Landau's Theorem Revisited

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Abstract

Two new elementary proofs are given of Landau's Theorem on necessary and sufficient conditions for a sequence of integers to be the score sequence for some tournament. The first is related to existing proofs by majorization, but it avoids depending on any facts about majorization. The second is natural and direct, but a bit more basic than existing proofs. Both proofs are constructive, so they each provide an algorithm for obtaining a tournament realizing a sequence satisfying Landau's conditions.

I. Introduction.

In 1953 H. G. Landau [2] proved that some rather obvious necessary conditions for a non-decreasing sequence of n integers to be the score sequence for some n-tournament are, in fact, also sufficient. Namely, the sequence is a score sequence if and only if, for each k, $1 \le k \le n$, the sum of the first k terms is at least $\binom{k}{2}$, with equality when $k = n$. There are now several proofs of this fundamental result in tournament theory, ranging from clever arguments involving gymnastics with subscripts, arguments involving arc reorientations of properly chosen arcs, arguments by contradiction, arguments involving the idea of majorization, to a constructive argument utilizing network flows and another one involving systems of distinct representatives. Many of these existing proofs are discussed in a 1996 survey by Reid [4]. The notation and terminology here will be as in that survey, except that for vertices x and y, $x \rightarrow y$ will be used to denote both an arc from x to y and the fact that x dominates y, where the context makes clear which use is intended.

In this paper we give a basic self-contained proof and algorithm that is related to known proofs by majorization (Aigner [1] in 1984 and Li [3] in 1986), but it is faster and not dependent on any appeals to chains and covers in lattices. And, we give a new direct proof (and algorithm) that is as basic as any in the literature, and perhaps more natural.

First we give the statement of Landau's Theorem.

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[1] Research supported in part by NSA Grant MSPF-94S-202.
[2] Research supported in part by Office of Naval Research Grant N00014-92-J-1400.

Theorem (Landau [2]). A sequence of integers **s** = ($s_1 \le s_2 \le \ldots \le s_n$), $n \ge 1$, is a score sequence if and only if

$$\sum_{i=1}^{k} s_i \ge \binom{k}{2},\ 1 \le k \le n,\ \text{ with equality for } k = n. \tag{1}$$

All of the published proofs concern the sufficiency of conditions (1) since the necessity follows easily from the observation that if **s** is a score sequence of some n-tournament T, then any k vertices of T form a subtournament W and, hence, the sum of the scores in T of these k vertices must be at least the sum of their scores in W which is just the total number of arcs in W, $\binom{k}{2}$.

II. A Majorization Proof.

Let **s** be an integer sequence satisfying conditions (1). Starting with the transitive n-tournament, denoted TT_n, we successively reverse the orientation of the two arcs in selected 2-paths until we construct a tournament with score sequence **s**.

Suppose that at some stage we have obtained n-tournament U with score sequence **u** = ($u_1, u_2, \ldots, u_{n-1}, u_n$), such that, for $1 \le k \le n$, $\sum_{i=1}^{k} s_i \ge \sum_{i=1}^{k} u_i$ (with equality for $k = n$). This holds initially, when $U = TT_n$, by our hypothesis concerning **s**, since TT_n has score sequence $\mathbf{t}_n = (0, 1, 2, \ldots, n\text{-}1)$. If **u** = **s**, we are done (**s** is the score sequence of U), so suppose that $\mathbf{u} \ne \mathbf{s}$. Let α denote the smallest index such that $u_\alpha < s_\alpha$. Let β denote the largest index such that $u_\beta = u_\alpha$. Since $\sum_{i=1}^{n} s_i = \sum_{i=1}^{n} u_i$ ($= \binom{n}{2}$), by (1) there exists a smallest index $\gamma > \beta$ such that $u_\gamma > s_\gamma$. By maximality of β, $u_{\beta+1} > u_\beta$, and by minimality of γ, $u_\gamma > u_{\gamma\text{-}1}$. We have $(u_1, \ldots u_{\alpha\text{-}1}) = (s_1, \ldots, s_{\alpha\text{-}1})$, $u_\alpha = \ldots = u_\beta < s_\alpha \le \ldots \le s_\beta \le s_{\beta+1}$, $s_{\beta+1} \ge u_{\beta+1}, \ldots, s_{\gamma\text{-}1} \ge u_{\gamma\text{-}1}$, $s_\gamma < u_\gamma$, and, of course, $u_\gamma \le \ldots \le u_n$ and $s_\gamma \le \ldots \le s_n$. Then $u_\gamma > s_\gamma \ge s_\beta > u_\beta$, or $u_\gamma \ge u_\beta + 2$. So, if vertex v_i in U has score u_i, $1 \le i \le n$, there must be a vertex v_λ, $\lambda \ne \beta, \gamma$, such that $v_\gamma \to v_\lambda \to v_\beta$ in U. Reversing this 2-path yields an n-tournament U′ with score sequence **u′** = ($u_1', u_2', \ldots, u_n'$), where

$$u_i' = \begin{cases} u_\gamma - 1 & ,\text{if } i = \gamma \\ u_\beta + 1 & ,\text{if } i = \beta \\ u_i & ,\text{otherwise.} \end{cases}$$

By choice of indices, $u_1' \le u_2' \le \ldots \le u_n'$. It is easy to check that for $1 \le k \le n$, $\sum_{i=1}^{k} s_i \ge \sum_{i=1}^{k} u_i'$.

For n-tuples of real numbers **a** and **b** recall the "Manhattan" metric d(**a**,**b**) = $\sum_{i=1}^{n} |a_i - b_i|$. Then, for the sequences **s**, **u**, **u′** above, d(**u′**, **s**) = d(**u**,**s**) - 2. Now, modulo 2, d(**u**,**s**) $\equiv \sum_{i=1}^{n}(u_i - s_i) = \sum_{i=1}^{n} u_i - \sum_{i=}^{n} s_i = 0$. So, eventually, after (1/2)d($\mathbf{t_n}$,**s**) such steps, we arrive at **u** = **s** and U realizes **s**. Û

III. A New Basic Proof.

The specific sequence $\mathbf{t}_n = (0, 1, 2, \ldots, n-1)$ satisfies conditions (1) as it is the score sequence of the transitive n-tournament. If sequence $\mathbf{s} \neq \mathbf{t}_n$ satisfies (1), then $s_1 \geq 0$ and $s_n \leq n-1$, so $\mathbf{s}$ must contain a repeated term. The object of this proof is to produce a new sequence $\mathbf{s}'$ from $\mathbf{s}$ which also satisfies (1), is "closer" to $\mathbf{t}_n$ than is $\mathbf{s}$, and is a score sequence if and only if $\mathbf{s}$ is a score sequence. Toward this end, define *k* to be the smallest index for which $s_k = s_{k+1}$, and define *m* to be the number of occurrences of the term s_k in $\mathbf{s}$. Note that $k \geq 1$ and $m \geq 2$, and that either $k + m - 1 = n$ or $s_k = s_{k+1} = \ldots = s_{k+m-1} < s_{k+m}$.

Define $\mathbf{s}'$ as follows: for $1 \leq i \leq n$, $$s_i' = \begin{cases} s_i - 1 & \text{if } i = k, \\ s_i + 1 & \text{if } i = k+m-1, \\ s_i & \text{otherwise.} \end{cases}$$

Then $s_1' \leq s_2' \leq \ldots \leq s_n'$.

If $\mathbf{s}'$ is the score sequence of some n-tournament T in which vertex v_i has score s_i', $1 \leq i \leq n$, then, since $s_{k+m-1}' > s_k'+1$, there is a vertex in T, say v_p, for which $v_{k+m-1} \rightarrow v_p$ and $v_p \rightarrow v_k$. Reversal of those two arcs in T yields an n-tournament with with score sequence $\mathbf{s}$. On the other hand, if $\mathbf{s}$ is the score sequence of some n-tournament W in which vertex v_i has score s_i, $1 \leq i \leq n$, then we may suppose that $v_k \rightarrow v_{k+m-1}$ in W, for otherwise, interchanging the labels on v_k and v_{k+m-1} does not change $\mathbf{s}$. Reversal of the arc $v_k \rightarrow v_{k+m-1}$ in W yields an n-tournament with score sequence $\mathbf{s}'$. That is, $\mathbf{s}'$ is a score sequence if and only if $\mathbf{s}$ is a score sequence.

Next, we show that $\sum_{i=1}^{j} s_i > \binom{j}{2}$, $k \leq j \leq k+m-2$. The proof is by induction on $m \geq 2$. The case $m = 2$ is very similar to the induction step and is omitted. Suppose that for some j, $k \leq j < k+m-2$, $\sum_{i=1}^{p} s_i > \binom{p}{2}$ for $k \leq p \leq j$. Conditions (1) imply that $\sum_{i=1}^{j+1} s_i \geq \binom{j+1}{2}$, but we want strict inequality. So, suppose that equality holds in this inequality. Recall that $k < j+2 \leq k+m-1$, so $s_{j+2} = s_{j+1} = \ldots = s_k$. Also, $s_{j+2} = \sum_{i=1}^{j+2} s_i - \sum_{i=1}^{j+1} s_i \geq \binom{j+2}{2} - \binom{j+1}{2} = j+1$. So, $s_{j+1} \geq j+1$. Consequently, by the induction assumption, $\sum_{i=1}^{j+1} s_i = s_{j+1} + \sum_{i=1}^{j} s_i > s_{j+1} + \binom{j}{2} \geq (j+1) + \binom{j}{2} = \binom{j+1}{2} + 1 > \binom{j+1}{2}$, a contradiction to our assumption that $\sum_{i=1}^{j+1} s_i = \binom{j+1}{2}$. So, strict inequality must hold above. This completes the induction step and the proof of the claim.

Now we can show that **s** satisfies (1) if and only if **s′** satisfies (1). If **s** satisfies (1), then

$$\sum_{i=1}^{j} s_i{}' = \begin{cases} \displaystyle\sum_{i=1}^{j} s_i & \text{if } j \le k-1, \\ \displaystyle\sum_{i=1}^{k-1} s_i + (s_k - 1) + \sum_{i=k+1}^{j} s_i & \text{if } k \le j \le k+m-2, \\ \displaystyle\sum_{i=1}^{k-1} s_i + (s_k - 1) + \sum_{i=k+1}^{k+m-2} s_i + (s_{k+m-1} + 1) + \sum_{i=k+m}^{j} s_i & \text{if } j \ge k+m-1. \end{cases}$$

In cases $j \le k-1$ and $j \ge k+m-1$, we see that $\sum_{i=1}^{j} s_i{}' = \sum_{i=1}^{j} s_i \ge \binom{j}{2}$. In cases $k \le j \le k+m-2$, the strict inequality established above implies that $\sum_{i=1}^{j} s_i{}' = (\sum_{i=1}^{j} s_i) - 1 > \binom{j}{2} - 1$. So, **s′** satisfies (1). On the other hand, if **s′** satisfies (1), then it is clear that **s** satisfies (1).

Let us define an order $\prec$ on integer sequences that satisfy (1) as follows: $\mathbf{a} = (a_1, a_2, \ldots, a_n) \prec \mathbf{b} = (b_1, b_2, \ldots, b_n)$ if $a_n = b_n$, $a_{n-1} = b_{n-1}, \ldots, a_{i+1} = b_{i+1}$, $a_i < b_i$, for some i, $1 \le i \le n$. Note that, for any sequence $\mathbf{s} \ne \mathbf{t}_n$ satisfying (1), $\mathbf{s} \prec \mathbf{t}_n$, where $\mathbf{t}_n$ is the fixed sequence (0, 1, 2, . . . ,n-1), the score sequence for the transitive n-tournament. We have shown that for every sequence $\mathbf{s} \ne \mathbf{t}_n$ satisfying (1) we can produce another sequence **s′** satisfying (1) such that $\mathbf{s} \prec \mathbf{s}'$. Moreover, **s** is a score sequence if and only if **s′** is a score sequence. So, by repeated application of this transformation starting from the original sequence satisfying (1) we must eventually reach $\mathbf{t}_n$. Thus, **s** is a score sequence, as required. Û

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