the maximum and minimum is 500, so the cosine curve has been shifted up 500 units, so $h(t) = 500 + 200\cos(Bt)$. The period is 12, so $B \cdot 12 = 2\pi$. Thus, $B = \pi/6$. The function $h(t) = 500 + 200\cos(\pi t/6)$ has the graph shown.

Example 7 On February 10, 1990, high tide in Boston was at midnight. The height of the water in the harbor is a periodic function, since it oscillates between high and low tide. If t is in hours since midnight, the height (in feet) is approximated by the formula

$$y = 5 + 4.9\cos\left(\frac{\pi}{6}t\right).$$

(a) Graph this function from $t = 0$ to $t = 24$.
(b) What was the water level at high tide?
(c) When was low tide, and what was the water level at that time?
(d) What is the period of this function, and what does it represent in terms of tides?
(e) What is the amplitude of this function, and what does it represent in terms of tides?

Solution

(a) See Figure 1.104.
(b) The water level at high tide was 9.9 feet (given by the y-intercept on the graph).
(c) Low tide occurs at $t = 6$ (6 am) and at $t = 18$ (6 pm). The water level at this time is 0.1 feet.
(d) The period is 12 hours and represents the interval between successive high tides or successive low tides. Of course, there is something wrong with the assumption in the model that the period is 12 hours. If so, the high tide would always be at noon or midnight, instead of progressing slowly through the day, as it in fact does. The interval between successive high tides actually averages about 12 hours 24 minutes, which could be taken into account in a more precise mathematical model.
(e) The maximum is 9.9, and the minimum is 0.1, so the amplitude is $(9.9 - 0.1)/2$, which is 4.9 feet. This represents half the difference between the depths at high and low tide.

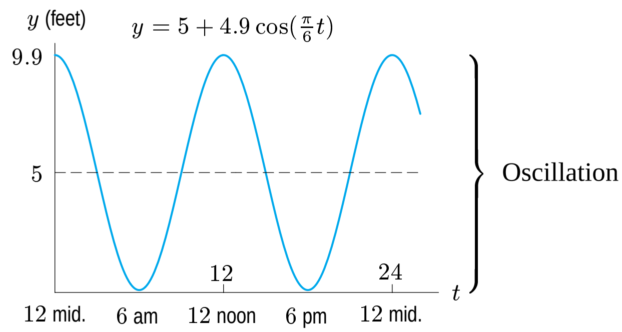


Figure 1.104: Graph of the function approximating the depth of the water in Boston on February 10, 1990

Problems for Section 1.10

1. Sketch a possible graph of sales of sunscreen in the northeastern US over a 3-year period, as a function of months since January 1 of the first year. Explain why your graph should be periodic. What is the period?

2. During the summer of 1988, one of the hottest on record in the Midwest, a graduate student in environmental science studied the temperature fluctuations of a river. Figure 1.105 shows the temperature of the river (in °C) every hour, with 0 being midnight of the first day.

   (a) Explain why a periodic function could be used to model these data.
   (b) Approximately when does the maximum occur? The minimum? Why does this make sense?
   (c) What is the period for these data? What is the amplitude?

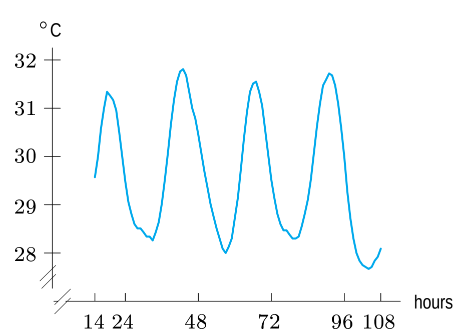


Figure 1.105

3. A population of animals varies periodically between a low of 700 on January 1 and a high of 900 on July 1. Graph the population against time.

4. Use the equation in Example 7 on page 66 to estimate the water level in Boston Harbor at 3:00 am, 4:00 am, and 5:00 pm on February 10, 1990.

5. Values of a function are given in the following table. Explain why this function appears to be periodic. Approximately what are the period and amplitude of the function? Assuming that the function is periodic, estimate its value at $t = 15$, at $t = 75$, and at $t = 135$.

t	20	25	30	35	40	45	50	55	60
$f(t)$	1.8	1.4	1.7	2.3	2.0	1.8	1.4	1.7	2.3

For Problems 6–11, sketch graphs of the functions. What are their amplitudes and periods?

6. $y = 3\sin x$
7. $y = 3\sin 2x$
8. $y = -3\sin 2\theta$
9. $y = 4\cos 2x$
10. $y = 4\cos(\frac{1}{2}t)$
11. $y = 5 - \sin 2t$

12. A person breathes in and out every three seconds. The volume of air in the person's lungs varies between a minimum of 2 liters and a maximum of 4 liters. Which of the following is the best formula for the volume of air in the person's lungs as a function of time?

(a) $y = 2 + 2\sin\left(\frac{\pi}{3}t\right)$ **(b)** $y = 3 + \sin\left(\frac{2\pi}{3}t\right)$

(c) $y = 2 + 2\sin\left(\frac{2\pi}{3}t\right)$ **(d)** $y = 3 + \sin\left(\frac{\pi}{3}t\right)$

13. Figure 1.106 shows the levels of the hormones estrogen and progesterone during the monthly ovarian cycles in females.[40] Is the level of both hormones periodic? What is the period in each case? Approximately when in the monthly cycle is estrogen at a peak? Approximately when in the monthly cycle is progesterone at a peak?

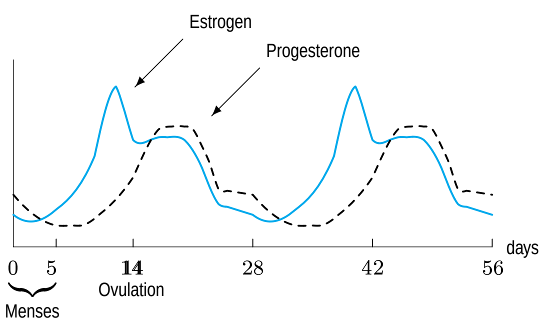


Figure 1.106

14. The following table shows values of a periodic function $f(x)$. The maximum value attained by the function is 5.

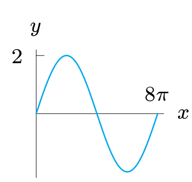
(a) What is the amplitude of this function?
(b) What is the period of this function?
(c) Find a formula for this periodic function.

x	0	2	4	6	8	10	12
$f(x)$	5	0	-5	0	5	0	-5

15. When a car's engine makes less than about 200 revolutions per minute, it stalls. What is the period of the rotation of the engine when it is about to stall?

For Problems 16–27, find a possible formula for each graph.

16.



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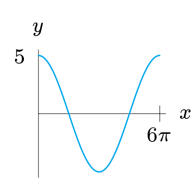

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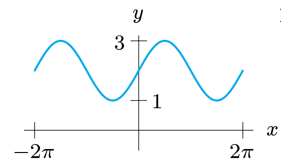

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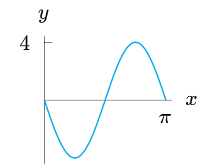

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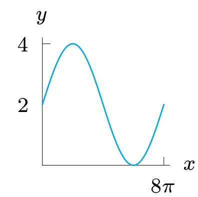

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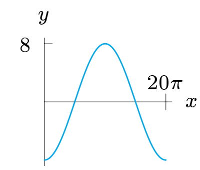

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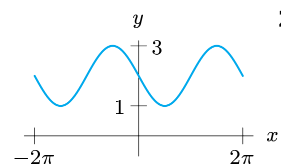

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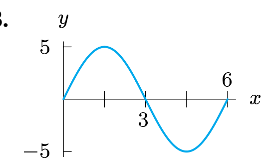

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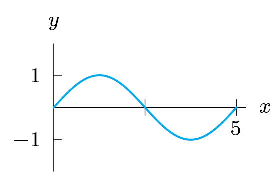

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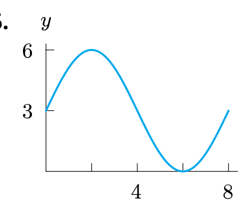

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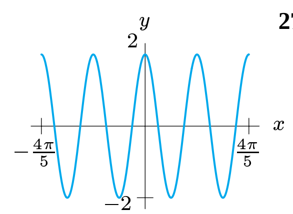

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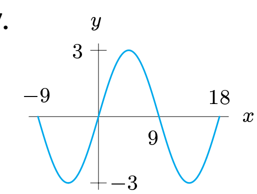


28. Most breeding birds in the northeast US migrate elsewhere during the winter. The number of bird species in an Ohio forest preserve oscillates between a high of 28 in June and a low of 10 in December.[41]

(a) Graph the number of bird species in this preserve as a function of t, the number of months since June. Include at least three years on your graph.
(b) What are the amplitude and period of this function?
(c) Find a formula for the number of bird species, B, as a function of the number of months, t since June.

29. The desert temperature, H, oscillates daily between 40° F at 5 am and 80° F at 5 pm. Write a possible formula for H in terms of t, measured in hours from 5 am.

[40] Robert M. Julien, *A Primer of Drug Action*, Seventh Edition, p. 360, (W. H. Freeman and Co., New York: 1995).
[41] Rosenzweig, M.L., *Species Diversity in Space and Time*, p. 71, (Cambridge University Press, 1995).

30. Figure 1.107 shows quarterly beer production during the period 1990 to 1993. Quarter 1 reflects production during the first three months of the year, etc.

(a) Explain why a periodic function should be used to model these data.
(b) Approximately when does the maximum occur? The minimum? Why does this make sense?
(c) What are the period and amplitude for these data?

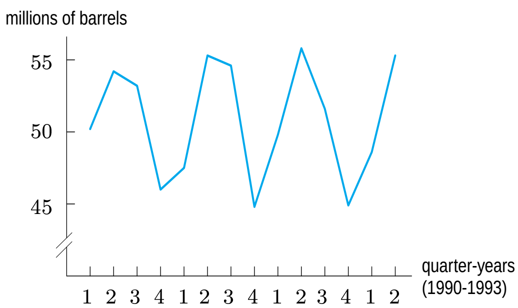


Figure 1.107

31. The Bay of Fundy in Canada has the largest tides in the world. The difference between low and high water levels is 15 meters (nearly 50 feet). At a particular point the depth of the water, y meters, is given as a function of time, t, in hours since midnight by

$$y = D + A\cos\left(B(t - C)\right).$$

(a) What is the physical meaning of D?
(b) What is the value of A?
(c) What is the value of B? Assume the time between successive high tides is 12.4 hours.
(d) What is the physical meaning of C?

32. In a US household, the voltage in volts in an electric outlet is given by

$$V = 156\sin(120\pi t),$$

where t is in seconds. However, in a European house, the voltage is given (in the same units) by

$$V = 339\sin(100\pi t).$$

Compare the voltages in the two regions, considering the maximum voltage and number of cycles (oscillations) per second.

CHAPTER SUMMARY

- **Function terminology**
  Domain/range, increasing/decreasing, concavity, intercepts.
- **Linear functions**
  Slope, y-intercept. Grow by equal amounts in equal times.
- **Economic applications**
  Cost, revenue, and profit functions. Supply and demand curves, equilibrium point. Depreciation function. Budget constraint. Present and future value.
- **Change, average rate of change, relative rate of change**
- **Exponential functions**
  Exponential growth and decay, growth rate, the number e, continuous growth rate, doubling time, half life, compound interest. Grow by equal percentages in equal times.
- **The natural logarithm function**
- **New functions from old**
  Composition, shifting, stretching.
- **Power functions and proportionality**
- **Polynomials**
- **Periodic functions**
  Sine, cosine, amplitude, period.

REVIEW PROBLEMS FOR CHAPTER ONE

1. Let $W = f(t)$ represent wheat production in Argentina, in millions of metric tons, with t is in years since 1990. Interpret the statement $f(9) = 14$ in terms of wheat production.

2. The yield, Y, of an apple orchard (in bushels) as a function of the amount, a, of fertilizer (in pounds) used on the orchard is shown in Figure 1.108.

(a) Describe the effect of the amount of fertilizer on the yield of the orchard.
(b) What is the vertical intercept? Explain what it means in terms of apples and fertilizer.
(c) What is the horizontal intercept? Explain what it means in terms of apples and fertilizer.
(d) What is the range of this function for $0 \le a \le 80$?
(e) Is the function increasing or decreasing at $a = 60$?
(f) Is the graph concave up or down near $a = 40$?

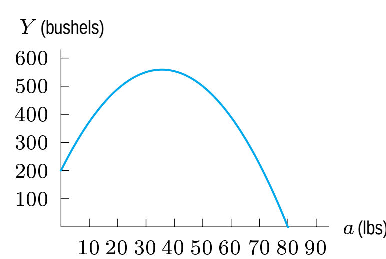


Figure 1.108

3. Let $y = f(x) = 3x - 5$.

(a) What is $f(1)$?
(b) Find the value of y when x is 5.
(c) Find the value of x when y is 4.
(d) Find the average rate of change of f between $x = 2$ and $x = 4$.

4. Table 1.36 gives sales of Pepsico, which operates two major businesses: beverages (including Pepsi) and snack foods.[42]

(a) Find the change in sales between 1991 and 1993.
(b) Find the average rate of change in sales between 1991 and 1993. Give units and interpret your answer.

Table 1.36 *Pepsico sales, in millions of dollars*

Year	1991	1992	1993	1994	1995	1996	1997
Sales	19608	21970	25021	28472	30421	31645	21000

5. Table 1.37 gives the revenues, R, of General Motors, the world's largest auto manufacturer.[43]

(a) Find the change in revenues between 1989 and 1997.
(b) Find the average rate of change in revenues between 1989 and 1997. Give units and interpret your answer.
(c) From 1987 to 1997, were there any one-year intervals during which the average rate of change was negative? If so, which?

Table 1.37

Year	R ($ m)	Year	R ($ m)	Year	R ($ m)
1987	101,782	1991	123,056	1995	168,829
1988	120,388	1992	132,429	1996	164,069
1989	123,212	1993	138,220	1997	172,000
1990	122!021	1994	154,951		

6. Suppose that $q = f(p)$ is the demand curve for a product, where p is the selling price in dollars and q is the quantity sold at that price.

(a) What does the statement $f(12) = 60$ tell you about demand for this product?
(b) Do you expect this function to be increasing or decreasing? Why?

Find the equation of the line passing through the points in Problems 7–10.

7. $(0, -1)$ and $(2, 3)$

8. $(-1, 3)$ and $(2, 2)$

9. $(0, 2)$ and $(2, 2)$

10. $(-1, 3)$ and $(-1, 4)$

11. Values of $F(t)$, $G(t)$, and $H(t)$ are in the Table 1.38. Which graph is concave up and which is concave down? Which function is linear?

Table 1.38

t	$F(t)$	$G(t)$	$H(t)$
10	15	15	15
20	22	18	17
30	28	21	20
40	33	24	24
50	37	27	29
60	40	30	35

12. The gross national product, G, of Iceland was 6 billion dollars in 1998. Give a formula for G (in billions of dollars) t years after 1998 if G increases by

(a) 3% per year **(b)** 0.2 billion dollars per year

13. A product costs $80 today. How much will the product cost in t days if the price is reduced by

(a) $4 a day **(b)** 5% a day

14. Table 1.39 gives values for three functions. Which functions could be linear? Which could be exponential? Which are neither? For those which could be linear or exponential, give a possible formula for the function.

Table 1.39

x	$f(x)$	$g(x)$	$h(x)$
0	25	30.8	15,000
1	20	27.6	9,000
2	14	24.4	5,400
3	7	21.2	3,240

15. A movie theater has fixed costs of $5000 per day and variable costs averaging $2 per customer. The theater charges $7 per ticket.

(a) How many customers per day does the theater need in order to make a profit?
(b) Find the cost and revenue functions and graph them on the same axes. Mark the break-even point.

16. Table 1.40 shows the concentration, c, of creatinine in the bloodstream of a dog.[44]

(a) Including units, find the average rate at which the concentration is changing between the
(i) 6^{th} and 8^{th} minutes. (ii) 8^{th} and 10^{th} minutes.
(b) Explain the sign and relative magnitudes of your results in terms of creatinine.

Table 1.40

t (minutes)	2	4	6	8	10
c (mg/ml)	0.439	0.383	0.336	0.298	0.266

[42] From *Value Line Investment Survey*, November 14, 1997, (New York: Value Line Publishing, Inc.) p. 1547.
[43] From *Value Line Investment Survey*, December 12, 1997, (New York: Value Line Publishing, Inc.) p. 105.
[44] From Cullen, M.R., *Linear Models in Biology*, (Chichester: Ellis Horwood, 1985).

17. For tax purposes, you may have to report the value of your assets, such as cars or refrigerators. The value you report drops with time. "Straight-line depreciation" assumes that the value is a linear function of time. If a $950 refrigerator depreciates completely in seven years, find a formula for its value as a function of time.

18. The six graphs in Figure 1.109 show frequently observed patterns of age-specific cancer incidence rates, in number of cases per 1000 people, as a function of age.[45] The scales on the vertical axes are equal.

(a) For each of the six graphs, write a sentence explaining the effect of age on the cancer rate.
(b) Which graph shows a relatively high incidence rate for children? Suggest a type of cancer that behaves this way.
(c) Which graph shows a brief decrease in the incidence rate at around age 50? Suggest a type of cancer that might behave this way.
(d) Which graph or graphs might represent a cancer that is caused by toxins which build up in the body over time? (For example, lung cancer.) Explain.

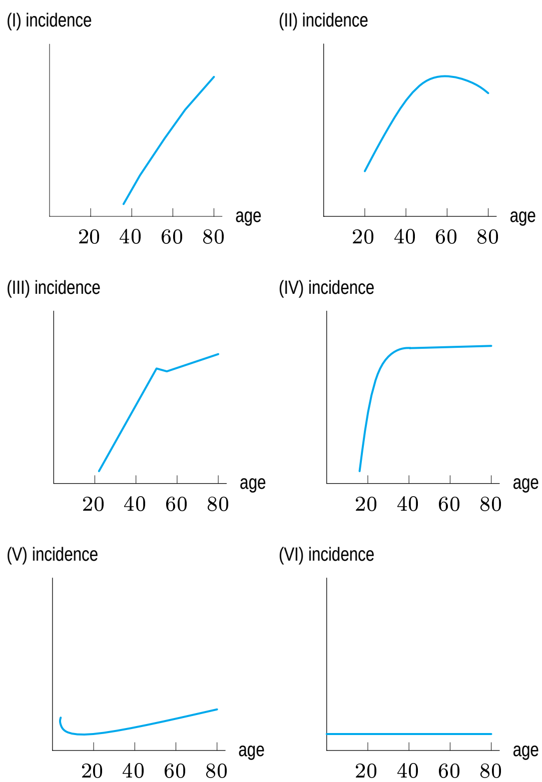


Figure 1.109

19. Figure 1.110 shows the age-adjusted death rates from different types of cancer among US males.[46]

(a) Discuss how the death rate has changed for the different types of cancers.
(b) For which type of cancer has the average rate of change between 1930 and 1967 been the largest? Estimate the average rate of change for this cancer type. Interpret your answer.
(c) For which type of cancer has the average rate of change between 1930 and 1967 been the most negative? Estimate the average rate of change for this cancer type. Interpret your answer.

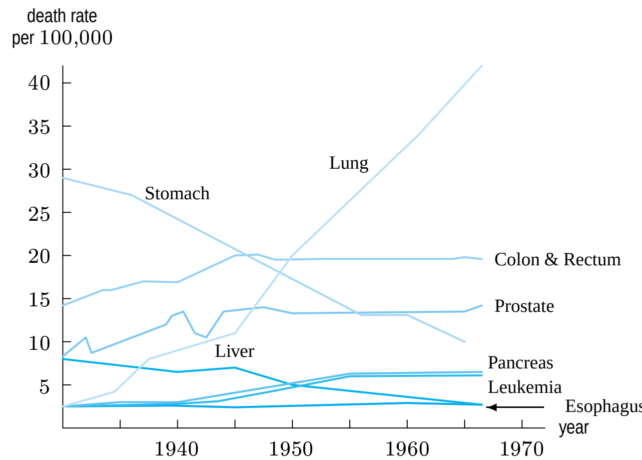
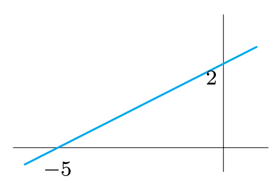


Figure 1.110

Find possible formulas for the graphs in Problems 20–25.

20.



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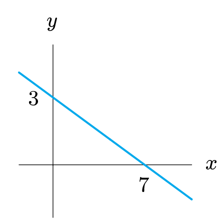

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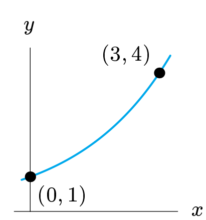

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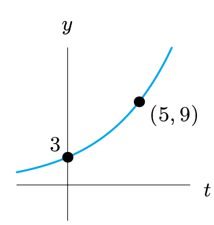

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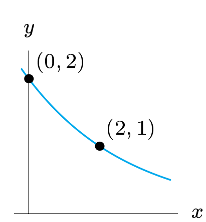

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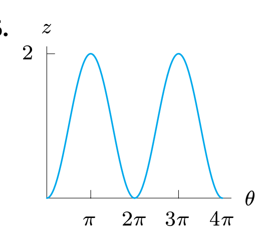


[45] Abraham M. Lilienfeld, *Foundations of Epidemiology*, p. 155, (New York: Oxford University Press, 1976).
[46] Abraham M. Lilienfeld, *Foundations of Epidemiology*, p. 67, (New York; Oxford University Press, 1976).

For Problems 26–29, solve for x using logs.

26. $3^x = 11$

27. $20 = 50(1.04)^x$

28. $e^{5x} = 100$

29. $25e^{3x} = 10$

30. If you need $20,000 in your bank account in 6 years, how much must be deposited now? The interest rate is 10%, compounded continuously.

31. If a bank pays 6% per year interest compounded continuously, how long does it take for the balance in an account to double?

32. Worldwide, wind energy[47] generating capacity, W, was 1930 megawatts in 1990 and 18,100 megawatts in 2000.

(a) Use the values given to write W, in megawatts, as a linear function of t, the number of years since 1990.
(b) Use the values given to write W as an exponential function of t.
(c) Graph the functions found in parts (a) and (b) on the same axes. Label the given values.

33. The concentration, C, of carbon dioxide in the atmosphere was 338.5 parts per million (ppm) in 1980 was 369.4 ppm in 2000.[48] Find a formula for the concentration C in t years after 1980 if:

(a) C is a linear function of t. What is the absolute yearly rate of increase in carbon dioxide concentration?
(b) C is an exponential function of t. What is the relative yearly rate of increase in carbon dioxide concentration?

34. Pregnant women metabolize some drugs at a slower rate than the rest of the population. The half-life of caffeine is about 4 hours for most people. In pregnant women, it is 10 hours.[49] (This is important because caffeine, like all psychoactive drugs, crosses the placenta to the fetus.) If a pregnant woman and her husband each have a cup of coffee containing 100 mg of caffeine at 8 am, how much caffeine does each have left in the body at 10 pm?

35. The half-life of radioactive strontium-90 is 29 years. In 1960, radioactive strontium-90 was released into the atmosphere during testing of nuclear weapons, and was absorbed into people's bones. How many years does it take until only 10% of the original amount absorbed remains?

36. If the world's population increased exponentially from 2.564 billion in 1950 to 4.478 billion in 1980 and continued to increase at the same percentage rate between 1980 and 1991, calculate what the world's population would have been in 1991. How does this compare to the actual population of 5.423 billion, and what conclusions, if any, can you draw?

37. Let $f(x) = x^2 + 1$ and $g(x) = \ln x$. Find

(a) $f(g(x))$ **(b)** $g(f(x))$ **(c)** $f(f(x))$

38. Let $f(x) = 2x + 3$ and $g(x) = 5x^2$. Find

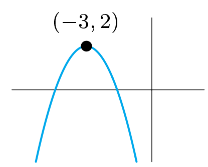
(a) $f(g(x))$ **(b)** $g(f(x))$ **(c)** $f(f(x))$

39. The blood mass of a mammal is proportional to its body mass. A rhinoceros with body mass 3000 kilograms has blood mass of 150 kilograms. Find a formula for the blood mass of a mammal as a function of the body mass and estimate the blood mass of a human with body mass 70 kilograms.

40. The number of species of lizards, N, found on an island off Baja California is proportional to the fourth root of the area, A, of the island.[50] Write a formula for N as a function of A. Graph this function. Is it increasing or decreasing? Is the graph concave up or concave down? What does this tell you about lizards and island area?

In Problems 41–42, use shifts of a power function to find a possible formula for the graph.

41.



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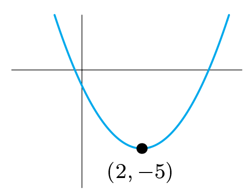


43. A company produces and sells shirts. The fixed costs are $7000 and the variable costs are $5 per shirt.

(a) Shirts are sold for $12 each. Find cost and revenue as functions of the quantity of shirts, q.
(b) The company is considering changing the selling price of the shirts. Demand is $q = 2000 - 40p$, where p is price in dollars and q is the number of shirts. What quantity is sold at the current price of $12? What profit is realized at this price?
(c) Use the demand equation to write cost and revenue as functions of the price, p. Then write profit as a function of price.
(d) Graph profit against price. Find the price that maximizes profits. What is this profit?

[47] The Worldwatch Institute, *Vital Signs* 2001, p. 45, (New York: W.W. Norton, 2001).
[48] The Worldwatch Institute, *Vital Signs* 2001, p. 53, (New York: W.W. Norton, 2001).
[49] From Robert M. Julien, *A Primer of Drug Action*, 7th ed., p. 159, (New York: W. H. Freeman, 1995).
[50] Rosenzweig, M.L., *Species Diversity in Space and Time*, p. 143, (Cambridge: Cambridge University Press, 1995).

44. Figure 1.111 shows supply and demand curves.

(a) What is the equilibrium price for this product? At this price, what quantity is produced?

(b) Choose a price above the equilibrium price—for example, $p = 300$. At this price, how many items are suppliers willing to produce? How many items do consumers want to buy? Use your answers to these questions to explain why, if prices are above the equilibrium price, the market tends to push prices lower (towards the equilibrium).

(c) Now choose a price below the equilibrium price—for example, $p = 200$. At this price, how many items are suppliers willing to produce? How many items do consumers want to buy? Use your answers to these questions to explain why, if prices are below the equilibrium price, the market tends to push prices higher (towards the equilibrium).

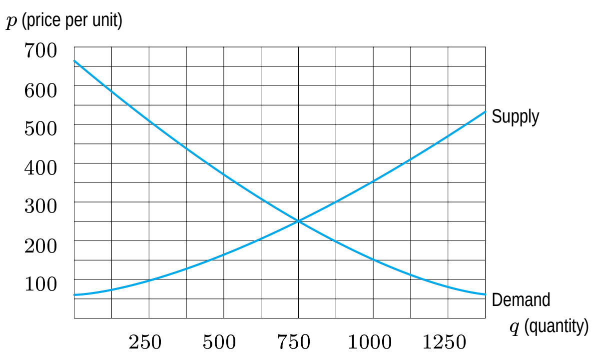


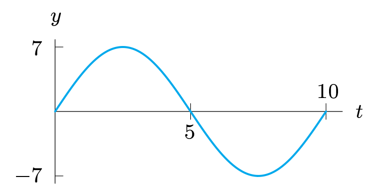
Figure 1.111

45. You win \$38,000 in the state lottery to be paid in two installments—\$19,000 now and \$19,000 one year from now. A friend offers you \$36,000 in return for your two lottery payments. Instead of accepting your friend's offer, you take out a one-year loan at an interest rate of 8.25% per year, compounded annually. The loan will be paid back by a single payment of \$19,000 (your second lottery check) at the end of the year. Which is better, your friend's offer or the loan?

46. You are considering whether to buy or lease a machine whose purchase price is \$12,000. Taxes on the machine will be \$580 due in one year, \$464 due in two years, and \$290 due in three years. If you buy the machine, you expect to be able to sell it after three years for \$5,000. If you lease the machine for three years, you make an initial payment of \$2650 and then three payments of \$2650 at the end of each of the next three years. The leasing company will pay the taxes. The interest rate is 7.75% per year, compounded annually. Should you buy or lease the machine? Explain.

Find a possible formula for the functions in Problems 47–48.

47.



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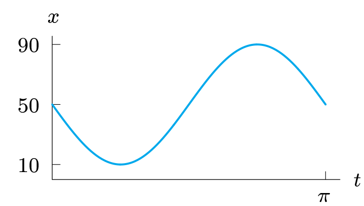

49. Figure 1.112 shows the number of reported[51] cases of mumps by month, in the US, for 1972–73.

(a) Find the period and amplitude of this function, and interpret each in terms of mumps.

(b) Predict the number of cases of mumps 30 months and 45 months after January 1, 1972.

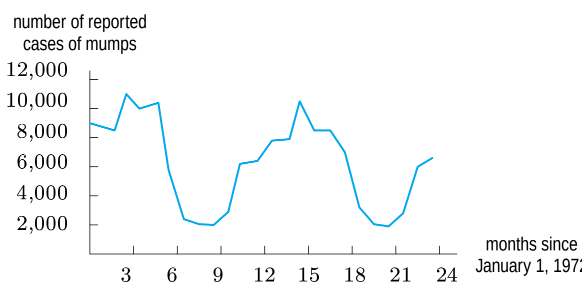

Figure 1.112

50. The depth of water in a tank oscillates once every 6 hours. If the smallest depth is 5.5 feet and the largest depth is 8.5 feet, find a possible formula for the depth in terms of time in hours.

51. Table 1.41 gives values for $g(t)$, a periodic function.

(a) Estimate the period and amplitude for this function.

(b) Estimate $g(34)$ and $g(60)$.

Table 1.41

t	0	2	4	6	8	10	12	14
$g(t)$	14	19	17	15	13	11	14	19
t	16	18	20	22	24	26	28	
$g(t)$	17	15	13	11	14	19	17	

[51] Center for Disease Control, 1974, *Reported Morbidity and Mortality in the United States 1973*, Vol. 22, No. 53.

PROJECTS FOR CHAPTER ONE

1. Compound Interest

The newspaper article below is from *The New York Times*, May 27, 1990. Fill in the three blanks. (For the first blank, assume that daily compounding is essentially the same as continuous compounding. For the last blank, assume the interest has been compounded yearly, and give your answer in dollars. Ignore the occurrence of leap years.)

213 Years After Loan, Uncle Sam Is Dunned

By LISA BELKIN

Special to The New York Times

SAN ANTONIO, May 26 — More than 200 years ago, a wealthy Pennsylvania merchant named Jacob DeHaven lent $450,000 to the Continental Congress to rescue the troops at Valley Forge. That loan was apparently never repaid.

So Mr. DeHaven's descendants are taking the United States Government to court to collect what they believe they are owed. The total: ____ in today's dollars if the interest is compounded daily at 6 percent, the going rate at the time. If compounded yearly, the bill is only ____.

Family Is Flexible

The descendants say that they are willing to be flexible about the amount of a settlement and that they might even accept a heartfelt thank you or perhaps a DeHaven statue. But they also note that interest is accumulating at ____ a second.

2. Population Center of the US

Since the opening up of the West, the US population has moved westward. To observe this, we look at the "population center" of the US, which is the point at which the country would balance if it were a flat plate with no weight, and every person had equal weight. In 1790 the population center was east of Baltimore, Maryland. It has been moving westward ever since, and in 1990 it crossed the Mississippi river to Steelville, Missouri (southwest of St. Louis). During the second half of the 20^{th} century, the population center has moved about 50 miles west every 10 years.

(a) Let us measure position westward from Steelville along the line running through Baltimore. For the years since 1990, express the approximate position of the population center as a function of time in years from 1990.

(b) The distance from Baltimore to Steelville is a bit over 700 miles. Could the population center have been moving at roughly the same rate for the last two centuries?

(c) Could the function in part (a) continue to apply for the next four centuries? Why or why not? [Hint: You may want to look at a map. Note that distances are in air miles and are not driving distances.]

FOCUS ON MODELING

FITTING FORMULAS TO DATA

In this section we see how the formulas that are used in a mathematical model can be developed. Some of the formulas we use are exact. However, many formulas we use are approximations, often constructed from data.

Fitting a Linear Function To Data

A company wants to understand the relationship between the amount spent on advertising, a, and total sales, S. The data they collect might look like that found in Table 1.42.

Table 1.42 *Advertising and sales: Linear relationship*

a (advertising in \$1000s)	3	4	5	6
S (sales in \$1000s)	100	120	140	160

The data in Table 1.42 are linear, so a formula fits it exactly. The slope of the line is 20 and we can determine that the vertical intercept is 40, so the line is

$$S = 40 + 20a.$$

Now suppose that the company collected the data in Table 1.43. This time the data are not linear. In general, it is difficult to find a formula to fit data exactly. We must be satisfied with a formula that is a good approximation to the data.

Table 1.43 *Advertising and sales: Nonlinear relationship*

a (advertising in \$1000s)	3	4	5	6
S (sales in \$1000s)	105	117	141	152

Figure 1.113 shows the data in Table 1.43. Since the relationship is nearly, though not exactly, linear, it is well approximated by a line. Figure 1.114 shows the line $S = 40 + 20a$ and the data.

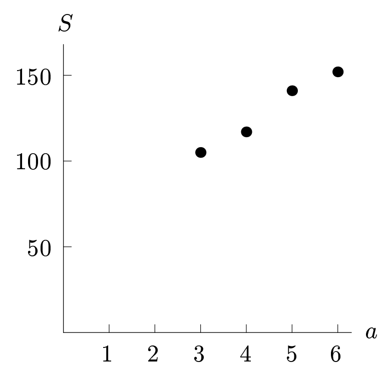

Figure 1.113: The sales data from Table 1.43

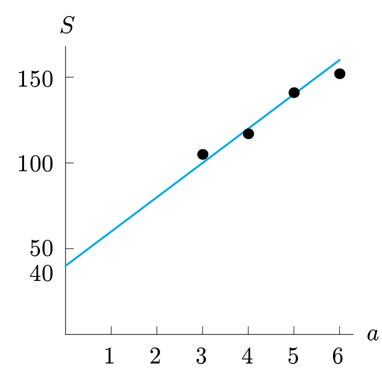

Figure 1.114: The line $S = 40 + 20a$ and the data from Table 1.43

The Regression Line

Is there a line that fits the data better than the one in Figure 1.114? If so, how do we find it? The process of fitting a line to a set of data is called *linear regression* and the line of best fit is called the *regression line*. (See page 77 for a discussion of what "best fit" means.) Many calculators and computer programs calculate the regression line from the data points. Alternatively, the regression line can be estimated by plotting the points on paper and fitting a line "by eye." In Chapter 9, we derive the formulas for the regression line. For the data in Table 1.43, the regression line is

$$S = 54.5 + 16.5a.$$

This line is graphed with the data in Figure 1.115.

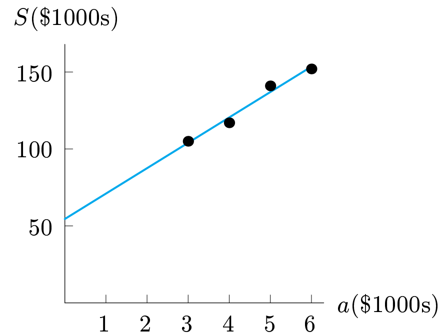

Figure 1.115: The regression line $S = 54.5 + 16.5a$ and the data from Table 1.43

Using the Regression Line to Make Predictions

We can use the formula for sales as a function of advertising to make predictions. For example, to predict total sales if $3500 is spent on advertising, substitute $a = 3.5$ into the regression line:

$$S = 54.5 + 16.5(3.5) = 112.25.$$

The regression line predicts sales of $112,250. To see that this is reasonable, compare it to the entries in Table 1.43. When $a = 3$, we have $S = 105$, and when $a = 4$, we have $S = 117$. Predicted sales of $S = 112.25$ when $a = 3.5$ makes sense because it falls between 105 and 117. See Figure 1.116. Of course, if we spent $3500 on advertising, sales would probably not be exactly $112,250. The regression equation allows us to make predictions, but does not provide exact results.

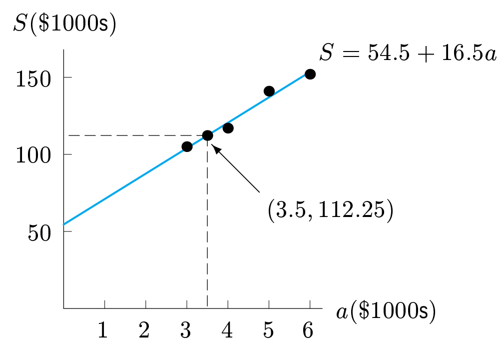

Figure 1.116: Predicting sales when spending $3,500 on advertising

Example 1 Predict total sales given advertising expenditures of $4800 and $10,000.

Solution When $4800 is spent on advertising, $a = 4.8$, so

$$S = 54.5 + 16.5(4.8) = 133.7.$$

Sales are predicted to be $133,700. When $10,000 is spent on advertising, $a = 10$, so

$$S = 54.5 + 16.5(10) = 219.5.$$

Sales are predicted to be $219,500.

Consider the two predictions made in Example 1 at $a = 4.8$ and $a = 10$. We have more confidence in the accuracy of the prediction when $a = 4.8$, because we are *interpolating* within an interval we already know something about. The prediction for $a = 10$ is less reliable, because we are *extrapolating* outside the interval defined by the data values in Table 1.43. In general, interpolation is safer than extrapolation.

Interpreting the Slope of the Regression Line

The slope of a linear function is the change in the dependent variable divided by the change in the independent variable. For the sales and advertising regression line, the slope is 16.5. This tells us that S increases by about 16.5 whenever a increases by 1. If advertising expenses increase by $\$1000$, sales increase by about $\$16{,}500$. In general, the slope tells us the expected change in the dependent variable for a unit change in the independent variable.

Example 2 A company collects the data in Table 1.44. Find the regression line and interpret its slope.

Table 1.44 *Cost to produce various quantities of a product*

q (quantity in units)	25	50	75	100	125
C (cost in dollars)	500	625	689	742	893

Solution A calculator or computer gives the regression line

$$C = 418.9 + 3.612q.$$

The regression line fits the data well; see Figure 1.117. The slope of the line is 3.612, which means that the cost increases about $\$3.612$ for every additional unit produced; that is, the marginal cost is $\$3.612$ per unit.

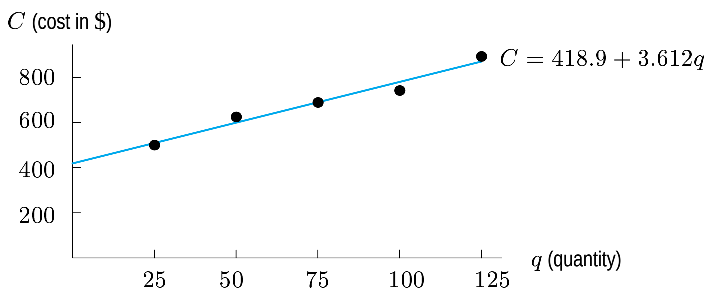

Figure 1.117: The regression line for the data in Table 1.44

How Regression Works: What "Best Fit" Means

Figure 1.118 illustrates how a line is fitted to a set of data. We assume that the value of y is in some way related to the value of x, although other factors could influence y as well. Thus, we assume that we can pick the value of x exactly but that the value of y may be only partially determined by this x-value.

A calculator or computer finds the line that minimizes the sum of the squares of the vertical distances between the data points and the line. See Figure 1.118. The regression line is also called a *least-squares line*, or the *line of best fit*.

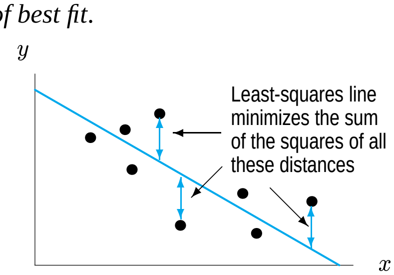

Figure 1.118: Data and the corresponding least-squares regression line

Regression When the Relationship Is Not Linear

Table 1.45 shows the population of the US (in millions) from 1790 to 1860. These points are plotted in Figure 1.119. Do the data look linear? Not really. It appears to make more sense to fit an exponential function than a linear function to this data. Finding the exponential function of best fit is called *exponential regression*. One algorithm used by a calculator or computer gives the exponential function that fits the data as

$$P = 3.9(1.03)^t,$$

where P is the US population in millions and t is years since 1790. Other algorithms may give different answers. See Figure 1.120.

Since the base of this exponential function is 1.03, the US population was increasing at the rate of about 3% per year between 1790 and 1860. Is it reasonable to expect the population to continue to increase at this rate? It turns out that this exponential model does not fit the population of the US well beyond 1860. In Section 4.7, we see another function that is used to model the US population.

Table 1.45 *US Population in millions, 1790–1860*

Year	1790	1800	1810	1820	1830	1840	1850	1860
Population	3.9	5.3	7.2	9.6	12.9	17.1	23.2	31.4

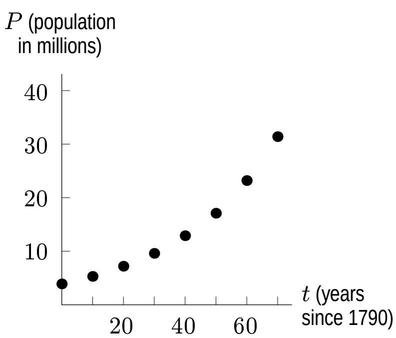

Figure 1.119: US Population 1790–1860

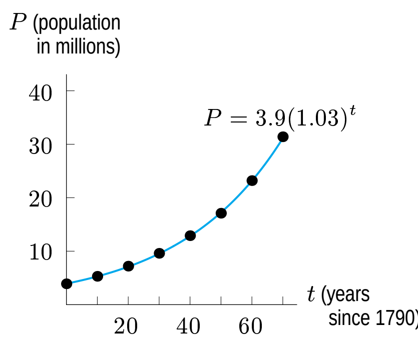

Figure 1.120: US population and an exponential regression function

Calculators and computers can do linear regression, exponential regression, logarithmic regression, quadratic regression, and more. To fit a formula to a set of data, the first step is to graph the data and identify the appropriate family of functions.

Example 3 The average fuel efficiency (miles per gallon of gasoline) of US automobiles declined until the 1960s and then started to rise as manufacturers made cars more fuel efficient.[52] See Table 1.46.

(a) Plot the data. What family of functions should be used to model the data: linear, exponential, logarithmic, power function, or a polynomial? If a polynomial, state the degree and whether the leading coefficient is positive or negative.

(b) Use quadratic regression to fit a quadratic polynomial to the data; graph it with the data.

Table 1.46 *What function fits these data?*

Year	1940	1950	1960	1970	1980	1986
Average miles per gallon	14.8	13.9	13.4	13.5	15.5	18.3

[52] C. Schaufele and N. Zumoff, *Earth Algebra, Preliminary Version*, p. 91, (New York: Harper Collins, 1993).

Solution

(a) The data are shown in Figure 1.121, with time t in years since 1940. Miles per gallon decreases and then increases, so a good function to model the data is a quadratic (degree 2) polynomial. Since the parabola opens up, the leading coefficient is positive.

(b) If $f(t)$ is average miles per gallon, one algorithm for quadratic regression tells us that the quadratic polynomial that fits the data is

$$f(t) = 0.00617t^2 - 0.225t + 15.10.$$

In Figure 1.122, we see that this quadratic does fit the data reasonably well.

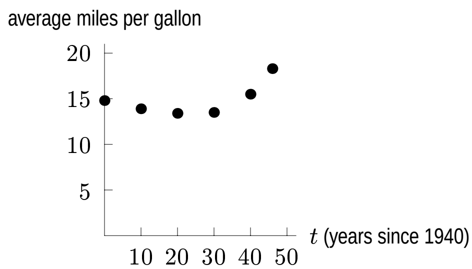

Figure 1.121: Data showing fuel efficiency of US automobiles over time

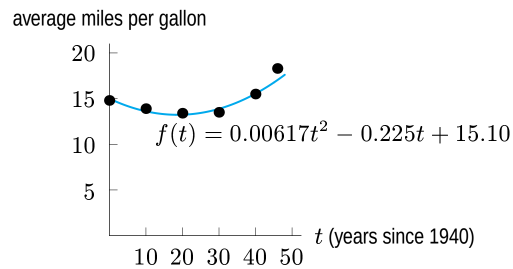

Figure 1.122: Data and best quadratic polynomial, found using regression

Problems on Fitting Formulas to Data

1. Table 1.47 gives the gross world product, G, which measures global output of goods and services.[53] If t is in years since 1950, the regression line for these data is

$$G = 3.543 + 0.734t.$$

(a) Plot the data and the regression line on the same axes. Does the line fit the data well?
(b) Interpret the slope of the line in terms of gross world product.
(c) Use the regression line to estimate gross world product in 2005 and in 2020. Comment on your confidence in the two predictions.

Table 1.47 *G, in trillions of 1999 dollars*

Year	1950	1960	1970	1980	1990	2000
G	6.4	10.0	16.3	23.6	31.9	43.2

2. Table 1.48 shows worldwide cigarette production as a function of t, the number of years since 1950.[54]

(a) Find the regression line for this data.
(b) Use the regression line to estimate world cigarette production in the year 2010.
(c) Interpret the slope of the line in terms of cigarette production.
(d) Plot the data and the regression line on the same axes. Does the line fit the data well?

Table 1.48 *Cigarette production, P, in billions*

t	0	10	20	30	40	50
P	1686	2150	3112	4388	5419	5564

3. Table 1.49 shows the US Gross National Product (GNP).

(a) Plot GNP against years since 1960. Does a line fit the data well?
(b) Find the regression line and graph it with the data.
(c) Use the regression line to estimate the GNP in 1985 and in 2020. Which estimate do you have more confidence in? Why?

Table 1.49 *GNP in 1982 dollars*

Year	1960	1970	1980	1989
GNP (billions)	1665	2416	3187	4118

[53] The Worldwatch Institute, *Vital Signs* 2001, p. 57, (New York: W.W. Norton, 2001).
[54] The Worldwatch Institute, *Vital Signs* 2001, p. 77, (New York: W.W. Norton, 2001).

4. The acidity of a solution is measured by its pH, with lower pH values indicating more acidity. A study of acid rain was undertaken in Colorado between 1975 and 1978, in which the acidity of rain was measured for 150 consecutive weeks. The data followed a generally linear pattern and the regression line was determined to be

$$P = 5.43 - 0.0053t,$$

where P is the pH of the rain and t is the number of weeks into the study.[55]

(a) Is the pH level increasing or decreasing over the period of the study? What does this tell you about the level of acidity in the rain?
(b) According to the line, what was the pH at the beginning of the study? At the end of the study ($t = 150$)?
(c) What is the slope of the regression line? Explain what this slope is telling you about the pH.

5. In a 1977 study[56] of 21 of the best American female runners, researchers measured the average stride rate, S, at different speeds, v. The data are given in Table 1.50.

(a) Find the regression line for these data, using stride rate as the dependent variable.
(b) Plot the regression line and the data on the same axes. Does the line fit the data well?
(c) Use the regression line to predict the stride rate when the speed is 18 ft/sec and when the speed is 10 ft/sec. Which prediction do you have more confidence in? Why?

Table 1.50 *Stride rate, S, in steps/sec, and speed, v, in ft/sec*

v	15.86	16.88	17.50	18.62	19.97	21.06	22.11
S	3.05	3.12	3.17	3.25	3.36	3.46	3.55

6. Table 1.51 shows the atmospheric concentration of carbon dioxide, CO_2, (in parts per million, ppm) at the Mauna Loa Observatory in Hawaii.[57]

(a) Find the average rate of change of the concentration of carbon dioxide between 1960 and 1990. Give units and interpret your answer in terms of carbon dioxide.
(b) Plot the data, and find the regression line for carbon dioxide concentration against years since 1960. Use the regression line to predict the concentration of carbon dioxide in the atmosphere in the year 2000.

Table 1.51

Year	1960	1965	1970	1975	1980	1985	1990
CO_2	316.8	319.9	325.3	331.0	338.5	345.7	354.0

7. In Problem 6, carbon dioxide concentration was modeled as a linear function of time. However, if we include data for carbon dioxide concentration from as far back as 1900, the data appear to be more exponential than linear. (They looked linear in Problem 6 because we were only looking at a small piece of the graph.) If C is the CO_2 concentration in ppm and t is in years since 1900, an exponential regression function to fit the data is

$$C = 272.27(1.0026)^t.$$

(a) What is the annual percent growth rate during this period? Interpret this rate in terms of CO_2 concentration.
(b) What CO_2 concentration is given by the model for 1900? For 1980? Compare the 1980 estimate to the actual value in Table 1.51.

8. Table 1.52 shows the average yearly per capita (i.e., per person) health care expenditures for various years. Does a linear or an exponential model appear to fit these data best? Find a formula for the regression function you decide is best. Graph the function with the data and assess how well it fits the data.

Table 1.52 *Health expenditures, $ per capita*

Years since 1970, t	0	5	10	15	20
Expenditure, C	349	591	1055	1596	2714

9. Table 1.53 shows the number of cars in the US.[58]

(a) Plot the data, with number of passenger cars as the dependent variable.
(b) Does a linear or exponential model appear to fit the data better?
(c) Use a linear model first: Find the regression line for these data. Graph it with the data. Use the regression line to predict the number of passenger cars in the year 2010 ($t = 70$).
(d) Interpret the slope of the regression line found in part (c) in terms of passenger cars.
(e) Now use an exponential model: Find the exponential regression function for these data. Graph it with the data. Use the exponential function to predict the number of passenger cars in the year 2010 ($t = 70$). Compare your prediction with the prediction obtained from the linear model.
(f) What annual percent growth rate in number of US passenger cars does your exponential model show?

[55]William M. Lewis and Michael C. Grant, "Acid Precipitation in the Western United States," *Science* 207 (1980) pp. 176-177.
[56]R.C. Nelson, C.M. Brooks, and N.L. Pike, "Biomechanical Comparison of Male and Female Distance Runners." *The Marathon: Physiological, Medical, Epidemiological, and Psychological Studies*, ed. P. Milvy, pp. 793-807, (New York: New York Academy of Sciences, 1977).
[57]Lester R. Brown, et al., *Vital Signs 1994*, p. 67, (New York: W. W. Norton, 1994).
[58]*Statistical Abstracts of the United States.*

Table 1.53 *Number of passenger cars, in millions*

t (years since 1940)	0	10	20	30	40	50
N (millions of cars)	27.5	40.3	61.7	89.3	121.6	133.7

10. Table 1.54 gives the population of the world in billions.

(a) Plot these data. Does a linear or exponential model seem to fit the data best?
(b) Find the exponential regression function.
(c) What annual percent growth rate does the exponential function show?
(d) Predict the population of the world in the year 2000 and in the year 2050. Comment on the relative confidence you have in these two estimates.

Table 1.54 *World population*

Year (since 1950)	0	10	20	30	40	44
World population (billions)	2.6	3.1	3.7	4.5	5.4	5.6

11. In 1969, all field goal attempts were analyzed in the National Football League and American Football League. See Table 1.55. (The data has been summarized: all attempts between 10 and 19 yards from the goal post are listed as 14.5 yards out, etc.)

(a) Graph the data, with success rate as the dependent variable. Discuss whether a linear or an exponential model fits best.
(b) Find the linear regression function; graph it with the data. Interpret the slope of the regression line in terms of football.
(c) Find the exponential regression function; graph it with the data. What success rate does this function predict from a distance of 50 yards?
(d) Using the graphs in parts (b) and (c), decide which model seems to fit the data best.

Table 1.55 *Successful fraction of field goal attempts*

Distance from goal, x yards	14.5	24.5	34.5	44.5	52.0
Fraction successful, Y	0.90	0.75	0.54	0.29	0.15

12. Table 1.56 shows the number of Japanese cars imported into the US.[59]

(a) Plot the number of Japanese cars imported against the number of years since 1964.
(b) Does the data look more linear or more exponential?
(c) Fit an exponential function to the data and graph it with the data.
(d) What annual percentage growth rate does the exponential model show?
(e) Do you expect this model to give accurate predictions beyond 1971? Explain.

Table 1.56 *Imported Japanese cars, 1964–1971*

Year since 1964	0	1	2	3	4	5	6	7
Cars (thousands)	16	24	56	70	170	260	381	704

13. Figure 1.123 shows oil production in the Middle East.[60] If you were to model this function with a polynomial, what degree would you choose? Would the leading coefficient be positive or negative?

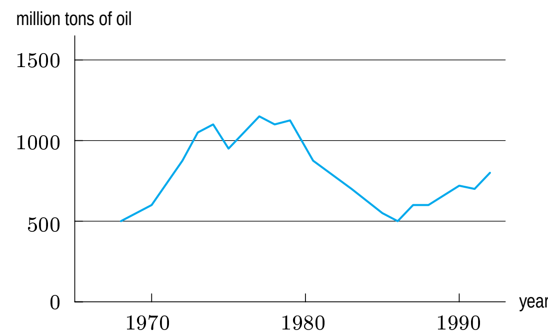

Figure 1.123

In Problems 14–16, tables of data are given.[61]

(a) Use a plot of the data to decide whether a linear, exponential, logarithmic, or quadratic model fits the data best.
(b) Use a calculator or computer to find the regression equation for the model you chose in part (a). If the equation is linear or exponential, interpret the absolute or relative rate of change.
(c) Use the regression equation to predict the value of the function in the year 2005.
(d) Plot the regression equation on the same axes as the data, and comment on the fit.

14. *World solar power, S, in megawatts; t in years since 1990*

t	0	1	2	3	4	5	6	7	8	9	10
S	46	55	58	60	69	79	89	126	153	201	288

15. *Nuclear warheads, N, in thousands; t in years since 1960*

t	0	5	10	15	20	25	30	35	40
N	20	39	40	52	61	69	60	43	32

16. *Carbon dioxide, C, in ppm; t in years since 1970*

t	0	5	10	15	20	25	30
C	325.5	331.0	338.5	345.7	354.0	360.9	369.40

[59] *The World Almanac 1995*.
[60] Lester R. Brown, et. al., *Vital Signs*, p. 49, (New York: W. W. Norton and Co., 1994).
[61] The Worldwatch Institute, *Vital Signs* 2001, (New York: W.W. Norton & Company, 2001), p. 47.

17. Table 1.57 gives the area of rain forest destroyed for agriculture and development.[62]

(a) Plot these data.
(b) Are the data increasing or decreasing? Concave up or concave down? In each case, interpret your answer in terms of rain forest.
(c) Use a calculator or computer to fit a logarithmic function to this data. Plot this function on the axes in part (a).
(d) Use the curve you found in part (c) to predict the area of rain forest destroyed in 2010.

Table 1.57 *Destruction of rain forest*

x (year)	1960	1970	1980	1988
y (million hectares)	2.21	3.79	4.92	5.77

18. For each graph in Figure 1.124, decide whether the best fit for the data appears to be a linear function, an exponential function, or a polynomial.

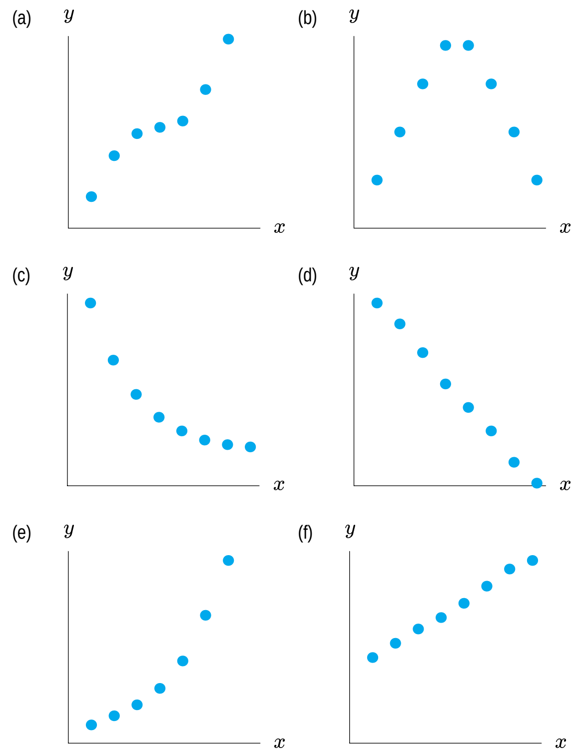

Figure 1.124

COMPOUND INTEREST AND THE NUMBER e

If you have some money, you may decide to invest it to earn interest. The interest can be paid in many different ways—for example, once a year or many times a year. If the interest is paid more frequently than once per year and the interest is not withdrawn, there is a benefit to the investor since the interest earns interest. This effect is called *compounding*. You may have noticed banks offering accounts that differ both in interest rates and in compounding methods. Some offer interest compounded annually, some quarterly, and others daily. Some even offer continuous compounding.

What is the difference between a bank account advertising 8% compounded annually (once per year) and one offering 8% compounded quarterly (four times per year)? In both cases 8% is an annual rate of interest. The expression 8% *compounded annually* means that at the end of each year, 8% of the current balance is added. This is equivalent to multiplying the current balance by 1.08. Thus, if \$100 is deposited, the balance, B, in dollars, will be

$$\begin{aligned} B &= 100(1.08) && \text{after one year,} \\ B &= 100(1.08)^2 && \text{after two years,} \\ B &= 100(1.08)^t && \text{after } t \text{ years.} \end{aligned}$$

The expression 8% *compounded quarterly* means that interest is added four times per year (every three months) and that $\frac{8}{4} = 2\%$ of the current balance is added each time. Thus, if \$100 is deposited, at the end of one year, four compoundings have taken place and the account will contain $\$100(1.02)^4$. Thus, the balance will be

$$\begin{aligned} B &= 100(1.02)^4 && \text{after one year,} \\ B &= 100(1.02)^8 && \text{after two years,} \\ B &= 100(1.02)^{4t} && \text{after } t \text{ years.} \end{aligned}$$

[62]C. Schaufele and N. Zumoff, *Earth Algebra, Preliminary Version*, p. 131, (New York: Harper Collins, 1993).

Note that 8% is *not* the rate used for each three month period; the annual rate is divided into four 2% payments. Calculating the total balance after one year under each method shows that

$$\begin{aligned}\text{Annual compounding:}\quad & B = 100(1.08) = 108.00,\\ \text{Quarterly compounding:}\quad & B = 100(1.02)^4 = 108.24.\end{aligned}$$

Thus, more money is earned from quarterly compounding, because the interest earns interest as the year goes by. In general, the more often interest is compounded, the more money will be earned (although the increase may not be very large).

We can measure the effect of compounding by introducing the notion of *effective annual yield*. Since $\$100$ invested at 8% compounded quarterly grows to $\$108.24$ by the end of one year, we say that the *effective annual yield* in this case is 8.24%. We now have two interest rates that describe the same investment: the 8% compounded quarterly and the 8.24% effective annual yield. Banks call the 8% the *annual percentage rate*, or APR. We may also call the 8% the *nominal rate* (nominal means "in name only"). However, it is the effective yield that tells you exactly how much interest the investment really pays. Thus, to compare two bank accounts, simply compare the effective annual yields. The next time you walk by a bank, look at the advertisements, which should (by law) include both the APR, or nominal rate, and the effective annual yield. We often abbreviate *annual percentage rate* to *annual rate*.

Using the Effective Annual Yield

Example 4 Which is better: Bank X paying a 7% annual rate compounded monthly or Bank Y offering a 6.9% annual rate compounded daily?

Solution We find the effective annual yield for each bank.

Bank X: There are 12 interest payments in a year, each payment being $0.07/12 = 0.005833$ times the current balance. If the initial deposit were $\$100$, then the balance B would be

$$\begin{aligned}B &= 100(1.005833) &&\text{after one month,}\\ B &= 100(1.005833)^2 &&\text{after two months,}\\ B &= 100(1.005833)^t &&\text{after } t \text{ months.}\end{aligned}$$

To find the effective annual yield, we look at one year, or 12 months, giving $B = 100(1.005833)^{12} = 100(1.072286)$, so the effective annual yield $\approx 7.23\%$.

Bank Y: There are 365 interest payments in a year (assuming it is not a leap year), each being $0.069/365 = 0.000189$ times the current balance. Then the balance is

$$\begin{aligned}B &= 100(1.000189) &&\text{after one day,}\\ B &= 100(1.000189)^2 &&\text{after two days,}\\ B &= 100(1.000189)^t &&\text{after } t \text{ days.}\end{aligned}$$

so at the end of one year we have multiplied the initial deposit by

$$(1.000189)^{365} = 1.071413$$

so the effective annual yield for Bank Y $\approx 7.14\%$.

Comparing effective annual yields for the banks, we see that Bank X is offering a better investment, by a small margin.

Example 5 If \$1000 is invested in each bank in Example 4, write an expression for the balance in each bank after t years.

Solution For Bank X, the effective annual yield $\approx 7.23\%$, so after t years the balance, in dollars, will be

$$B = 100\left(1 + \frac{0.07}{12}\right)^{12t} = 1000(1.005833)^{12t} = 1000(1.0723)^t.$$

For Bank Y, the effective annual yield $\approx 7.14\%$, so after t years the balance, in dollars, will be

$$B = 1000\left(1 + \frac{0.069}{365}\right)^{365t} = 1000(1.0714)^t.$$

(Again, we are ignoring leap years.)

If interest at an annual rate of r is compounded n times a year, then r/n times the current balance is added n times a year. Therefore, with an initial deposit of P, the balance t years later is

$$B = P\left(1 + \frac{r}{n}\right)^{nt}.$$

Note that r is the nominal rate; for example, $r = 0.05$ when the annual rate is 5%.

Increasing the Frequency of Compounding: Continuous Compounding

Let us look at the effect of increasing the frequency of compounding. How much effect does it have?

Example 6 Find the effective annual yield for a 7% annual rate compounded
(a) 1000 times a year. (b) 10,000 times a year.

Solution (a) In one year, a deposit is multiplied by

$$\left(1 + \frac{0.07}{1000}\right)^{1000} \approx 1.0725056,$$

giving an effective annual yield of about 7.25056%.

(b) In one year, a deposit is multiplied by

$$\left(1 + \frac{0.07}{10{,}000}\right)^{10{,}000} \approx 1.0725079,$$

giving an effective annual yield of about 7.25079%.

You can see that there's not a great deal of difference between compounding 1000 times each year (about three times per day) and 10,000 times each year (about 30 times per day). What happens if we compound more often still? Every minute? Every second? You may be surprised to know that the effective annual yield does not increase indefinitely, but tends to a finite value. The benefit of increasing the frequency of compounding becomes negligible beyond a certain point.

For example, if you were to compute the effective annual yield on a 7% investment compounded n times per year for values of n larger than 100,000, you would find that

$$\left(1+\frac{0.07}{n}\right)^n \approx 1.0725082.$$

So the effective annual yield is about 7.25082%. Even if you take $n = 1{,}000{,}000$ or $n = 10^{10}$, the effective annual yield does not change appreciably. The value 7.25082% is an upper bound that is approached as the frequency of compounding increases.

When the effective annual yield is at this upper bound, we say that the interest is being *compounded continuously*. (The word *continuously* is used because the upper bound is approached by compounding more and more frequently.) Thus, when a 7% nominal annual rate is compounded so frequently that the effective annual yield is 7.25082%, we say that the 7% is compounded *continuously*. This represents the most one can get from a 7% nominal rate.

Where Does the Number e Fit In?

It turns out that e is intimately connected to continuous compounding. To see this, use a calculator to check that $e^{0.07} \approx 1.0725082$, which is the same number we obtained by compounding 7% a large number of times. So you have discovered that for very large n

$$\left(1+\frac{0.07}{n}\right)^n \approx e^{0.07}.$$

As n gets larger, the approximation gets better and better, and we write

$$\lim_{n\to\infty}\left(1+\frac{0.07}{n}\right)^n = e^{0.07},$$

meaning that as n increases, the value of $(1+0.07/n)^n$ approaches $e^{0.07}$.

If P is deposited at an annual rate of 7% compounded continuously, the balance, B, after t years, is given by

$$B = P(e^{0.07})^t = Pe^{0.07t}.$$

> If interest on an initial deposit of P is *compounded continuously* at an annual rate r, the balance t years later can be calculated using the formula
>
> $$B = Pe^{rt}.$$

In working with compound interest, it is important to be clear whether interest rates are nominal rates or effective yields, as well as whether compounding is continuous or not.

Example 7 Find the effective annual yield of a 6% annual rate, compounded continuously.

Solution In one year, an investment of P becomes $Pe^{0.06}$. Using a calculator, we see that

$$Pe^{0.06} = P(1.0618365).$$

So the effective annual yield is about 6.18%.

Example 8 You invest money in a certificate of deposit (CD) for your child's education, and you want it to be worth \$120,000 in 10 years. How much should you invest if the CD pays interest at a 9% annual rate compounded quarterly? Continuously?

Solution Suppose you invest P initially. A 9% annual rate compounded quarterly has an effective annual yield given by $(1 + 0.09/4)^4 = 1.0930833$, or 9.30833%. So after 10 years you will have

$$P(1.0930833)^{10} = 120{,}000.$$

Therefore, you should invest

$$P = \frac{120{,}000}{(1.0930833)^{10}} = \frac{120{,}000}{2.4351885} = 49{,}277.50.$$

On the other hand, if the CD pays 9% compounded continuously, after 10 years you will have

$$Pe^{(0.09)10} = 120{,}000.$$

So you would need to invest

$$P = \frac{120{,}000}{e^{(0.09)10}} = \frac{120{,}000}{2.4596031} = 48{,}788.36.$$

Notice that to achieve the same result, continuous compounding requires a smaller initial investment than quarterly compounding. This is to be expected since the effective annual yield is higher for continuous than for quarterly compounding.

Problems on Compound Interest and the Number e

1. A department store issues its own credit card, with an interest rate of 2% per month. Explain why this is not the same as an annual rate of 24%. What is the effective annual rate?

2. A deposit of \$10,000 is made into an account paying a nominal yearly interest rate of 8%. Determine the amount in the account in 10 years if the interest is compounded:

   (a) Annually **(b)** Monthly **(c)** Weekly
   (d) Daily **(e)** Continuously

3. A deposit of \$50,000 is made into an account paying a nominal yearly interest rate of 6%. Determine the amount in the account in 20 years if the interest is compounded:

   (a) Annually **(b)** Monthly **(c)** Weekly
   (d) Daily **(e)** Continuously

4. Use a graph of $y = (1 + 0.07/x)^x$ to estimate the number that $(1 + 0.07/x)^x$ approaches as $x \to \infty$. Confirm that the value you get is $e^{0.07}$.

5. Find the effective annual yield of a 6% annual rate, compounded continuously.

6. What nominal annual interest rate has an effective annual yield of 5% under continuous compounding?

7. What is the effective annual yield, under continuous compounding, for a nominal annual interest rate of 8%?

8. **(a)** Find the effective annual yield for a 5% annual interest rate compounded n times/year if

   (i) $n = 1000$ (ii) $n = 10{,}000$
   (iii) $n = 100{,}000$

   (b) Look at the sequence of answers in part (a), and predict the effective annual yield for a 5% annual rate compounded continuously.

   (c) Compute $e^{0.05}$. How does this confirm your answer to part (b)?

9. **(a)** Find $(1 + 0.04/n)^n$ for $n = 10{,}000$, and $100{,}000$, and $1{,}000{,}000$. Use the results to predict the effective annual yield of a 4% annual rate compounded continuously.

   (b) Confirm your answer by computing $e^{0.04}$.

10. A bank account is earning interest at 6% per year compounded continuously.

    (a) By what percentage has the bank balance in the account increased over one year? (This is the effective annual yield.)

    (b) How long does it take the balance to double?

    (c) For an interest rate of r, find a formula giving the doubling time in terms of the interest rate.

11. Explain how you can match the interest rates (a)–(e) with the effective annual yields I–V without calculation.

(a) 5.5% annual rate, compounded continuously.
(b) 5.5% annual rate, compounded quarterly.
(c) 5.5% annual rate, compounded weekly.
(d) 5% annual rate, compounded yearly.
(e) 5% annual rate, compounded twice a year.

I. 5%	II. 5.06%	III. 5.61%
IV. 5.651%	V. 5.654%	

Countries with very high inflation rates often publish monthly rather than yearly inflation figures, because monthly figures are less alarming. Problems 12–13 involve such high rates, which are called *hyperinflation*.

12. In 1989, US inflation was 4.6% a year. In 1989 Argentina had an inflation rate of about 33% a month.

(a) What is the yearly equivalent of Argentina's 33% monthly rate?
(b) What is the monthly equivalent of the US 4.6% yearly rate?

13. Between December 1988 and December 1989, Brazil's inflation rate was 1290% a year. (This means that between 1988 and 1989, prices increased by a factor of $1 + 12.90 = 13.90$.)

(a) What would an article which cost 1000 cruzados (the Brazilian currency unit) in 1988 cost in 1989?
(b) What was Brazil's monthly inflation rate during this period?

FOCUS ON THEORY

LIMITS TO INFINITY AND END BEHAVIOR

Comparing Power Functions

As x gets large, how do different power functions compare? For positive powers, Figure 1.125 shows that the higher the power of x, the faster the function climbs. For large values of x (in fact, for all $x > 1$), $y = x^5$ is above $y = x^4$, which is above $y = x^3$, and so on. Not only are the higher powers larger, but they are *much* larger. This is because if $x = 100$, for example, 100^5 is one hundred times as big as 100^4 which is one hundred times as big as 100^3. As x gets larger (written as $x \to \infty$), any positive power of x completely swamps all lower powers of x. We say that, as $x \to \infty$, higher powers of x *dominate* lower powers.

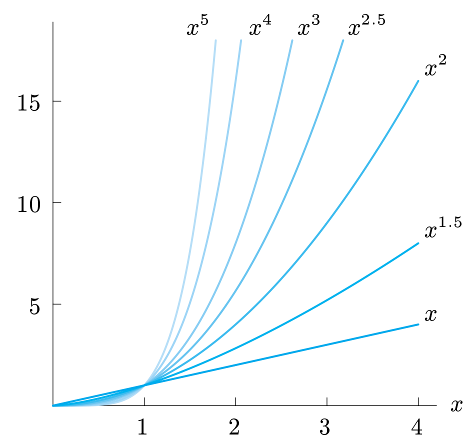

Figure 1.125: Powers of x: Which is largest for large values of x?

Limits to Infinity

When we consider the values of a function $f(x)$ as $x \to \infty$, we are looking for the *limit* as $x \to \infty$. This is abbreviated

$$\lim_{x\to\infty} f(x).$$

The notation $\lim_{x\to\infty} f(x) = L$ means that the values of the function approach L as the values of x get larger and larger. We have $f(x) \to L$ as $x \to \infty$. The behavior of a function as $x \to \infty$ and as $x \to -\infty$ is called the *end behavior* of the function.

Example 9 Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ in each case.

(a) $f(x) = x^2$ (b) $f(x) = -x^3$ (c) $f(x) = e^x$

Solution (a) As x gets larger and larger without bound, the function x^2 gets larger and larger without bound, so as $x \to \infty$, we have $x^2 \to \infty$. Thus,

$$\lim_{x\to\infty} (x^2) = \infty.$$

The square of a negative number is positive, so as $x \to -\infty$, we have $x^2 \to +\infty$. Thus,

$$\lim_{x \to -\infty} (x^2) = \infty.$$

To see this graphically, look at Figure 1.126. As $x \to \infty$ or as $x \to -\infty$, the function values get bigger and bigger and the "ends" of the graph go up.

(b) The graph of $f(x) = -x^3$ is in Figure 1.127. As $x \to \infty$, the function values get more and more negative; as $x \to -\infty$, the function values are positive and get larger and larger. We have

$$\lim_{x \to \infty} (-x^3) = -\infty \quad \text{and} \quad \lim_{x \to -\infty} (-x^3) = \infty.$$

(c) The graph of $f(x) = e^x$ is in Figure 1.128. As $x \to \infty$, the function values get larger without bound, and as $x \to -\infty$, the function values get closer and closer to zero. We have

$$\lim_{x \to \infty} (e^x) = \infty \quad \text{and} \quad \lim_{x \to -\infty} (e^x) = 0.$$

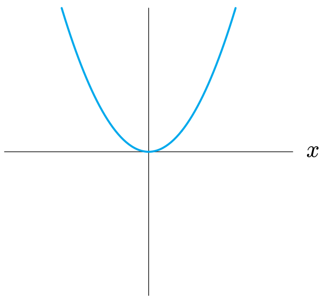

Figure 1.126: End behavior of $f(x) = x^2$

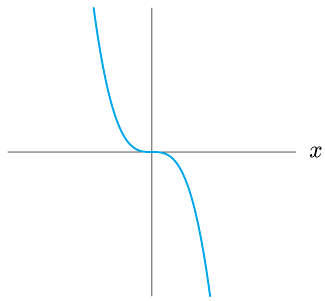

Figure 1.127: End behavior of $f(x) = -x^3$

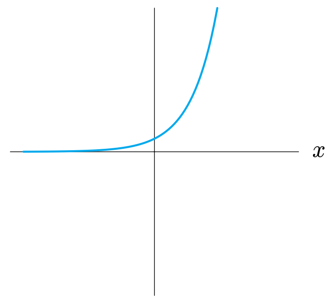

Figure 1.128: End behavior of $f(x) = e^x$

Exponential Functions and Power Functions: Which Dominate?

How does the growth of power functions compare to the growth of exponential functions, as x gets large? In everyday language, the word exponential is often used to imply very fast growth. But do exponential functions always grow faster than power functions? To determine what happens "in the long run," we often want to know which functions dominate as $x \to \infty$. Let's compare functions of the form $y = a^x$ for $a > 1$, and $y = x^n$ for $n > 0$.

First we consider $y = 2^x$ and $y = x^3$. The close-up, or local, view in Figure 1.129(a) shows that between $x = 2$ and $x = 4$, the graph of $y = 2^x$ lies below the graph of $y = x^3$. But Figure 1.129(b) shows that the exponential function $y = 2^x$ eventually overtakes $y = x^3$. The far-away, or global, view in Figure 1.129(c) shows that, for large x, the value of x^3 is insignificant compared to 2^x. Indeed, 2^x is growing so much faster than x^3 that the graph of 2^x appears almost vertical in comparison to the more leisurely climb of x^3.

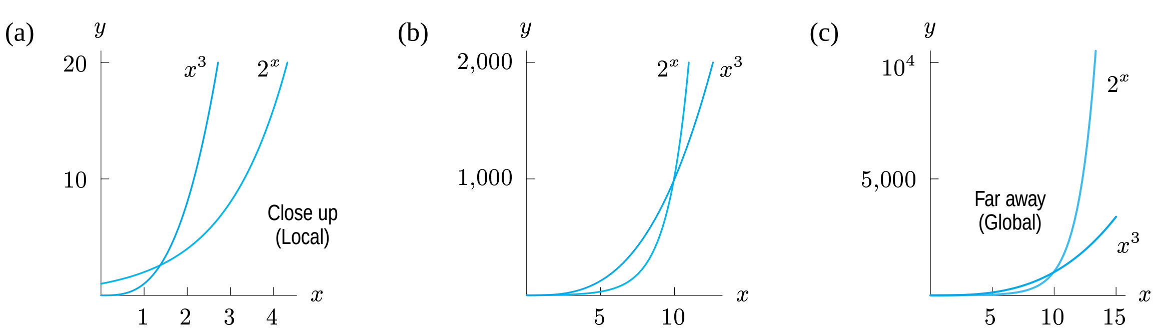

Figure 1.129: Local and global views of $y = 2^x$ and $y = x^3$: Notice that $y = 2^x$ eventually dominates $y = x^3$

In fact, *every* exponential growth function eventually dominates *every* power function. Although an exponential function may be below a power function for some values of x, if we look at large enough x-values, a^x eventually dominates x^n, no matter what n is (provided $a > 1$).

What about the logarithm function? We know that exponential functions grow very quickly and logarithm functions grow very slowly. See Figure 1.130. Logarithm functions grow so slowly, in fact, that every power function x^n eventually dominates $\ln x$ (provided n is positive).

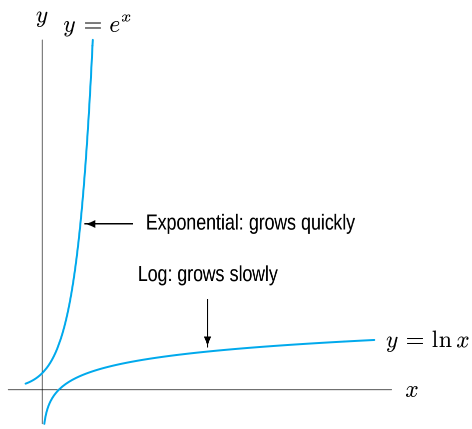

Figure 1.130: Exponential and logarithmic growth

End Behavior of Polynomials

We saw in Section 1.9 that if a polynomial is viewed in a large enough window, it has approximately the same shape as the power function given by the leading term. Provided $a_n \neq 0$, we have

$$\lim_{x\to\infty} (a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0) = \lim_{x\to\infty} (a_n x^n).$$

A similar result holds as $x \to -\infty$. The end behavior of a polynomial is the same as the end behavior of its leading term.

Example 10 Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ in each case. (a) $f(x) = 5x^3 - 20x^2 + 15x - 100$ (b) $f(x) = 25 + 10x - 15x^2 - 3x^4$

Solution (a) We have

$$\lim_{x\to\infty} (5x^3 - 20x^2 + 15x - 100) = \lim_{x\to\infty} (5x^3) = \infty.$$

Similarly,

$$\lim_{x\to-\infty} (5x^3 - 20x^2 + 15x - 100) = \lim_{x\to-\infty} (5x^3) = -\infty.$$

(b) We have

$$\lim_{x\to\infty} (25 + 10x - 15x^2 - 3x^4) = \lim_{x\to\infty} (-3x^4) = -\infty.$$

Similarly,

$$\lim_{x\to-\infty} (25 + 10x - 15x^2 - 3x^4) = \lim_{x\to-\infty} (-3x^4) = -\infty.$$

Problems on Limits to Infinity and End Behavior

1. As $x \to \infty$, which of the three functions $y = 1000x^2$, $y = 20x^3$, $y = 0.1x^4$ has the largest values? Which has the smallest values? Sketch a global picture of these three functions (for $x \geq 0$) on the same axes.

2. As $x \to \infty$, which of the two functions $y = 5000x^3$ and $y = 0.2x^4$ dominates? The two graphs intersect at the origin. Are there any other points of intersection? If so, find their x values.

3. Graph $y = x^{1/2}$ and $y = x^{2/3}$ for $x \geq 0$ on the same axes. Which function has larger values as $x \to \infty$?

4. By hand, graph $f(x) = x^3$ and $g(x) = 20x^2$ on the same axes. Which function has larger values as $x \to \infty$?

5. By hand, graph $f(x) = x^5$, $g(x) = -x^3$, and $h(x) = 5x^2$ on the same axes. Which has the largest positive values as $x \to \infty$? As $x \to -\infty$?

In Problems 6–7, graph $f(x)$ and $g(x)$ in two windows. Describe what you see.

(a) Window $-7 \leq x \leq 7$ and $-15 \leq y \leq 15$
(b) Window $-50 \leq x \leq 50$ and $-10{,}000 \leq y \leq 10{,}000$

6. $f(x) = 0.2x^3 - 5x + 3$ and $g(x) = 0.2x^3$

7. $f(x) = 3 - 5x + 5x^2 + x^3 - x^4$ and $g(x) = -x^4$

In Problems 8–10, draw a possible graph for $f(x)$.

8. $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$

9. $\lim_{x\to\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = +\infty$

10. $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = +\infty$

11. A continuous function has $\lim_{x\to\infty} f(x) = 3$.

(a) In words, explain what this limit means.
(b) In each part, (i)–(iv), graph a function $f(x)$ with this limit and which is

(i) Increasing (ii) Decreasing
(iii) Concave up (iv) Oscillating

12. A continuous function has $\lim_{x\to-\infty} g(x) = 3$.

(a) In words, explain what this limit means.
(b) In each part, (i)–(iv), graph a function $g(x)$ with this limit and which is

(i) Increasing (ii) Decreasing
(iii) Concave up (iv) Oscillating

13. Estimate $\lim_{x\to\infty} \frac{1}{x}$. Explain your reasoning.

14. If $f(x) = -x^2$, what is $\lim_{x\to\infty} f(x)$? What is $\lim_{x\to-\infty} f(x)$?

In Problems 15–18, find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

15. $f(x) = -10x^4$

16. $f(x) = 2^x$

17. $f(x) = 8(1 - e^{-x})$

18. $f(x) = 4x^5 - 25x^3 - 60x^2 + 1000x + 5000$

Which function in Problems 19–24 has larger values as $x \to \infty$?

19. $3x^5$ or $58x^4$

20. $12x^6$ or $(1.06)^x$

21. $x^{1/2}$ or $\ln x$

22. $x^3 + 2x^2 + 25x + 100$ or $10 - 6x^2 + x^4$

23. $5x^3 + 20x^2 + 150x + 200$ or $0.5x^4$

24. $5x^3 + 20x^2 + 150x + 200$ or $e^{0.2x}$

25. Match $y = 70x^2$, $y = 5x^3$, $y = x^4$, and $y = 0.2x^5$ with their graphs in Figure 1.131.

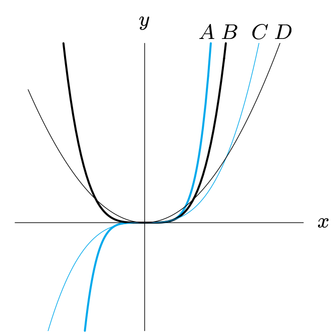

Figure 1.131

26. Match $y = e^x$, $y = \ln x$, $y = x^2$, and $y = x^{1/2}$ with their graphs in Figure 1.132.

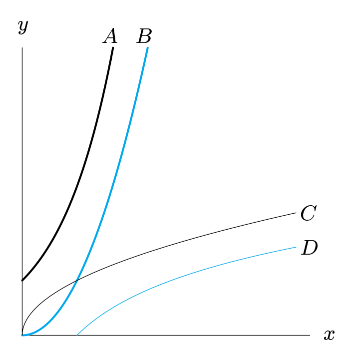

Figure 1.132

27. Global graphs of $y = x^5, y = 100x^2$, and $y = 3^x$ are in Figure 1.133. Which function corresponds to which curve?

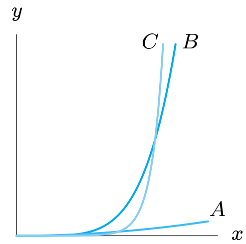

Figure 1.133

28. Use a graphing calculator or a computer to graph $y = x^4$ and $y = 3^x$. Determine approximate domains and ranges that give each of the graphs in Figure 1.134.

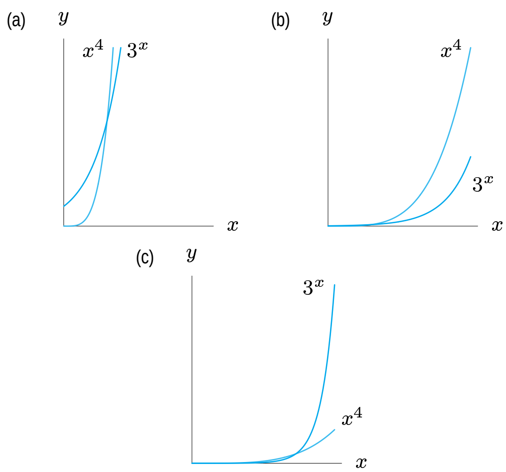

Figure 1.134

29. Graph $f(x) = e^{-x^2}$ using a window that includes both positive and negative values of x.

(a) For what values of x is f increasing? For what values is it decreasing?

(b) Is the graph of f concave up or concave down near $x = 0$?

(c) As $x \to \infty$, what happens to the value of $f(x)$? As $x \to -\infty$, what happens to $f(x)$?

30. Graph $f(x) = \ln(x^2 + 1)$ using a window that includes positive and negative values of x.

(a) For what values of x is f increasing? For what values is it decreasing?

(b) Is f concave up or concave down near $x = 0$?

(c) As $x \to \infty$, what happens to the value of $f(x)$? As $x \to -\infty$,what happens to $f(x)$?