Permutations with Low Discrepancy Consecutive k-sums

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August 22, 2002

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Abstract

Consider the permutation $\pi = (\pi_1, \ldots, \pi_n)$ of $1, 2, \ldots, n$ as being placed on a circle with indices taken modulo n. For given $k \leq n$ there are n sums of k consecutive entries. We say the maximum difference of any consecutive k-sum from the average k-sum is the *discrepancy* of the permutation. We seek a permutation of minimum discrepancy. We find that in general the discrepancy is small, never more than $k+6$, independent of n. For $g = gcd(n,k) > 1$, we show that the discrepancy is $\leq 7/2$. For $g = 1$ it is more complicated. Our constructions show that the discrepancy never exceeds $k/2$ by more than 9 for large n, while it is at least $k/2$ for infinitely many n.
We also give an analysis for the easier case of linear permutations, where we view the permutation as written on a line. The analogous discrepancy is at most 2 for all n, k.

Key words: permutations, discrepancy, k-sums.

1 Introduction

Is it possible to arrange the integers 1 through n on a circle so that, for a given k, any sum of k consecutive integers on the circle is close to the expected value of $k(n+1)/2$? We can do remarkably well.

An exercise in the discrete math text by C.L.Liu [2] gives the following problem for a modified roulette wheel: Show that if numbers 1 through 36 are placed on a wheel in any order, some three consecutive numbers add up to at least 56.

The average of the 36 numbers is $37/2$, so the average of the sums of the 36 consecutive triples is $111/2 = 55.5$ for any ordering. Since the sums are all integer, some triple sum is at least 56.

Not as obvious is whether this value, 56, is best possible. When Liu's problem was assigned by one of the authors to a class several years ago, students were asked to construct a circular permutation of 1 through 36 with the maximum of the sums of consecutive triples as small as possible. Keith Morris produced one with maximum just 57. This is optimal, since a more in-depth analysis shows that 56 is not possible.

The natural extension of the problem is to investigate the minimum, over all circular permutations of 1 through n, of the maximum sum of any k consecutive terms. This was the subject of a Master's thesis by Morris [3], who determined exact values of the minimum maximum consecutive k-sum for small n, k, and obtained various bounds [3]. Note that the minimum maximum consecutive k-sum will always be at least the average k-sum, $k(n+1)/2$. On the other hand, a permutation that achieves the minimum maximum consecutive k-sum may have k-sums far below the average. Indeed, this is the case with some of the constructions of Morris.

Here we investigate permutations where all consecutive k-sums are close to the average. This seemed the correct measure of evenness and is in line with most discrepancy measures

Let S_n denote the set of permutations $\pi = (\pi_1, \ldots, \pi_n)$ of $\{1, \ldots, n\}$, viewed circularly, so that indices are always evaluated modulo n. For $k \leq n$ we define the *discrepancy* of $\pi \in S_n$ by

$$\operatorname{disc}(\pi, k) := \max_{1 \leq i \leq n} \left| \sum_{j=1}^{k} \pi_{i+j} - \frac{k(n+1)}{2} \right|.$$

The discrepancy is the minimum absolute difference of the k-sums from their average. We are here interested in minimizing the discrepancy, so we wish to compute

$$\mathrm{disc}(n,k) := \min_{\pi\in S_n} \mathrm{disc}(\pi,k).$$

We have succeeded in constructing π with small discrepancy, so that the consecutive k-sums are all quite close to the average. This is in contrast to most discrepancy problems where a very non trivial lower bound on discrepancy often arises. In fact, for Liu's roulette case, Theorem 4 shows that $\mathrm{disc}(36,3) = 3/2$.

Let us begin with a few simple observations. First, the complement of the terms in a consecutive k-sum form a complementary consecutive $(n-k)$-sum, so it suffices to analyze the case that $n \geq 2k$, and this is assumed throughout the paper unless stated otherwise. Second, the discrepancy is strictly positive for $n > k$. Third, by looking at the average sum, $k(n+1)/2$, we see that the discrepancy is integer, when k is even or n is odd, while it is a half-integer (an integer plus $1/2$) when k is odd and n is even.

To analyze the rise and fall of k-sums for a given permutation π, we define the n-tuple

$$\mathbf{d} = (d_1, d_2, \ldots, d_n),$$

where

$$d_i = \pi_{i+k} - \pi_i$$

is the difference between consecutive k-sums:

$$d_i = (\pi_{i+1} + \cdots \pi_{i+k}) - (\pi_i + \cdots \pi_{i+k-1}).$$

Adopting notation for strings, we use a^j to mean the string a is repeated j times, such as $(2,(1,-1)^3,-2)$ for the string $(2,1,-1,1,-1,1,-1,-2)$.

The d_i's give us the following bounds on the discrepancy of π:

Proposition 1

$$\frac{1}{2}\max_{s,t}\sum_{j=s}^{t} d_j \leq \mathrm{disc}(\pi,k) < \max_{s,t}\sum_{j=s}^{t} d_j \tag{1}$$

Proof: Observe that $\sum_{j=s}^{t} d_j = \sum_{i=1}^{k} \pi_{t+i} - \sum_{i=0}^{k-1} \pi_{s+i}$ is the difference between two k-sums. The first inequality follows, since, at best, the average k-sum, $k(n+1)/2$, lies halfway in between.

The difference between the maximum and minimum consecutive k-sums is the sum on the right, because we maximize over the choices of s,t. The average k-sum lies strictly between the minimum and the maximum, which yields the strict inequality. ■

For example with $\pi = (1,6,3,2,5,4,7)$ with $n = 7$ and $k = 3$ we find that $3 = \frac{1}{2}\max_{s,t}\sum_{j=s}^{t} d_j \leq 4 = \text{disc}(\pi,k) < 6 = \max_{s,t}\sum_{j=s}^{t} d_j$

Consider a permutation $\pi \in S_n$. It is now natural to look at the sequence $\pi_i, \pi_{i+k}, \pi_{i+2k}, \ldots$ since in this order the values of d_i are made more obvious. When the indices are reduced modulo n, what happens depends on $\gcd(n,k) = g$. We obtain g finite *loops* of length n/g:

$$\text{loop}(i) = (\pi_i, \pi_{i+k}, \ldots, \pi_{i+(n/g-1)k}), \qquad 1 \leq i \leq g.$$

Note that loop(i) occupies the n/g positions $\pi_i, \pi_{i+g}, \pi_{i+2g}, \ldots \pi_{i+(n/g-1)g}$, but we wish to use the order above. To describe π, we need to specify the g loops loop(i).

The results we obtain can be summarized by saying that, apart from the number theoretic conditions of Theorem 11, we can achieve unexpectedly small discrepancy. Section 2 contains a number of special constructions. We show that $\text{disc}(n,2) = 1$ and $\text{disc}(n,3) \leq 2$. Section 3 provides upper bounds when n and k have a common factor. For even k, we find $\text{disc}(mk,k) = 1$, and for odd k, $\text{disc}(mk,k) \leq 2$. If $g := \gcd(n,k)$ is even, we have $\text{disc}(n,k) \leq 2$, while for odd $g > 1$, we have $\text{disc}(n,k) \leq 7/2$. This section also contains a pretty number-theoretical lemma that is interesting on its own (Lemma 8).

Section 4 contains the main general results when $g := \gcd(n,k) = 1$. Lower bounds are provided in Theorem 11: Let $n = ak + r$, where $a > 1$, $\gcd(k,r) = 1$, and $1 \leq r < n$; Let s, b be the smallest positive integers such that $rs = bk \pm 1$; Then $\text{disc}(n,k) \geq \frac{1}{2}\frac{k}{s}$. A special case yielding the largest lower bound is when $n \equiv \pm 1 (\text{mod } k)$, in which case $\text{disc}(n,k) \geq k/2$. Using the result $\text{disc}(n,2) = 1$, it follows that $\text{disc}(n,k) = \frac{k}{2}$ for such n when k is even. For $g = 1$ and odd k, we show that $\text{disc}(n,k) \leq k+6$. In fact, for fixed odd k and sufficiently large n, $\text{disc}(n,k) \leq k/2 + 9$. We offer the conjecture that the lower bound in Theorem 11 may be essentially exact.

In Section 5, we consider the simpler problem of *linear discrepancy*, restricting to linear permutations (not reducing indices modulo n). As before, we can define the linear discrepancy of $\pi \in S_n$ by

$$\text{ldisc}(\pi,k) := \max_{0 \leq i \leq n-k} \left| \sum_{j=1}^{k} \pi_{i+j} - \frac{k(n+1)}{2} \right|.$$

We must impose the restriction $k \le n$ for this to make sense. The definition can be a little suspect since the average of these restricted consecutive k-sums may not be $\frac{k(n+1)}{2}$. In continuous versions of these k-sums, there can be interesting counterintuitive examples [1]. We define $\mathrm{ldisc}(n, k)$ in analogy to $\mathrm{disc}(n, k)$

$$\mathrm{ldisc}(n, k) := \min_{\pi \in S_n} \mathrm{ldisc}(\pi, k).$$

We are able to show that for any n, k, the linear discrepancy can be made extremely small, $\mathrm{ldisc}(n, k) \le 2$.

2 Some Special Cases

The following are some special cases with exact or nearly exact bounds.

Theorem 2 *Let k be odd. Then $\mathrm{disc}(2k, k) = 1/2$ and $\mathrm{disc}(3k, k) = 1$.*

Proof: Note that $g = k$. First consider the case $n = 2k$. There will be k loops $\mathrm{loop}(i)$, $1 \le i \le k$, depending on the parity of i, as follows:

$$\mathrm{loop}(2j-1) = (2j-1, 2j) \qquad \mathrm{loop}(2j) = (2k-2j+2, 2k-2j+1).$$

That is,

$$\pi = (1, 2k, 3, 2k-2, \ldots, 2, 2k-1, 4, 2k-3, \ldots, k+1).$$

We verify that $\mathbf{d} = (1, -1, 1, -1, 1, \ldots)$, so $\max_{s,t} \sum_{j=s}^{t} d_j = 1$ and so, by Proposition 1, $\mathrm{disc}(\pi, k) = 1/2$. Thus $\mathrm{disc}(2k, k) = 1/2$.

For the case $n = 3k$, we again have k loops, with

$$\mathrm{loop}(2j-1) = (3j-2, 3j-1, 3j) \qquad \mathrm{loop}(2j) = (3k-3j+3, 3k-3j+2, 3k-3j+1).$$

We verify that $\mathbf{d} = ((1, -1)^{\lfloor k/2 \rfloor}, 1, (1, -1)^{\lfloor k/2 \rfloor}, 1, (-2, 2)^{\lfloor k/2 \rfloor}, -2)$, and so $\max_{s,t} \sum_{j=s}^{t} d_j = 2$ and as above $\mathrm{disc}(\pi, k) = 1$. Since n is odd, $\mathrm{disc}(3k, k)$ is an integer and so, by Proposition 1, $\mathrm{disc}(3k, k) = 1$. ∎

Theorem 3 *For $n \ge 3$, $\mathrm{disc}(n, 2) = 1$.*

Proof: For $n = 2t + 1$, let

$$\pi = (1, 2t, 3, 2t - 2, 5, 2t - 4, \ldots, 2, 2t + 1).$$

Thus, $\mathbf{d} = ((2, -2)^{\lfloor n/2 \rfloor - 1}, 2, -1, -1)$, and we get $\mathrm{disc}(n, 2) = 1$, since $\mathrm{disc}(n, k) > 0$ for $n > k$.

Similarly, for $n = 4t$, let π be given by the loops

$$\mathrm{loop}(1) = (1, 3, 5, 7, \ldots, 2t - 1, 2t, 2t - 2, \ldots, 4, 2)$$

$$\mathrm{loop}(2) = (4t - 1, 4t - 3, 4t - 5, 4t - 7, \ldots, 2t + 1, 2t + 2, 2t + 4, \ldots, 4t - 2, 4t).$$

Thus, $\mathbf{d} = ((2, -2)^{(n/4)-1}, 1, 1, (-2, 2)^{(n/4)-1}, -1, -1)$.

For $n = 4t + 2$ let π be given by the two loops

$$\mathrm{loop}(1) = (1, 3, 5, 7, \ldots, 2t - 1, 2t + 1, 2t, 2t - 2, \ldots, 4, 2)$$

$$\mathrm{loop}(2) = (4t+1, 4t-1, 4t-3, 4t-5, \ldots, 2t+3, 2t+2, 2t+4, 2t+6 \ldots, 4t, 4t+2).$$

Thus, $\mathbf{d} = ((2, -2)^{\lfloor n/4 \rfloor - 1}, 2, -1, -1, (2, -2)^{\lfloor n/4 \rfloor - 1}, 2, -1, -1)$. ■

The following theorem nearly solves all cases with $k = 3$. It gives a remarkably small upper bound, just 2, on $\mathrm{disc}(n, 3)$. For even n, it shows that the discrepancy is either $1/2$ or $3/2$, since the average triple sum is a half-integer. But a simple argument shows that it cannot be $1/2$ for $n \geq 8$, and we get that $\mathrm{disc}(n, 3) = 3/2$ for all even $n \geq 8$. We find that $\mathrm{disc}(6, 3) = 1/2$ thanks to the (unique up to dihedral group) permutation $\pi = (1, 6, 3, 2, 5, 4)$. For odd n, the situation is a little more complicated. The discrepancy is integral, and it must be either 1 or 2 but Theorem 2 yields $\mathrm{disc}(9, 3) = 1$ and Theorem 11 yields $\mathrm{disc}(7, 3) = 2$.

The roulette wheel example can now be solved! We obtain $\mathrm{disc}(36, 3) = 3/2$ from the permutation, obtained from Case 1 (3 divides n) of the proof below. The permutation is:

$$(1, 18, 35, 4, 16, 34, 7, 15, 32, 10, 13, 31, 12, 14, 29, 11, 17, 28,$$
$$9, 20, 26, 8, 23, 25, 6, 24, 27, 5, 22, 30, 3, 21, 33, 2, 19, 36)$$

Theorem 4 *Let* $n \geq 6$. *Then* $\mathrm{disc}(n, 3) \leq 2$.

Proof: First consider the case $\gcd(n,3) = 1$. There will be a single loop which, informally, starts at 1, rises by 3's until it reaches the top, and then falls back to 2 picking up the missing entries:For $n \equiv 1 (\mathrm{mod}\ 3)$,

$$\text{loop} = (1, 4, 7, \ldots, n, n-1, n-2, n-4, \ldots, 3, 2), \quad n \equiv 1 (\mathrm{mod}\ 3)$$

$$\text{loop} = (1, 4, 7, \ldots, n-1, n, n-2, n-3 \ldots, 3, 2), \quad n \equiv 2 (\mathrm{mod}\ 3).$$

After placing loop into π and computing $\mathbf{d}$ we obtain 4 cases:

$n \equiv 1 (\mathrm{mod}\ 6):\ \pi = (1, (n-1)/2, n-1, \ldots)$

$$\mathbf{d} = ((3, -1, -1, 3, -2, -2)^{(n-7)/6}, 3, -1, -1, 3, -1, -2, -1),$$

$n \equiv 2 (\mathrm{mod}\ 6):\ \pi = (1, n, n/2 - 1, \ldots)$

$$\mathbf{d} = ((3, -2, -1)^{(n-5)/3}, 3, -1, -1, 1, -2),$$

$n \equiv 4 (\mathrm{mod}\ 6):\ \pi = (1, n/2, n-1, \ldots)$

$$\mathbf{d} = ((3, -2, -1, 3, -1, -2)^{(n-4)/6}, 3, -1, -1, -1),$$

$n \equiv 5 (\mathrm{mod}\ 6):\ \pi = (1, n, (n-1)/2, \ldots)$

$$\mathbf{d} = ((3, -2, -2, 3, -1, -1)^{(n-5)/6}, 3, -2, -1, 1, -1).$$

We can check that for even n, $\mathrm{disc}(n,3) \leq 3/2$ and for odd n, $\mathrm{disc}(n,3) \leq 2$. (Note that Theorem 11 shows that these are tight). Let us give as an example the first case, with $\mathbf{d} = ((3, -1, -1, 3, -2, -2)^{(n-7)/6}, 3, -1, -1, 3, -1, -2, -1)$. Let $\pi_1 + \pi_2 + \cdots + \pi_k = x$. With $d_1 = 3$ we get $\pi_2 + \pi_2 + \cdots + \pi_{k+1} = x + 3$. Thus using $\mathbf{d}$, the successive k-sums are $(x+3, x+2, x+1, x+4, x+2, x)^{(n-7)/6}, x+3, x+2, x+1, x+4, x+3, x+1, x$. The sum of all k-sums is now $(4x+8)(n-7)/6 + (2x+4)(n-7)/6 + 7x + 14$. We compute $x + 2 = (n+1)k/2$ and all k-sums are in $\{x, x+1, x+2, x+3, x+4\}$ and so $\mathrm{disc}(\pi, k) = 2$.

For the remainder, suppose 3 divides n. We split this into six cases modulo 18. In each case we must specify 3 loops $\text{loop}(i) = \sigma^{(i)}$, where $\sigma^{(i)} = (\sigma_1^{(i)}, \sigma_2^{(i)}, \ldots)$, $1 \leq i \leq 3$, is built from the sets $\{1, 2, \ldots, n/3\}$, $\{n/3 + 1, n/3 + 2, \ldots, 2n/3\}$, $\{2n/3 + 1, 2n/3 + 2, \ldots, n\}$, for $1 \leq i \leq 3$, respectively. Each $\sigma^{(i)}$ starts near the bottom, rises by 3's to near the top and then returns to the bottom using the remaining symbols. However, for the loops, we rotate the starting points of the $\sigma^{(i)}$'s to reduce the discrepancy when the three loops are interwoven. In each case, loop(1) starts at the bottom of its range, with 1, loop(2) starts at the middle of its range, $\lfloor n/2 \rfloor$, and loop(3) starts near its top, at $n-1$. As an example for $n = 18$, we have $\pi = (1, 9, 17, 4, 7, 16, 6, 8, 14, 5, 11, 13, 3, 12, 15, 2, 10, 18)$. Let $f(x_1, x_2, x_3, \ldots)$ denote the vector of successive differences for $(x_1, x_2, x_3, \ldots)$, namely $f(x_1, x_2, x_3, \ldots) = (x_2 - x_1, x_3 - x_2, \ldots)$. We give only these vectors since they determine $\mathrm{disc}(\pi, k)$. For our particular π we have $f(\mathrm{loop}(1)) = (3, 2, -1, -2, -1, -1)$, $f(\mathrm{loop}(2)) = (-2, 1, 3, 1, -2, -1)$ and $f(\mathrm{loop}(3)) = (-1, -2, -1, 2, 3, -1))$

Case 1. $n = 18p$. We have

$$\begin{aligned} f(\mathrm{loop}(1)) &= (3^{2p-1}, 2, (-1, -2)^{2p-1}, -1, -1) \\ f(\mathrm{loop}(2)) &= ((-2, -1)^{p-1}, -2, 1, 3^{2p-1}, 1, (-2, -1)^{p-1}, -2, -1) \\ f(\mathrm{loop}(3)) &= ((-1, -2)^{2p-1}, -1, 2, 3^{2p-1}, -1). \end{aligned}$$

Our choice yields $\mathbf{d} = ((3, -2, -1, 3, -1, -2)^{p-1}, 3, -2, -1, 2, 1, -2, -1,$ $(3, -1, -2, 3, -2, -1)^{p-1}, 3, -1, -2, 1, 2, -1, -2, (3, -2, -1, 3, -1, -2)^{p-1}, 3, (-1)^3)$, and we can check $\mathrm{disc}(\pi, 3) = 3/2$.

Case 2. $n = 18p + 3$. We have

$$\begin{aligned} f(\mathrm{loop}(1)) &= (3^{2p}, -1, (-1, -2)^{2p-1}, -1, -1) \\ f(\mathrm{loop}(2)) &= ((-1, -2)^{p-1}, -1, -1, 3^{2p}, -1, (-1, -2)^{p}) \\ f(\mathrm{loop}(3)) &= ((-1, -2)^{2p-1}, -1, -1, 3^{2p}, -1). \end{aligned}$$

Our choice yields $\mathbf{d} = ((3, -1, -1, 3, -2, -2)^{p-1}, 3, -1, -1, 3,$ $-1, -2, -1, (3, -1, -1, 3, -2, -2)^{p-1}, 3, -1, -1, 3, -1, 2, -1,$ $3, -1, -1, (3, -2, -2, 3, -1, -1)^{p-1}, 3, -1, -2, -1)$, and we can check $\mathrm{disc}(\pi, 3) = 2$.

Case 3. $n = 18p + 6$. We have

$$\begin{aligned} f(\mathrm{loop}(1)) &= (3^{2p}, 1, (-2, -1)^{2p}, -1) \\ f(\mathrm{loop}(2)) &= ((-2, -1)^{p}, 2, 3^{2p-1}, 2, (-1, -2)^{p}, -1) \\ f(\mathrm{loop}(3)) &= ((-1, -2)^{2p}, 1, 3^{2p}, -1). \end{aligned}$$

Our choice yields $\mathbf{d} = ((3, -2, -1, 3, -1, -2)^{p}, 1, 2, -1, -2,$ $(3, -2, -1, 3, -1, -2)^{p-1}, 3, -2, -1, 2, 1, -2, -1,$ $(3, -1, -2, 3, -2, -1)^{p-1}, 3 - 1, -2, 3, (-1)^3)$ and we can check $\mathrm{disc}(\pi, 3) = 3/2$.

Case 4. $n = 18p + 9$. We have

$$\begin{aligned} f(\text{loop}(1)) &= (3^{2p}, 2, (-1,-2)^{2p}, -1, -1) \\ f(\text{loop}(2)) &= ((-1,-2)^{p}, 1, 3^{2p}, 1, (-2,-1)^{p}, -2) \\ f(\text{loop}(3)) &= ((-1,-2)^{2p}, -1, 3^{2p}, 2, -1). \end{aligned}$$

Our choice yields $\mathbf{d} = ((3,-1,-1,3,-2,-2)^p, 2, 1, -1, -1,$ $(3,-2,-2,3,-1,-1)^p, 1, 2, (-2,-2,3,-1,-1,3)^p, -1, -2, -1)$ and we can check $\text{disc}(\pi, 3) = 2$.

Case 5. $n = 18p + 12$. We have

$$\begin{aligned} f(\text{loop}(1)) &= (3^{2p+1}, -1, (-1,-2)^{2p}, -1, -1) \\ f(\text{loop}(2)) &= ((-2,-1)^{p}, -1, 3^{2p+1}, -1, (-1,-2)^{p}, -1) \\ f(\text{loop}(3)) &= ((-1,-2)^{2p}, -1, -1, 3^{2p+1}, -1). \end{aligned}$$

Our choice yields $\mathbf{d} = ((3,-2,-1,3,-1,-2)^p, 3, -1, -1, -1,$ $(3,-2,-1,3,-1,-2)^p, 3, -1, -1, -1, (3,-2,-1,3,-1,-2)^p, 3, (-1)^3)$ and we can check $\text{disc}(\pi, 3) = 3/2$.

Case 6. $n = 18p + 15$. We have

$$\begin{aligned} f(\text{loop}(1)) &= (3^{2p+1}, 1, (-2,-1)^{2p+1}, -1) \\ f(\text{loop}(2)) &= (-1, (-2,-1)^{p}, 2, 3^{2p}, 2, (-1,-2)^{p+1}) \\ f(\text{loop}(3)) &= ((-1,-2)^{2p+1}, 1, 3^{2p+1}, -1). \end{aligned}$$

Our choice yields $\mathbf{d} = ((3,-1,-1,3,-2,-2)^p, 3, -1, -1, 1, 2,$ $-2, -2, (3,-1,-1,3,-2,-2)^p, 2, 1, -1, -1, (3,-2,-2,3,-1,-1)^p, 3, -1, -2, -1)$ and we can check $\text{disc}(\pi, 3) = 2$. ■

Here is a simple upper bound when k is factored:

Proposition 5 $\text{disc}(n, pq) \leq p \cdot \text{disc}(n, q)$.

Proof: A permutation with $\text{disc}(\pi, q) = t$ will have $\text{disc}(\pi, pq) \leq pt$. ■

Corollary 6 *For even* k, $\text{disc}(n, k) \leq k/2$.

Proof: For $p = k/2$ and $q = 2$, simply use the permutations π given in Theorem 3. ■

3 Bounds When n and k Have a Common Factor

As a warm up for the construction in Theorem 9 consider the following.

Theorem 7 *If k is even, then* $\mathrm{disc}(mk, k) = 1$*. If k is odd, then* $\mathrm{disc}(mk, k) \leq 2$.

Proof: To construct π we need k loops.

First consider even k. Let $\sigma(1), \sigma(2)$ be $\mathrm{loop}(1), \mathrm{loop}(2)$ from the constructions in Theorem 3, with n replaced by $2m$, depending on the parity of m. Then, $\sigma(1)$ has the symbols $1, 2, \ldots, m$, while $\sigma(2)$ has the symbols $m+1, m+2, \ldots, 2m$. Use the notation $\vec{1} = (1)^m$. We define

$$\mathrm{loop}(1) = \sigma(1), \quad \mathrm{loop}(2) = \sigma(2),$$
$$\mathrm{loop}(3) = (4m+1) \cdot \vec{1} - \sigma(2), \quad \mathrm{loop}(4) = 2m \cdot \vec{1} + \sigma(2),$$
$$\mathrm{loop}(5) = (6m+1) \cdot \vec{1} - \sigma(2), \quad \mathrm{loop}(6) = 4m \cdot \vec{1} + \sigma(2), \quad \ldots.$$

For the resulting π, compute $\mathbf{d}$ by using the vector $\mathbf{d}$ computed in Theorem 3 which combines the two vectors $f(\mathrm{loop}(1)), f(\mathrm{loop}(2))$ alternately starting with the entry of $f(\mathrm{loop}(1))$ for $\mathrm{loop}(1)$. Let i^* refer to the vector of $k-1$ entries $(i, -i, i, -i, \ldots, i) = (i, (-i, i)^{(k/2)-1})$. For even m,

$$\mathbf{d} = ((2, -2^*)^{(m/2)-1}, 1, 1^*, (-2, 2^*)^{(m/2)-1}, -1, -1^*).$$

For odd m,

$$\mathbf{d} = ((2, -2^*)^{\lfloor m/2 \rfloor - 1}, 2, -1^*, -1, 2^*(-2, 2^*)^{\lfloor m/2 \rfloor - 1}, -1, -1^*).$$

We can now check that $\mathrm{disc}(mk, k) = 1$.

For the remainder, assume k is odd. Let $n = 3m$. Then let $\sigma(i)$, $1 \leq i \leq 3$, be the three loops $\mathrm{loop}(i)$ from the constructions in the cases, 3 divides n, of the proof of Theorem 4. In particular, $\sigma(3)$ has the numbers $2m+1, 2m+2, \ldots, 3m$. Now define the rest of the loops (for $k > 3$):

$$\mathrm{loop}(4) = (5m+1) \cdot \vec{1} - \sigma(3), \quad \mathrm{loop}(5) = 2m \cdot \vec{1} + \sigma(3),$$

$$\mathrm{loop}(6) = (7m+1) \cdot \vec{1} - \sigma(3), \quad \mathrm{loop}(7) = (4m) \cdot \vec{1} + \sigma(3), \quad \ldots.$$

We check as before that the discrepancy is bounded as promised. ■

The following easy lemma shows that a particular construction for placing 1/2's and $-1/2$'s around a circle result in low discrepancy for consecutive b-sums and it is used in Case 2 in Theorem 9.

Lemma 8 *Let a, b be integers with a even, b odd, and* $\gcd(a, b) = 1$. *Form a sequence $s = (s_1, s_2, \ldots, s_a)$ with $s_i = -1/2$ if $i \equiv e \cdot b (mod\ a)$ for $1 \leq e \leq a/2$ and $s_i = 1/2$ if $i \equiv e \cdot b(mod\ a)$ for $c/2 + 1 \leq e \leq a$. Then $\sum_{j=t}^{t+b-1} s_j \in \{-1/2, 1/2\}$ for any t where indices are taken modulo a.*

Proof: The role of k is taken by b. Note that with two exceptions, $s_{i+b} = s_i$. As a result, the numbers $d_i = s_{i+b} - s_i$ are 0 for all but two values and the other two values are $1/2, -1/2$. The sum of all the entries of s is 0 and so the sum of all consecutive b-sums is 0. We deduce that half the consecutive b-sums on s are 1/2 and half are $-1/2$. ■

Here is the main upper bound theorem when $\gcd(n, k) > 1$.

Theorem 9 *Let $g = \gcd(n, k)$ and assume $g > 1$. Then $\text{disc}(n, k) \leq 2$ for even g, and $\text{disc}(n, k) \leq 7/2$ for odd g.*

Proof: To give π we need to specify the g loops $\text{loop}(1), \text{loop}(2), \ldots, \text{loop}(g)$. A special permutation $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{n/g})$, satisfying

$$\sigma_{i+1} - \sigma_i \in \{-2, -1, 1, 2\},$$

can be given for even n/g by

$$\sigma = (1, 3, 5, \ldots, n/g - 1, n/g, n/g - 2, n/g - 4, \ldots, 2)$$

and for odd n/g by

$$\sigma = (1, 3, 5, \ldots, n/g, n/g - 1, n/g - 3, n/g - 5, \ldots, 2).$$

Case 1. g is even (and so k is even).

We obtain π by placing $\sigma + (ln/g) \cdot \vec{1}$ in $\text{loop}(1+2l)$ and placing $(n + 1) \cdot \vec{1} - (\sigma + (ln/g) \cdot \vec{1})$ in $\text{loop}(2 + 2l)$, for $0 \leq 2l \leq g - 2$. We note that π is indeed a permutation of $1, 2, \ldots, n$, and for odd i, $\pi_i + \pi_{i+1} = n + 1$. Thus for any t, $\sum_{j=1}^{k} \pi_{2t+j} = k(n+1)/2$. Now the entries d_i come from successive differences in a given loop and by construction $d_i \in \{-2,-1,1,2\}$. Hence $|d_i| \le 2$. We deduce $\mathrm{disc}(\pi,k) \le 2$.

Case 2. g is odd.

We begin as in Case 1 by obtaining π by placing $\sigma+(ln/g)\cdot\vec{1}$ in loop$(1+2l)$ and placing $(n+1)\cdot\vec{1}-(\sigma+(ln/g)\cdot\vec{1})$ in loop$(2+2l)$, for $0 \le 2l \le g-3$. As above $\pi_{gs+i}+\pi_{gs+i+1} = n+1$ for odd i, and $i \le g-4$ and any s. We have yet to assign the $3n/g$ numbers j with $(g-3)/2\cdot n/g < j \le (g+3)/2\cdot n/g$ into the three loops loop$(g-2)$, loop$(g-1)$, loop(g). We use a special scheme for $k=3$. For odd n/g with $3n/g = 6p+3$ we define

$$
\begin{array}{lcccccccc}
\sigma(1) = & 3p+1 & 3p-2 & 3p-5 & \dots & 1 & 2 & 5 & \dots & 3p-1 \\
\sigma(2) = & 3 & 9 & 15 & \dots & 6p+3 & 6p & 6p-6 & \dots & 6 \\
\sigma(3) = & 6p+2 & 6p-1 & 6p-5 & \dots & 3p+2 & 3p+4 & 3p+7 & \dots & 6p+1
\end{array}
$$

and for even n/g with $3n/g = 6p$ we define

$$
\begin{array}{lcccccccc}
\sigma(1) = & 3p-1 & 3p-4 & 3p-7 & \dots & 2 & 1 & 4 & \dots & 3p-2 \\
\sigma(2) = & 3 & 9 & 15 & \dots & 6p-3 & 6p & 6p-6 & \dots & 6 \\
\sigma(3) = & 6p-1 & 6p-4 & 6p-7 & \dots & 3p+2 & 3p+1 & 3p+4 & \dots & 6p-2
\end{array}
$$

We place $\sigma(i)+(g-3)/2\cdot(n/g)\cdot\vec{1}$ in loop$(g-3+i)$ for $1 \le i \le 3$.

For odd n/g we check that $\pi_{gs+g-2}+\pi_{gs+g-1}+\pi_{gs+g} = 3(n+1)/2$ and so $\pi_{gs+1}+\pi_{gs+2}+\pi_{gs+3}+\cdots+\pi_{gs+g} = g(n+1)/2$. And so the $k-sum$ beginning with π_{gs+1} is $k(n+1)/2$. Then using $d_{gs+2l+1}+d_{gs+2l+2} = 0$ for $0 \le 2l \le g-3$ and $|d_{gs+i}| \le 2$ for $1 \le i \le g-3$, and $|d_{gs+g-2}|$ and $|d_{gs+g-2}+d_{gs+g-1}|$ are at most 3, we conclude $\mathrm{disc}(\pi,k) \le 3$.

For even n/g, we compute $\pi_{gs+g-2}+\pi_{gs+g-1}+\pi_{gs+g} = 3(n+1)/2-1/2$ for $0 \le s \le p-1$ and $\pi_{gs+g-2}+\pi_{gs+g-1}+\pi_{gs+g} = 3(n+1)/2+1/2$ for $p \le s \le 2p-1$. We use Lemma 8 with $a = n/g$ and $b = k/g$ to deduce that for any s, $\sum_{j=1}^{k}\pi_{gs+j}$ is either $k(n+1)/2+1/2$ or $k(n+1)/2-1/2$. Knowing that $d_{gs+2l+1}+d_{gs+2l+2} = 0$ and $|d_{gs+2l+1}| = 2$ for $0 \le 2l \le g-3$ and $|d_{gs+g-2}| \le 3$, and $|d_{gs+g-2}+d_{gs+g-1}| \le 3$ we conclude $\mathrm{disc}(\pi,k) \le 7/2$. ■

The special scheme $\sigma(1),\sigma(2),\sigma(3)$ for $k=3$ is not taken from Theorem 4, but the loops yield a permutation π with $\mathrm{disc}(\pi,3) \le 7/2$ (i.e., not enough for Theorem 4, but where $\sigma(1)+\sigma(2)+\sigma(3)$ has special properties).

The following result shows that improvements to Theorem 9 are sometimes possible if $\mathrm{disc}(n/g,k/g) < 2$.

Theorem 10 *Let n, k be given with $g = \gcd(n, k) > 1$ being odd. Then*

$$\text{disc}(n, k) \leq \text{disc}(n/g, k/g).$$

Proof: To construct a $\pi \in S_n$ with $\text{disc}(\pi, k) \leq \text{disc}(n/g, k/g)$ we start with a permutation $\tau \in S_{n/g}$ with $\text{disc}(\tau, k/g) = \text{disc}(n/g, k/g)$. Compute $\mathbf{e} = (e_1, e_2, \ldots, e_{n/g})$ as $e_i = \tau_{i+k/g} - \tau_i$ (where the indices are taken modulo n/g). Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_{n/g})$ where $\sigma_i = \tau_{1+(i-1)k/g}$. Thus σ is the loop generating τ. We form π by specifying g loops:

$$\text{loop}(1) = \sigma, \quad \text{loop}(2) = (\frac{2n}{g} + 1) \cdot \vec{1} - \sigma, \quad \text{loop}(3) = 2n/g \cdot \vec{1} + \sigma,$$

$$\text{loop}(4) = (\frac{4n}{g} + 1) \cdot \vec{1} - \sigma, \quad \text{loop}(5) = 4n/g \cdot \vec{1} + \sigma, \ldots$$

If we let a^* denote the vector $(a, -a, a, -a, \ldots, a)$ of length g, then we verify that $\mathbf{d}$ computed from π has $\mathbf{d} = (e_1^*, e_2^*, \ldots, e_{n/g}^*)$. Thus $\text{disc}(\pi, k) = \text{disc}(\tau, k/g)$. ∎

4 Bounds for Relatively Prime n and k

The following theorem identifies cases where non trivial lower bounds can be established. As we will conjecture later, these bounds may be exact or nearly so. An easy and important case is where $r = s = 1$ and $b = 0$.

Theorem 11 *Let n, k be given with $\gcd(n, k) = 1$, $n > 2k$. Set $n = ak + r$, where $1 \leq r \leq k - 1$, and s is the smallest positive integer such that $rs = \pm 1(mod\ k)$. Then*

$$\text{disc}(n, k) \geq \frac{k}{2s}.$$

Proof: For some $b \geq 0$, we have $rs = bk \pm 1$. This gives $sn = ask + rs = (as + b)k \pm 1$. We seek a lower bound on $\text{disc}(n, k)$ by seeking lower bounds on $\sum_{j=s}^{t} d_j$. Consider a permutation π with $\pi_1 = 1$ and assume $\pi_{1+pk} = n$. Since $(as + b)k \equiv \mp 1 (\text{mod } n)$, the $p + 1$ entries $\pi_1, \pi_{1+k}, \pi_{1+2k}, \ldots, \pi_{1+pk} = n$ will be in $as + b$ separate sets of consecutive positions, consecutive in π. We deduce from $\sum_{j=0}^{p-1} d_{1+jk} = n - 1$ that for one of the sets of consecutive entries, say $\pi_s, \pi_{s+1}, \ldots, \pi_t$ we have

$$\sum_{j=s}^{t} d_j \geq \frac{n-1}{as+b} = \frac{k(n-1)}{sn \mp 1} = \frac{k}{s} - \frac{1}{s}\left(\frac{k(s \mp 1)}{sn \mp 1}\right) = \frac{k}{s} - \frac{1}{s}\left(\frac{s \mp 1}{as+b}\right).$$

Note that for $n > 2k$, we have $a \geq 2$. Then where $rs \equiv 1 (\bmod\ k)$, we have $\frac{s-1}{as+b} \leq \frac{s-1}{2s} < 1$ and where $rs = bk - 1$, we have $b \geq 1$ and $\frac{s+1}{as+b} \leq \frac{s+1}{2s+1} < 1$. In either case, $\frac{k}{s} - \frac{1}{s}\left(\frac{s \mp 1}{as+b}\right) > \frac{k-1}{s}$. Because $\sum_{j=s}^{t} d_j$ is integral we have $\sum_{j=s}^{t} d_j \geq \frac{k}{s}$ and in fact $\sum_{j=s}^{t} d_j \geq \left\lceil \frac{k}{s} \right\rceil$.
Thus, $\operatorname{disc}(\pi, k) \geq \frac{k}{2s}$ using our bound (1) of Proposition 1. ■

Corollary 12 *Let n, k be given with k even and $n \equiv \pm 1 (\bmod\ k)$. Then* $\operatorname{disc}(n, k) = k/2$.

Proof: We use Theorem 11 combined with Corollary 6. ■

The following provides a general upper bound that is surely not tight. However, in conjunction with Theorem 9, it shows that the discrepancy does not grow with large n.

Theorem 13 *Let k be odd and* $\gcd(n, k) = 1$. *Then*

$$\operatorname{disc}(n, k) \leq k + 6.$$

Moreover for fixed k, the given construction shows for $n > n_o(k)$ that

$$\operatorname{disc}(n, k) \leq \frac{k}{2} + 9.$$

Proof: Consider a permutation $\pi \in S_n$. The infinite sequence $\pi_i, \pi_{i+k}, \pi_{i+2k}, \ldots$ becomes, when the indices are reduced modulo n, a finite 'loop' of length n and so a permutation $\sigma \in S_n$. Thus to specify π we need to specify σ. For $n = ak + b$ with $1 \leq b < k$ we construct the permutation σ as follows. Informally, it rises by 3's for a times and then rises by 2's until it reaches n and then drops back to 1 picking up the remaining numbers and hence initially goes down by 2's and then alternating goes down by 1's and 2's for the last $2a$ entries. To use up all the numbers 1 through n requires some constant number of alterations to the pattern. For even $n - a$ we set

$$\sigma = (1, 4, 7, \ldots, 3a+1, 3a+3, 3a+5, \ldots, n-1, n, n-2, n-4, \ldots$$

$$\ldots, 3a+2, 3a, 3a-1, 3a-3, \ldots, 5, 3, 2),$$

while if $n - a$ is odd, we set

$$\sigma = (1, 4, 7, \ldots, 3a+1, 3a+3, 3a+5, \ldots, n-2, n, n-1, n-3, \ldots$$

$$\ldots, 3a+2, 3a, 3a-1, 3a-3, \ldots, 5, 3, 2).$$

We obtain π by setting $\pi_{1+ik} = \sigma_{1+i}$. We have 2 cases. Either $n-a$ is even (and so b is even as well) or $n-a$ is odd (and b is odd as well). We will only deal with the case $n-a$ and b are both even in what follows, leaving the other case to the reader. We deduce:

- $d_i = 3$ for the a positions in π, $i = 1, 1+k, \ldots, n-b+1-k$. Recall $1+ak = n-b+1$.
- $d_i = 2$ for the $\frac{(k-3)}{2}a + \frac{b-2}{2}$ positions $i = -b+1, k-b+1, \ldots, 1-2k+\frac{b}{2}$.
- $d_i = 1$ for position $i = \frac{b}{2}+1-k$.
- $d_i = -2$ for the $\frac{(k-3)}{2}a + \frac{b-2}{2}$ positions $i = \frac{b}{2}+1, k+\frac{b}{2}+1, \ldots, 2b-2k+1$.
- $(d_i, d_{i+k}) = (-2, -1)$ for the a pairs $(i, i+k) = (2b-k+1, 2b+1), \ldots, (-3k+1, -2k+1)$.
- $d_i = -1$ for $i = 1-k$.

In general $d_i = d_{i+k}$ or (from the -2,-1 pairs) $d_i = d_{i+2k}$ with $|d_i| = 1$ or 2. Note that the transition positions i where this may not occur are in positions $-b-k+1, 1-2k+\frac{b}{2}, \frac{b}{2}-k+1, 2b-2k+1, -2k+1$ which are all close to the 1 position and are in an interval $[-2k+1, \max\{1+\frac{b}{2}-k, 2b-2k+1\}]$ that has $< 2k$ positions, independently of a.

For $S_i = \sum_{j=0}^{k-1} \pi_{i+j}$ and $D_i = \sum_{j=0}^{k-1} d_{i+j}$, we have $S_i - S_{i-k} = D_i$. For $\max\{1+\frac{b}{2}-k, 2b-2k+1\} < i < n-3k-2$ the terms in the sum D_i consist of 1 value of 3, $(k-3)/2$ values each of 2 and -2, and 2 values from $\{-2,-2\}$, $\{-2,-1\}$ or $\{-1,-1\}$. Thus $D_i = -1$,0 or 1. If $D_i = 0$ then $D_{i+k} = 0$ and if $D_i = \pm 1$ then $D_{i+k} = \mp 1$, as long as $i+2k-1$ is in the same range. Hence, as long as i and $i+(j+1)k-1$ are outside the specified transition range, $S_{i+jk} - S_i = -1, 0$ or 1.

In the transition range for the strings of d_i's, we have $d_i - d_{i-k} = -1$ in the 2 cases $3 \to 2$ and $2 \to 1$, $d_i - d_{i-k} = 1$ in the shift $(-2,-2) \to (-2,-1)$, $d_i - d_{i-k} = -3$ in the shift $1 \to -2$, and $d_i - d_{i-k} = 4$ in the shift $-1 \to 3$. Through this transition range we then have $|S_{i+jk} - S_i| \le 5$ for $j = 0, 1, 2$. Thus, as long as $j \le a$, we have $|S_{i+jk} - S_i| \le 7$. In particular, we have $|S_{i+b} - S_i| \le 7$. This is generous, since the differences described tend, more often than not, to cancel each other.

If S_u, S_l are the maximum and minimum k-sums, respectively, in π, then $S_u - S_l \leq S_u - S_t + 7$, where $t = l + jk$ with $|j| \leq a$ and $|u - t| < \frac{k}{2}$. Now, $S_u - S_t$ is a sum of $|u - t|$ of the d_i's, so that $|S_u - S_t| \leq 2|u - t| + 1 \leq k$, since in any span of k positions there is at most 1 value $d_i = 3$ and for the rest $|d_i| \leq 2$. Thus $S_u - S_l \leq k + 7$ and then the average lies between the maximum and the minimum.

We follow the same proof technique in the case $n - a$ is odd (and b is odd).

Now with k fixed and $n \to \infty$ we use the fact that $|(S_{k+1} + S_{k+2} + \cdots + S_{2k}) - (S_{pk+1} + S_{pk+2} + \cdots + S_{(p+1)k})|$ is at most k, for even p, and 0, for odd p, where $p \leq n/k - 1$. Then *for large* n the k k-sums $S_{k+1}, S_{k+2}, \ldots, S_{2k}$ will dominate as an estimate for the average for all k-sums since there will only be $2k$ sums we are not considering and their values are within $k + 6$ of the average value by our work above. Hence,

$$\left|(S_{k+1} + S_{k+2} + \cdots + S_{2k}) - \frac{k^2(n+1)}{2}\right| < k.$$

Thus, the average value of a k-sum among $S_{k+1}, S_{k+2}, \ldots, S_{2k}$ is within 1 of $k(n+1)/2$. Now $|S_{k+i} - S_{k+i-1}| = |d_{k+i}| \leq 2$ for $2 \leq i \leq k$ and $|S_{k+1} - S_{2k}| \leq |S_{k+1} - S_{2k+1}| + |S_{2k+1} - S_{2k}| \leq 1 + 3 = 4$. This restricts things, remembering that the average of $S_{k+1}, S_{k+2}, \ldots, S_{2k}$ is within 1 of $k(n+1)/2$ so that

$$\left(\max_{1 \leq i \leq k} S_{k+i}\right) - \frac{k(n+1)}{2} \leq \frac{k}{2} + 2, \qquad \frac{k(n+1)}{2} - \min_{1 \leq i \leq k} S_{k+i} \leq \frac{k}{2} + 2.$$

This could be achieved when the k k-sums start up by 2's and then decrease by 2's with the indices of the sums considered modulo k. We now deduce that no k sum is more than 7 away from the k k-sums $S_1, S_2, \ldots, S_k$ by our previous arguments and so $\text{disc}(n,k) \leq k/2 + 9$. ■

We can now say something about the discrepancy for fixed k as n grows, by applying Theorem 9, Corollary 12, and Theorem 13: $\text{disc}(n,k)$ never exceeds $k/2$ by more than 9 for large n, while it is at least $k/2$ for infinitely many n.

The following conjecture suggests that the exact values for the discrepancies may be within reach.

Conjecture 14 *Let n, k be given with $\gcd(n,k) = 1$. Then the bound of Theorem 11 is exact in the cases either k is even or both k and n are odd. In the case k is odd and n is even we believe the bound is off by at most 1.*

Computer experiments have been run for $k = 402$ and $2011 \leq n \leq 2209$ and as well for $k = 127$ and $382 \leq n \leq 507$ (for the cases $\gcd(n,k) = 1$) obtaining constructions that support this conjecture. It is reminiscent of design theory problems, and quite different from standard discrepancy results, that there seem to always be enough choices to find a sequence achieving the bound or getting within 1.

5 Linear Permutations

The case where we view the permutations on a line turns out to be easier. There are only $n-k+1$ consecutive k-sums to consider with no wraparound. The constructions are simpler and there seems to be more choices that give low discrepancy. Recall that we defined the linear discrepancy ldisc in the introduction.

Theorem 15 *Let n, k be given. Then* $\mathrm{ldisc}(n,k) \leq 2$.

Proof: Instead of loops we define π using k strands with $\mathrm{strand}(i) = (\pi_i, \pi_{i+k}, \pi_{i+2k}, \ldots \pi_{i+pk})$ for $1 \leq i \leq k$ and $p = \lfloor (n-i)/k \rfloor$. Also in place of $\mathbf{d}$ we define $\hat{\mathbf{d}}$ as

$$\hat{\mathbf{d}} = (\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_{n-k}) = (\pi_{k+1} - \pi_1, \pi_{k+2} - \pi_2, \ldots, \pi_n - \pi_{n-k})$$

where the missing k entries of $\mathbf{d}$ are not relevant to ldisc.

Case 1. n, k are even.

Let $n = ak + r$ with $0 \leq r \leq k-1$. Then r is even. If $r = 0$ we can use the result from Theorem 7 to achieve discrepancy 1. Assume $r \geq 1$. Use the notation $\vec{1}$ to denote the vector of 1's of appropriate size. Define

$$\begin{aligned}
\mathrm{strand}(1) &= (1, 2, \ldots, a+1) \\
\mathrm{strand}(2) &= (n+1) \cdot \vec{1} - \mathrm{strand}(1) \\
\mathrm{strand}(3) &= (a+1) \cdot \vec{1} + \mathrm{strand}(1) \\
\mathrm{strand}(4) &= (n+1) \cdot \vec{1} - \mathrm{strand}(3) \ldots \\
\mathrm{strand}(r-1) &= (r/2 - 1)(a+1) \cdot \vec{1} + \mathrm{strand}(1)
\end{aligned}$$

$$
\begin{aligned}
\text{strand}(r) &= (n+1)\cdot\vec{1} - \text{strand}(r-1) \\
\text{strand}(r+1) &= ((r/2)(a+1)+1, (r/2)(a+1)+2, \ldots, (r/2)(a+1)+a) \\
\text{strand}(r+2) &= (n+1)\cdot\vec{1} - \text{strand}(r+1), \ldots \\
\text{strand}(k-1) &= (((k-r)/2)-1)\cdot\vec{1} + \text{strand}(r-1) \\
\text{strand}(k) &= (n+1)\cdot\vec{1} - \text{strand}(k-1)
\end{aligned}
$$

We verify $\pi_1+\pi_2 = \pi_3+\pi_4 = \cdots = \pi_{n-1}+\pi_n = n+1$ and so $\pi_1+\pi_2+\cdots+\pi_k = k(n+1)/2$. Also $\hat{\mathbf{d}} = (1,-1,1,-1,\ldots,1,-1)$. Thus $\text{ldisc}(n,k) = 1$.

Case 2. k is even, n is odd.

Let $n = ak + r$, where r is odd, $1 \le r \le k-1$. We must be a little more careful with $\text{strand}(r), \text{strand}(r+1)$ which must use the middle $2a+1$ elements. We set $\text{strand}(r) = ((r/2-1/2)(a+1) + (k/2-r/2-1/2)a)\cdot\vec{1} + \text{strand}(1)$, $\text{strand}(r+1) = ((r/2+1/2)(a+1) + (k/2-r/2+1/2)a)\cdot\vec{1} + (-1,-2,\ldots,-a)$. As above, we verify $\pi_1+\pi_2+\cdots\pi_k = k(n+1)/2$ and $\hat{\mathbf{d}} = (1,-1,1,-1,\ldots,-1)$. Thus $\text{ldisc}(n,k) = 1$.

The solutions for odd k combine solutions for $k=3$ with solutions for even k. Where $n = ak+r$ with $0 \le r \le k-1$ we use constructions from Theorem 7 when $r=0$ and constructions from Theorem 6 otherwise.

Case 3. $n = ak + r$, with r even, k odd.

For $r = 0$, the bounds $\text{disc}(ak,k) \le \frac{3}{2}$ for even a and ≤ 2 for odd a follow from Theorem 7. For $2 \le r \le r-1$, we proceed as in Case 2 for strands 1, 2,…, $r-2$ and strands $r+2$, $r+3,\ldots,k$:

$$
\begin{aligned}
\text{strand}(1) &= (1,2,\ldots,a+1), \\
\text{strand}(2) &= (n+1)\cdot\vec{1} - \text{strand}(1) \\
\text{strand}(3) &= (a+1)\cdot\vec{1} + \text{strand}(1), \\
\text{strand}(4) &= (n+1)\cdot\vec{1} - \text{strand}(3), \ldots, \\
\text{strand}(r-3) &= (r/2-2)(a+1)\cdot\vec{1} + \text{strand}(1), \\
\text{strand}(r-2) &= (n+1)\cdot\vec{1} - \text{strand}(r-1).
\end{aligned}
$$

Then

$$
\begin{aligned}
\text{strand}(r+2) &= (r/2-1)(a+1)\cdot\vec{1} + (1,2,\ldots,a), \\
\text{strand}(r+3) &= (n+1)\cdot\vec{1} - \text{strand}(r+2), \\
\text{strand}(r+4) &= a\cdot\vec{1} + \text{strand}(r+2),
\end{aligned}
$$

$$\begin{aligned}
\text{strand}(r+5) &= (n+1)\cdot\vec{1} - \text{strand}(r+4),\ldots,\\
\text{strand}(k-1) &= (k/2 - r/2 - 3/2)a\cdot\vec{1} + \text{strand}(r+2),\\
\text{strand}(k) &= (n+1)\cdot\vec{1} - \text{strand}(k-1).
\end{aligned}$$

We are left with the $3a+2$ numbers starting with $(r/2-1)(a+1)+(k/2-r/2-1/2)a+1 = ((k-3)a+r-2)/2+1$. For strands $r-1, r, r+1$ we use the permutation π' constructed in Theorem 6 for $n' = 3a+2$. There are two cases depending upon the parity of a. For even a we have $n' \equiv 2 (\text{mod } 6)$ and $\pi' = (1, n, n/2-1, \ldots)$. Then we set

$$\begin{aligned}
\text{strand}(r-1) &= (((k-3)a+r-2)/2)\cdot\vec{1} + (\pi'_1, \pi'_4, \pi'_7, \ldots),\\
\text{strand}(r) &= (((k-3)a+r-2)/2)\cdot\vec{1} + (\pi'_2, \pi'_5, \ldots),\\
\text{strand}(r+1) &= (((k-3)a+r-2)/2)\cdot\vec{1} + (\pi'_3, \pi'_6, \ldots).
\end{aligned}$$

For the permutation π' we have its vector

$$\mathbf{d} = (-1, 1, -2, (3, -2, -1)^{a-1}, 3, -1)$$

and then for the permutation π, we derive

$$\hat{\mathbf{d}} = (\left((1,-1)^{(r-2)/2}, 3, -2, -1, (1,-1)^{(k-r-1)/2}\right)^{a-1}, (1,-1)^{(r-2)/2}, 3, -1).$$

Note that for the first k-sum

$$\begin{aligned}
\sum_1^k \pi_i - \frac{k(n+1)}{2} &= \frac{(k-3)}{2}(n+1) + \pi_{r-1} + \pi_r + \pi_{r+1} - \frac{k(n+1)}{2}\\
&= \pi'_1 + \pi'_2 + \pi'_3 + \frac{3((k-3)a+r-2)}{2} - \frac{3(ak+r+1)}{2} =\\
&= \pi'_1 + \pi'_2 + \pi'_3 - \frac{3}{2}(3a+3) = 1 + n' + \frac{n'}{2} - 1 - \frac{3}{2}(n'+1)\\
&= -\frac{3}{2}.
\end{aligned}$$

Checking the $\hat{\mathbf{d}}$ vector, we see that the k-sums will always be within $3/2$ of the average, giving $\text{ldisc}(n,k) \leq \frac{3}{2}$.
For odd a, we have $n' \equiv 5 (\text{mod } 6)$ and we add $(((k-3)a+r-2)/2)\cdot\vec{1}$ to the permutation $\pi' = (1, n/2, n-1, \ldots)$ to obtain strands $r-1, r, r+1$. For π' in this case we have

$$\mathbf{d} = (-1, 1, -1, (3, -2, -2, 3, -1, -1)^{(n-5)/6}, 3, -2).$$

We obtain $\text{ldisc}(n, k) \leq 2$.

Case 4. $n = ak + r$, with r odd, k odd.

We proceed as in the previous case for strands 1, 2,..., $r - 1$ and strands $r + 3$, $r + 4, \ldots, k$. We are left with the $3a + 1$ numbers starting with $(r/2 - 1/2)(a + 1) + (k/2 - r/2 - 1)a + 1 = ((k - 3)a + r - 1)/2 + 1$. For strands $r-1, r, r+1$ we use $((k-3)a+r-1)/2+\pi'$ where π' is the permutation constructed in Theorem 6 for $n' = 3a + 1$, with two cases depending upon the parity of a. We obtain $\text{disc}(n, k) \leq 2$ for even a and $\text{disc}(n, k) \leq \frac{3}{2}$ for odd a.

■

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