Sept. 7, 1999

On the Half-Half Case of the Zarankiewicz Problem

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Abstract

Consider the minimum number $f(m,n)$ of zeroes in a $2m \times 2n$ $(0,1)$-matrix M that contains no $m \times n$ submatrix of ones. This special case of the well-known Zarankiewicz problem was studied by Griggs and Ouyang, who showed, for $m \leq n$, that $2n+m+1 \leq f(m,n) \leq 2n+2m-\gcd(m,n)+1$. The lower bound is sharp when m is fixed for all large n. They proposed determining $\lim_{m\to\infty}\{f(m,m+1)/m\}$. In this paper, we show that this limit is 3. Indeed, we determine the actual value of $f(m,km+1)$ for all k, m. For general m, n, we derive a new upper bound on $f(m,n)$. We also give the actual value of $f(m,n)$ for all $m \leq 7$ and $n \leq 20$.

Running head: The Half-Half Problem

[1] Research supported in part by grant NSA/MSP MDA904–95H1024.

[2] Research supported in part by grant NSF DMS–9701211.

Section 1. Introduction

The terminology and notation in this paper are the same as in the paper [4] by Griggs and Ouyang. We consider rectangular matrices M with entries that are 0 or 1. The intersection of a rows and b columns of a matrix is called an $a \times b$ *submatrix*. We say that a $2m \times 2n$ matrix M has *Property Z* if every $m \times n$ submatrix has at least one zero, *i.e.,* M has no half-half all ones submatrix. An equivalent formulation of Property Z, that is typically more useful in our study, is to require that for every m rows of M at least $n+1$ columns contain a zero somewhere in those rows. We denote by $f(m,n)$ the minimum number of zeroes in such a matrix M with Property Z. For simplicity, we often assume that $m \leq n$, since we may switch to the transpose when $m > n$.

In general, we may ask the maximum number $Z = Z_{m,n}(k,l)$ of ones in a $k \times l$ matrix M avoiding $m \times n$ all ones submatrix. (Note that $f(m,n) = 4mn - Z_{m,n}(2m, 2n)$.) In 1951 Zarankiewicz [5] posed the problem of determining $Z_{m,m}(k,k)$ for $k \geq 4$, and the general problem concerning $Z_{m,n}(k,l)$ has also become known as the *problem of Zarankiewicz*.

By viewing M as the incidence matrix for a bipartite graph, we can obtain the graph-theoretic formulation of Zarankiewicz problem that asks for the maximum number of edges in a bipartite graph (K,L) with part sizes $|K| = k$, $|L| = l$ such that there is no complete bipartite subgraph $K_{m,n}$ with m vertices in K and n vertices in L.

A survey of work on the Zarankiewicz problem appears in [1, Sec. VI.2]. Some of the more recent work includes the papers [2, 3, 4].

For the half-half case of the Zarankiewicz problem, Griggs and Ouyang obtained the following results on $f(m,n)$:

Theorem 1.1. [4] *Assume $m \leq n$. Then*

$$f(m,n) \geq 2n + m + 1,$$

where the equality holds precisely when
(1) n is a multiple of m, or
(2) $k + r \geq m$, where $n = km + r$, and $0 < r < m$.

Theorem 1.2. [4] *Assume $m < n$. Then*
(1) $f(m,n) \leq 2km + f(r,m)$, where $n = km + r$, and $0 < r \leq m$,
(2) $f(m,n) \leq 2n + 2m - \gcd(m,n) + 1$, where $\gcd(m,n)$ is the greatest common divisor of m and n.

By Theorem 1.1 and Theorem 1.2(2), they observed that $3m + 4 \leq f(m, m+1) \leq 4m + 2$ and proposed determining $\lim_{m \to \infty} \{f(m, m+1)/m\}$. In this paper we show that this limit is 3. Indeed, we prove that for all k, m, $f(m, km+1) = 2(km+1) + m + i$, where i is the largest integer such that $\lfloor i^2/4 \rfloor k + i - 1 < m$. For general m, n, we also derive a new upper bound on $f(m,n)$.

In Section 2 we consider $n = km + 1$ and construct $2m \times 2n$ matrices M_t for $1 \le t \le m$ such that each matrix M_t has Property Z. Denoting the number of zeroes in M_t by $g(t)$, we prove $f(m,n) = \min\{g(t) : 1 \le t \le m\}$ and derive the formula for $f(m,n)$.

In Section 3 we consider an extension of matrices M_t for general m, n, and derive a new upper bound on $f(m,n)$. In Section 4 we give the actual value of $f(m,n)$ for small m, n. Some of these values are obtained by tedious analysis of several cases. Finally, in Section 5 we summarize what we now know.

Section 2. The Actual Value of $f(m, km+1)$

When $n = km+r$ with $0 < r < m$ and $k+r \ge m$, Griggs and Ouyang [4] presented a matrix achieving $f(m,n) = 2n + m + 1$. By permuting columns and rearranging the entries in the last row of this matrix, we obtain the matrix shown in Figure 1. (All the blank entries in this figure are ones.)

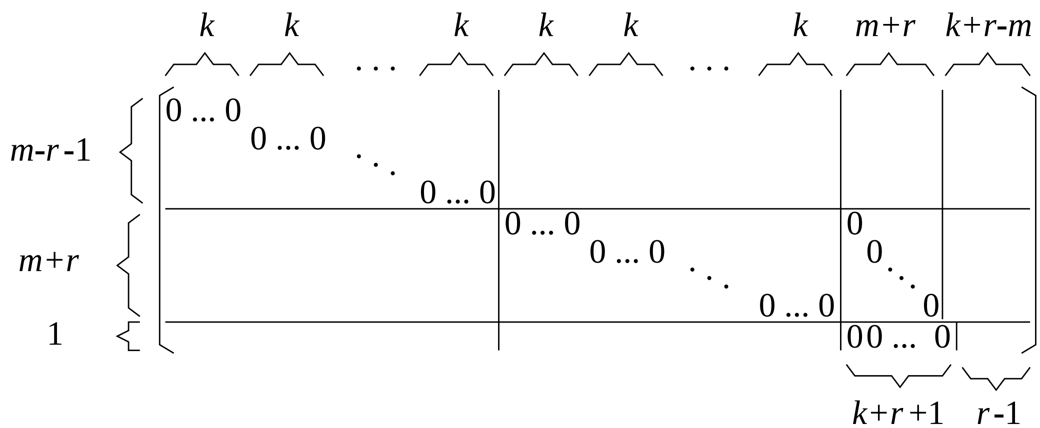

Figure 1. A matrix achieving $f(m,n) = 2n + m + 1$.

This matrix inspires us to consider the following construction: Assume $2 \le m < n$ and $n = km + 1$. For $1 \le t \le m$, we construct a $2m \times 2n$ $(0,1)$-matrix M_t illustrated in Figure 2. In this construction, q, α, and β are the integers satisfying $2n = k(m - t) + k(t-1)q + k\alpha + q + \beta$, *i.e.*,

$$km + kt + 2 = (kt - k + 1)q + k\alpha + \beta,$$

where $0 < k\alpha + \beta \le kt - k + 1$ and $0 < \beta \le k$. For example, when $m = 3$ and $n = 4$, Figure 3 displays the matrices M_1, M_2, and M_3.

Denote the number of zeroes in M_t by $g(t)$. Then we have

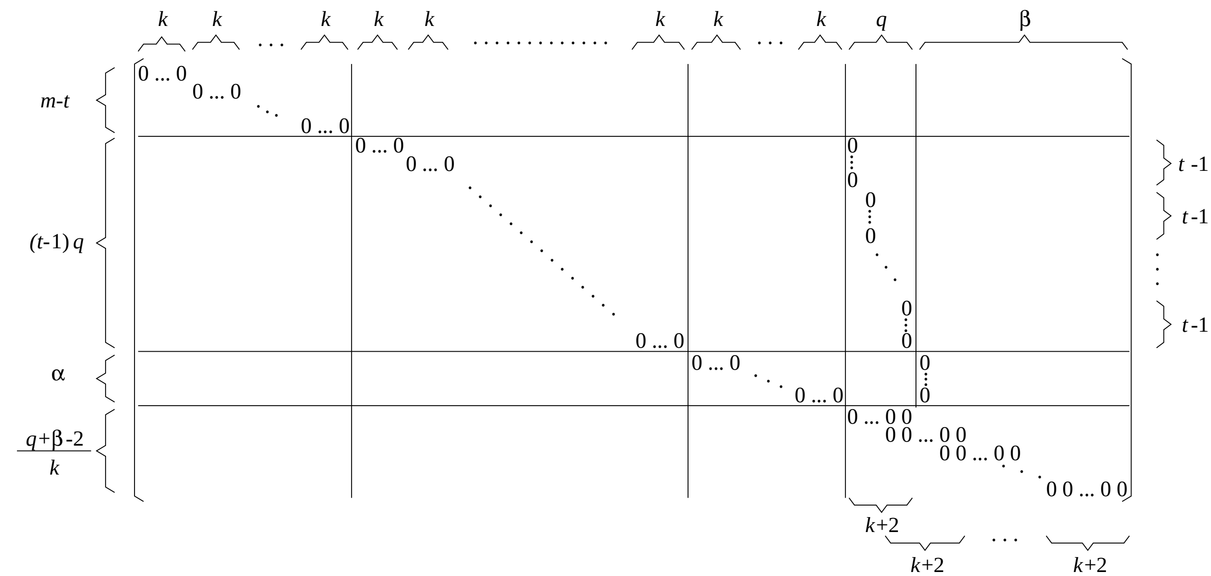

Figure 2. The matrix M_t for $n = km + 1$.

$$M_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Figure 3. The matrices M_t for $(m, n) = (3, 4)$.

Proposition 2.1. *Assume* $2 \leq m < n$, $n = km + 1$, *and* $1 \leq t \leq m$. *Then*
(1) The matrix M_t *has Property Z;*
(2) $g(t) = (2k + 1 + \frac{1}{kt-k+1})m + t + \frac{-t+2}{kt-k+1} + \frac{-\alpha+(t-1)\beta}{kt-k+1}$;
(3) $g(t) = \left\lceil (2k + 1 + \frac{1}{kt-k+1})m + t + \frac{-t+2}{kt-k+1} \right\rceil$.

Proof. (1) We consider a two-coloring on all zeroes in M_t: We assign blue to the first k zeroes in each row, and assign red to all the rest. Then in any m rows, we can find exactly km blue zeroes and at least 2 red zeroes such that all these zeroes are in different columns. Therefore, any m rows have zeroes in at least $km + 2 = n + 1$ columns, and Property Z holds for the matrix M_t.

(2) Note that the number of zeroes in M_t is $k(m - t) + (k + 1)((t - 1)q + \alpha) + (k + 2)\frac{q+\beta-2}{k}$. Since $km + kt + 2 = (kt - k + 1)q + k\alpha + \beta$, we can write q in terms of other variables and obtain the formula for $g(t)$.

(3) From the conditions $0 < k\alpha + \beta \leq kt - k + 1$ and $0 < \beta \leq k$, we have $0 \leq \alpha \leq t - 1$. Thus $0 \leq \frac{-\alpha+(t-1)\beta}{kt-k+1} < 1$ and the formula for $g(t)$ is verified. ∎

Lemma 2.2. *Assume* $2 \leq m < n$ *and* $n = km + 1$. *Then*

$$f(m, n) = \min\{g(t) : 1 \leq t \leq m\}.$$

Proof. Let $M = [M_{i,j}]$ be a $2m \times 2n$ $(0, 1)$-matrix with Property Z. By Proposition 2.1(1), it suffices to show that the number of zeroes in M is not less than $g(t)$ for some t, $1 \leq t \leq m$.

Let $R_0 = \emptyset$. For $i = 1, \ldots, 2m$, let $R_i = \{j : M_{i,j} = 0\}$ and $r_i = |R_i|$. Without loss of generality, we may assume $0 \leq r_1 \leq r_2 \leq \cdots \leq r_{2m}$. Choose the integer t as small as possible such that $1 \leq t \leq m$ and $|R_0 \cup R_1 \cup \cdots \cup R_{m-t}| \leq k(m - t)$. We consider three cases:

Case (1) $t = 1$: Since M has Property Z, we have $|R_1 \cup \cdots \cup R_m| \geq km + 2$. Then the condition "$t = 1$" forces $r_m \geq k + 2$. Thus the number of zeroes in $M \geq (km + 2) + m(k + 2) = g(1)$.

Case (2) $2 \leq t \leq m - 1$ and $|R_0 \cup R_1 \cup \cdots \cup R_{m-t}| < k(m - t)$: Note that $g(t) \leq (2k + 2)m + t$, since we have Proposition 2.1(3) and $t \geq 2$. Let $|R_1 \cup \cdots \cup R_{m-t}| = p$. Then the choice of t implies $r_{m-t+1} \geq k(m - t + 1) - p + 1$, and hence the number of zeroes in $M \geq p + (k(m - t + 1) - p + 1)(m + t)$. Replacing p with $k(m - t) - 1$, we obtain that the number of zeroes in $M \geq (2k + 2)m + 2t - 1 > (2k + 2)m + t \geq g(t)$.

Case (3) $2 \leq t \leq m$ and $|R_0 \cup R_1 \cup \cdots \cup R_{m-t}| = k(m - t)$: For $i = m - t + 1, m - t + 2, \ldots, 2m$, let $R'_i = R_i \backslash (R_0 \cup R_1 \cup \cdots \cup R_{m-t})$ and $r'_i = |R'_i|$. Then the choice of t implies $r'_{m-t+1} \geq k + 1$. Write $km + kt + 2 = (kt - k + 1)q + k\alpha + \beta$, where $0 < k\alpha + \beta \leq kt - k + 1$ and $0 < \beta \leq k$. Comparing M with M_t, we note that it is enough to show $r'_{m-t+(t-1)q+\alpha+1} \geq k + 2$.

Assume the contrary. Then $r'_{m-t+1} = \cdots = r'_{m-t+(t-1)q+\alpha+1} = k + 1$. Divide the index set $I = \{m - t + 1, \ldots, m - t + (t - 1)q + \alpha + 1\}$ into as many disjoint subsets

$I_1, \ldots, I_p$ as possible such that for all i, j in different subsets, $R'_i \cap R'_j = \emptyset$. Then the choice of t implies that for any index subset I_x, $|\cup_{i\in I_x} R'_i| = k|I_x|+1$ and $|I_x| \le t-1$.

Now we count $|\cup_{i\in I} R'_i|$ in two ways: On the one hand, we have $|\cup_{i\in I} R'_i| = \sum_{x=1}^{p}(k|I_x|+1) = k\sum_{x=1}^{p}|I_x| + p \ge k|I| + \lceil |I|/(t-1)\rceil \ge k|I|+q+1$; on the other hand, we note that $|\cup_{i\in I} R'_i| \le k((t-1)q+\alpha)+q+\beta \le k(|I|-1)+q+k \le k|I|+q$, a contradiction. ∎

Lemma 2.2 will facilitate our search for $f(m, km+1)$. It allows us to confine our analysis to the values of $g(t)$ only. Using some fundamental Calculus, we obtain the minimum of $g(t)$:

Theorem 2.3. *Assume $2 \le m < n$ and $n = km+1$. Let $t_0 = \frac{k-1+\sqrt{km+k+1}}{k}$. Then*

$$f(m, n) = \min\{g(\lfloor t_0 \rfloor), g(\lceil t_0 \rceil)\}.$$

Proof. It is easy to verify that $1 < t_0 \le m$. So $g(\lfloor t_0 \rfloor)$ and $g(\lceil t_0 \rceil)$ are well-defined. By Lemma 2.2, it suffices to show that $\min\{g(t) : 1 \le t \le m\} = \min\{g(\lfloor t_0 \rfloor), g(\lceil t_0 \rceil)\}$. Consider a continuous function $h(x) = (2k+1+\frac{1}{kx-k+1})m + x + \frac{-x+2}{kx-k+1}$, where $x \in (1-\frac{1}{k}, m+1)$. Then $h(t) = g(t)$ for $t = 1, \ldots, m$, since we have Proposition 2.1(3). By taking the first and second derivatives for $h(x)$, we verify that $h(t_0)$ is a minimum and the proof is complete. ∎

We provide in next theorem an alternative formula for $f(m, km+1)$.

Theorem 2.4. *Assume $2 \le m < n$, $n = km+1$, and i is the largest integer such that $\lfloor i^2/4 \rfloor k + i - 1 < m$. Then*

$$f(m, n) = g(\lfloor (i+3)/2 \rfloor) = 2n + m + i.$$

Proof. We assume that i is an odd number and let $i = 2\ell - 1$ for some integer ℓ. (The proof of the other case "i is even" is similar.)

First, we prove $g(\lfloor (i+3)/2 \rfloor) = 2n+m+i$, *i.e.*, $g(\ell+1) = 2n+m+2\ell-1$. By the choice of i, we have $(\ell^2-\ell)k+2\ell-2 < m \le \ell^2 k+2\ell-1$. Then $(k\ell+1)(k\ell+2) < km+k(\ell+1)+2 \le (k\ell+1)(k\ell+2)$. So we can write $km+k(\ell+1)+2 = (k\ell+1)q+k\alpha+\beta$, where $0 < k\alpha+\beta \le k\ell+1$, $0 < \beta \le k$, and $k\ell-k+2 \le q \le k\ell+1$. Therefore, $\frac{q+\beta-2}{k} = \ell$ and $g(\ell+1) = k(m-\ell-1)+(k+2)\frac{q+\beta-2}{k}+(k+1)(2m-(m-\ell-1)-\frac{q+\beta-2}{k}) = 2n+m+2\ell-1$.

By Lemma 2.2, it remains to prove that for $1 \le t \le m$, $g(t) \ge 2n+m+i$. Indeed, by Proposition 2.1(3), we only need to show $t + \frac{m-t+2}{kt-k+1} > 2\ell$. Since the choice of i gives $m > (\ell^2-\ell)k+2\ell-2$, it is enough to show that $kt^2-(2\ell k+k)t+\ell^2 k+\ell k \ge 0$. We note that this inequality is equivalent to $(t-\ell)(t-(\ell+1)) \ge 0$, which is verified for all integers t and ℓ. ∎

For general m, n with $n = km+1$, Theorem 1.2(2) gives $f(m, n) \le 2n+2m$. Now we can improve this upper bound:

Corollary 2.5. *Assume* $2 \le m < n$ *and* $n = km + 1$. *Then*

$$2n + m + 1 \le f(m,n) \le 2n + m + 2\left\lfloor \sqrt{m} \right\rfloor.$$

Proof. Let $i = 2\lfloor \sqrt{m} \rfloor$. By Theorem 2.4, it suffices to show that $m \le \left\lfloor (i+1)^2/4 \right\rfloor k + i$. Let $\ell = \lfloor \sqrt{m} \rfloor$. Then $\left\lfloor (i+1)^2/4 \right\rfloor k + i = (\ell^2 + \ell)k + 2\ell \ge (\ell+1)^2 > (\sqrt{m})^2 = m$. ∎

Section 3. An Upper Bound on $f(m,n)$ for General m, n

When n is a multiple of m, Theorem 1.1 gives $f(m,n) = 2n+m+1$. So we assume in this section that n is not a multiple of m.

We have constructed the matrix M_t for the case $n = km + 1$ in Section 2. Now we consider the following extension for general m, n: Let $2 \le m < n$ and $n = km + r$, where $0 < r < m$. For any integer t with $1 \le t \le m$ and $t = r\ell + 1$ for some integer ℓ, we construct a $2m \times 2n$ $(0,1)$-matrix M_t illustrated in Figure 4. In this construction, q, α, and β are the integers satisfying

$$km + kt + 2r = (k\ell + 1)q + k\alpha + \beta,$$

where $0 < k\alpha + \beta \le k\ell + 1$ and $0 < \beta \le k$. For example, when $m = 4$ and $n = 6$, Figure 5 displays the matrices M_1 and M_3.

In particular, when $k + r \ge m$, M_{r+1} is the same matrix as shown in Figure 1 that achieves $f(m,n) = 2n + m + 1$.

Denote the number of zeroes in M_t by $g(t)$. Similar to Proposition 2.1 and Theorem 2.4, we can prove the following results:

Proposition 3.1. *Assume* $2 \le m < n$ *and* $n = km + r$, *where* $0 < r < m$. *Let* t *be an integer such that* $1 \le t \le m$ *and* $t = r\ell + 1$. *Then*
(1) The matrix M_t *has Property Z;*
(2) $g(t) = (2k + 1 + \frac{r}{k\ell+1})m + r\ell + 1 + \frac{-r^2\ell + r}{k\ell+1} + \frac{-r\alpha + r\ell\beta}{k\ell+1}$. ∎

Theorem 3.2. *Assume* $2 \le m < n$ *and* $n = km + r$, *where* $0 < r < m$. *Let* i *be the largest integer such that* $\left\lfloor \frac{i^2}{4} \right\rfloor k + \left\lfloor \frac{i}{2} \right\rfloor r + \left\lfloor \frac{i-1}{2} \right\rfloor < m$.
(1) If $1 \le i \le 2\left\lfloor \frac{m-1}{r} \right\rfloor$, *then*

$$f(m,n) \le g\left(\left\lfloor \frac{i+1}{2} \right\rfloor r + 1 \right) \le 2n + m + 1 + (i-1)r;$$

(2) If $i > 2\left\lfloor \frac{m-1}{r} \right\rfloor$, *i.e.,* $g\left(\left\lfloor \frac{i+1}{2} \right\rfloor r + 1 \right)$ *is not defined, let* $\ell = \left\lfloor \frac{m-1}{r} \right\rfloor$, *then*

$$f(m,n) \le g(r\ell + 1) \le 2n + m + 1 + \left(\ell - 1 + \left\lfloor \left\lfloor \frac{kr + k - 1}{k\ell + 1} \right\rfloor / k \right\rfloor \right) r.$$

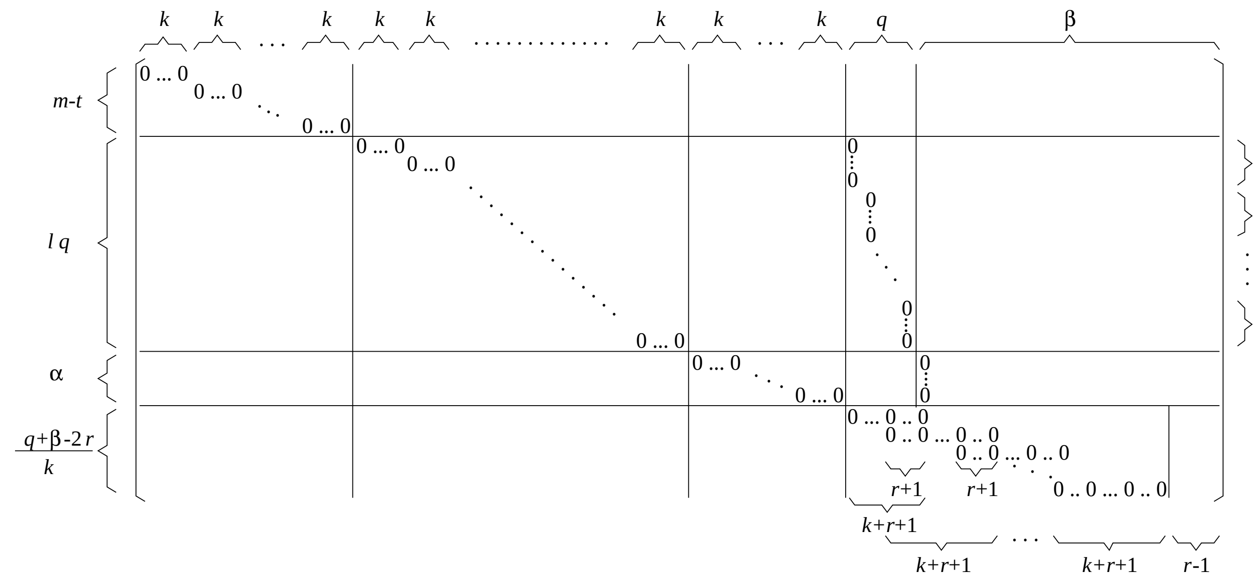

Figure 4. The matrix M_t for $n = km + r$, $r \neq 0$.

$$M_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Figure 5. The matrices M_1 and M_3 for $(m, n) = (4, 6)$.

Proof. The proof of (1) is similar to that of Theorem 2.4. To prove (2), we note that $km + k(r\ell + 1) + 2r \le (k\ell + 1)(2r) + kr + k$, since $m \le r(\ell + 1)$. So we can write $km + k(r\ell+1) + 2r = (k\ell+1)q + k\alpha + \beta$, where $0 < k\alpha + \beta \le k\ell + 1$, $0 < \beta \le k$, and $q \le 2r + \left\lfloor \frac{kr+k-1}{k\ell+1} \right\rfloor$. Therefore, $g(r\ell + 1) = k(m - r\ell - 1) + (k + r + 1)\frac{q+\beta-2r}{k} + (k + 1)(2m - (m - r\ell - 1) - \frac{q+\beta-2r}{k}) \le 2n + m + 1 + \left(\ell - 1 + \left\lfloor \left\lfloor \frac{kr+k-1}{k\ell+1} \right\rfloor /k \right\rfloor\right) r$. ∎

Note that each of Theorem 1.2 and Theorem 3.2 does not always provide a sharp bound for given m, n. For example, when $m = 4$ and $n = 6$, both theorems give $f(4, 6) \le 19$; however, the matrix in Figure 6 shows $f(4, 6) \le 18$. (Then it follows from Theorem 1.1 that $f(4, 6) = 18$.) We will check the performance of these two theorems for some small m, n in next section.

$$\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}$$

Figure 6. A matrix giving $f(4, 6) \le 18$.

By Theorem 1.1 and Theorem 1.2(2), Griggs and Ouyang [4] observed that $3m + 4 \le f(m, m+1) \le 4m + 2$ and proposed determining $\lim_{m\to\infty}\{f(m, m+1)/m\}$. From Corollary 2.5, we can show that this limit is 3. In general, we have the following extension:

Theorem 3.3. *For fixed positive integers k and r,*

$$\lim_{m\to\infty} \frac{f(m, km + r)}{m} = 2k + 1.$$

Proof. Note that if $m \ge r^2 + 2$, then $i = 2\lfloor\sqrt{m}\rfloor \le 2\lfloor(m-1)/r\rfloor$ in Theorem 3.2(1) gives the upper bound $f(m, km + r) \le (2k + 1)m + 2r\lfloor\sqrt{m}\rfloor + r + 1$. On the other hand, Theorem 1.1 gives the lower bound $f(m, km + r) \ge (2k + 1)m + 2r + 1$. Thus $f(m, km + r)/m \to 2k + 1$ as $m \to \infty$. ∎

Section 4. The Actual Value of $f(m, n)$ for Small m, n

By Theorems 1.1, 1.2, 2.4, 3.2, and tedious analysis of several cases, we have obtained in Figure 7 the actual value of $f(m, n)$ for $m \le 7$ and $n \le 20$. In this figure, B denotes the general lower bound $2n + m + 1$.

	n=1	n=2	n=3	n=4	n=5	n=6	n=7	n=8	n=9	n=10	n=11	n=12	n=13	n=14	n=15	n=16	n=17	n=18	n=19	n=20
m=1	B+0= 4	B+0= 6	B+0= 8	B+0= 10	B+0= 12	B+0= 14	B+0= 16	B+0= 18	B+0= 20	B+0= 22	B+0= 24	B+0= 26	B+0= 28	B+0= 30	B+0= 32	B+0= 34	B+0= 36	B+0= 38	B+0= 40	B+0= 42
m=2		B+0= 7	B+0= 9	B+0= 11	B+0= 13	B+0= 15	B+0= 17	B+0= 19	B+0= 21	B+0= 23	B+0= 25	B+0= 27	B+0= 29	B+0= 31	B+0= 33	B+0= 35	B+0= 37	B+0= 39	B+0= 41	B+0= 43
m=3			B+0= 10	B+1= 13	B+0= 14	B+0= 16	B+0= 18	B+0= 20	B+0= 22	B+0= 24	B+0= 26	B+0= 28	B+0= 30	B+0= 32	B+0= 34	B+0= 36	B+0= 38	B+0= 40	B+0= 42	B+0= 44
m=4				B+0= 13	B+1= 16	B+1= 18	B+0= 19	B+0= 21	B+1= 24	B+0= 25	B+0= 27	B+0= 29	B+0= 31	B+0= 33	B+0= 35	B+0= 37	B+0= 39	B+0= 41	B+0= 43	B+0= 45
m=5					B+0= 16	B+2= 20	B+2= 22	B+2= 24	B+0= 24	B+0= 26	B+1= 29	B+2= 32	B+0= 32	B+0= 34	B+0= 36	B+1= 39	B+0= 40	B+0= 42	B+0= 44	B+0= 46
m=6						B+0= 19	B+2= 23	B+2= 25	B+2= 27	B+2= 29	B+0= 29	B+0= 31	B+1= 34	B+2= 37	B+2= 39	B+0= 39	B+0= 41	B+0= 43	B+1= 46	B+2= 49
m=7							B+0= 22	B+2= 26	B+3= 29	B+3= 31	B+3= 33	B+2= 34	B+0= 34	B+0= 36	B+2= 40	B+2= 42	B+3= 45	B+3= 47	B+0= 46	B+0= 48

Figure 7. The actual value of $f(m,n)$ for $m \le 7$ and $n \le 20$.

Note that $f(5,6) > f(6,6)$ and $f(7,18) > f(7,19)$. Thus increasing m or n may actually decrease f.

When $n = km + r$ with $r \neq 0$, $r \neq 1$,and $k + r < m$, we may use Theorem 1.2 or Theorem 3.2 to find an upper bound for $f(m,n)$. For small m, n, the performance of these two theorems is displayed in Figure 8.

	$f(4,6)$	$f(5,7)$	$f(5,8)$	$f(5,12)$	$f(6,8)$	$f(6,9)$	$f(6,10)$	$f(6,14)$	$f(6,15)$
actual value	$= 18$	$= 22$	$= 24$	$= 32$	$= 25$	$= 27$	$= 29$	$= 37$	$= 39$
Theorem 1.2	≤ 19	≤ 23	≤ 24	≤ 33	≤ 27	≤ 28	≤ 31	≤ 39	≤ 40
Theorem 3.2	≤ 19	≤ 22	≤ 25	≤ 32	≤ 25	≤ 28	≤ 31	≤ 37	≤ 40

	$f(6,20)$	$f(7,9)$	$f(7,10)$	$f(7,11)$	$f(7,12)$	$f(7,16)$	$f(7,17)$	$f(7,18)$
actual value	$= 49$	$= 29$	$= 31$	$= 33$	$= 34$	$= 42$	$= 45$	$= 47$
Theorem 1.2	≤ 51	≤ 31	≤ 32	≤ 33	≤ 37	≤ 45	≤ 46	≤ 47
Theorem 3.2	≤ 49	≤ 30	≤ 31	≤ 34	≤ 37	≤ 42	≤ 45	≤ 48

Figure 8. The performance of two upper bound theorems.

Section 5. Conclusion

We summarize the results concerning the value of $f(m,n)$ here: Assume $m \le n$ and write $n = km + r$, where $0 \le r < m$.
Case (1) If $r = 0$ or $k + r \ge m$, then $f(m,n) = 2n + m + 1$;
Case (2) If $r = 1$, $f(m,n)$ can be evaluated by Theorem 2.4 (or Theorem 2.3);
Case (3) If $m \le 7$ and $n \le 20$, the value of $f(m,n)$ is given in Figure 7 in Section 4.

If (m, n) is not in any of these three cases, then $2n + m + 2 \leq f(m, n) \leq u$, where u is an upper bound obtained from Theorem 1.2 or 3.2. So the value of $f(m, n)$ for general m, n is still undetermined.

For Case (1), Griggs and Ouyang described in [4] all extremal matrices, *i.e.,* the matrices attaining $f(m, n)$. In this study we obtain the actual value of $f(m, n)$ for Case (2). So the extremal matrices for Case (2) deserve further investigation.

As we mentioned in Section 1, the problem of determining $f(m, n)$ is a special case of the famous problem of Zarankiewicz [5]. See [4] for more related open problems.

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