Some Recent Results about Cross Intersecting Families

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This talk is based on joint work with

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1. Set up and Weighed Erdős–Ko–Rado
\[ [n] := \{1, 2, \ldots, n\}, \quad p \in (0, 1). \]

A family of subsets \( \mathcal{A} \subset 2^{[n]} \).

\( \mathcal{A} \) is \( t \)-intersecting if \(|A \cap A'| \geq t\) for all \( A, A' \in \mathcal{A} \).
• \([n] := \{1, 2, \ldots, n\}, \quad p \in (0, 1)\).

• A family of subsets \(\mathcal{A} \subset 2^n\).

• \(\mathcal{A}\) is \(t\)-intersecting if \(|A \cap A'| \geq t\) for all \(A, A' \in \mathcal{A}\).

• The \(p\)-weight (or product measure) of \(\mathcal{A}\) is

\[
\mu_p(\mathcal{A}) := \sum_{A \in \mathcal{A}} p^{|A|}(1 - p)^{n - |A|}.
\]

• Ex. \(\mathcal{F}_0 := \{A \subset [n]: [t] \subset A\}\) is a \(t\)-intersecting family with \(\mu_p(\mathcal{F}_0) = p^t\).
Another example of $t$-intersecting family:

$$\mathcal{F}_1 := \{F \subset [n] : |F \cap [t + 2]| \geq t + 1\}.$$

It follows that

$$\mu_p(\mathcal{F}_1) = (t + 2)p^{t+1}q + p^{t+2}$$

and

$$\mu_p(\mathcal{F}_0) \geq \mu_p(\mathcal{F}_1) \text{ iff } p \leq \frac{1}{t + 1}.$$
For $i \geq 0$ the following family is also $t$-int:

$$\mathcal{F}_i := \{ F \subset [n] : |F \cap [t + 2i]| \geq t + i \}.$$
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$$\mathcal{F}_i := \{ F \subseteq [n] : |F \cap [t + 2i]| \geq t + i \}.$$

- $\mu_p(\mathcal{F}_i) \geq \mu_p(\mathcal{F}_{i+1})$ iff $p \leq \frac{i+1}{t+2i+1}$.

**Theorem** (Ahlsweide–Khachatrian, Bey–Engel, Dinur–Safra, T)

Let $\mathcal{A} \subseteq 2^{[n]}$ be $t$-intersecting. Then

$$\mu_p(\mathcal{A}) \leq \max_i \mu_p(\mathcal{F}_i).$$
Corollary

Let $\mathcal{A} \subset 2^{[n]}$ be $t$-intersecting.

1. If $p \leq \frac{1}{t+1}$ then
   \[ \mu_p(\mathcal{A}) \leq \mu_p(\mathcal{F}_0) = p^t. \]

2. If $\frac{1}{t+1} \leq p \leq \frac{2}{t+3}$ then
   \[ \mu_p(\mathcal{A}) \leq \mu_p(\mathcal{F}_1). \]
2. Extension to cross intersecting families
• Let $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$.

• $\mathcal{A}$ and $\mathcal{B}$ are cross $t$-intersecting if $|A \cap B| \geq t$ for all $A \in \mathcal{A}, B \in \mathcal{B}$. 
Let $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$.

$\mathcal{A}$ and $\mathcal{B}$ are **cross** $t$-intersecting if $|A \cap B| \geq t$ for all $A \in \mathcal{A}, B \in \mathcal{B}$.

**Theorem (Frankl–Lee–Siggers–T)**

Let $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ be cross $t$-intersecting.

1. If $t \geq 14$ and $p \leq \frac{1}{t+1}$ then (arXiv 1303.0657)
   $$\mu_p(\mathcal{A}) \mu_p(\mathcal{B}) \leq (\mu_p(\mathcal{F}_0))^2 = p^{2t}.$$

2. If $t \geq 52$ and $\frac{1}{t+1} \leq p \leq \frac{2}{t+3}$ then
   $$\mu_p(\mathcal{A}) \mu_p(\mathcal{B}) \leq (\mu_p(\mathcal{F}_1))^2.$$
Some ideas in the proof

- Let $\mathcal{A}$ and $\mathcal{B}$ be cross $t$-intersecting.
- Assign a walk in $\mathbb{Z}^2$ to each $A \in \mathcal{A}$:

  \begin{align*}
  n &= 6, 
  A &= \{2, 5\} \iff 
  \end{align*}
Some ideas in the proof

- Let $\mathcal{A}$ and $\mathcal{B}$ be cross $t$-intersecting.
- Assign a walk in $\mathbb{Z}^2$ to each $A \in \mathcal{A}$:

  \[ n = 6, \; A = \{2, 5\} \iff \]

- We may assume that $\mathcal{A}, \mathcal{B}$ are shifted.
- **Key fact:** there are $a, b$ such that
  1. all walks in $\mathcal{A}$ hit $y = x + a$,
  2. all walks in $\mathcal{B}$ hit $y = x + b$,
  3. $a + b \geq 2t$. 
Some ideas in the proof (continued)

- Consider the infinite random walk on $\mathbb{Z}^2$ where $i$-th step is “↑” with probability $p$, and “→” with probability $1 - p$.

- $\mu_p(\mathcal{A})$ is bounded as follows:

\[
\mu_p(\mathcal{A}) \leq \Pr \left( \text{the random walk hits } y = x + a \right) = \left( \frac{p}{1 - p} \right)^a.
\]
3. Different measures and algebraic approach
Let $G$ be a bi-regular bipartite graph with $V(G) = V_1 \cup V_2$.

$U_1 \subset V_1$ and $U_2 \subset V_2$ are cross independent if $uv \notin E(G)$ for all $u \in V_1$, $v \in V_2$.

For $i = 1, 2$ let $\tilde{\mu}_i$ be a general measure:

$$\tilde{\mu}_i : V_i \rightarrow [0, 1] \quad \text{and} \quad \sum_{v \in V_i} \tilde{\mu}_i(v) = 1.$$
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For $i = 1, 2$ let $\tilde{\mu}_i$ be a general measure:

$$\tilde{\mu}_i : V_i \to [0, 1] \text{ and } \sum_{v \in V_i} \tilde{\mu}_i(v) = 1.$$ 

(Key fact): Let $\sigma_1 \geq \sigma_2 \geq \cdots$ be singular values of a bip. adjacency matrix of $G$. Then

$$\sqrt{\tilde{\mu}_1(U_1)\tilde{\mu}_2(U_2)} \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}.$$
Recall $\mu_p(\mathcal{A}) := \sum_{A \in \mathcal{A}} p^{|A|} (1 - p)^{n-|A|}$.

Let $p_1, p_2 \in (0, 1)$ and let $q_i := 1 - p_i \ (i = 1, 2)$. 
Recall $\mu_p(\mathcal{A}) := \sum_{A \in \mathcal{A}} p^{|A|} (1 - p)^{n - |A|}$.

Let $p_1, p_2 \in (0, 1)$ and let $q_i := 1 - p_i$ ($i = 1, 2$).

**Theorem**

If \( (p_1 p_2) / (q_1 q_2) < \left( \sqrt{2} - 1 \right)^2 \cdots \cdots (\ast) \), and $\mathcal{A}, \mathcal{B} \subset 2^n$ are cross $t$-intersecting, then

$$\sqrt{\mu_{p_1}(\mathcal{A}) \mu_{p_2}(\mathcal{B})} \leq \left( \frac{\sqrt{p_1 p_2}}{\sqrt{p_1 p_2} + \sqrt{q_1 q_2}} \right)^t.$$

If $p_1 = p_2$, then the bound is sharp.

If $p_1, p_2 < \frac{\log 2}{t+1} < \frac{1}{t+1}$, then $(\ast)$ is satisfied.
For the case $t = 1$ we get the exact bound:

**Theorem (Suda–Tanaka–T)**

Let $p_1, p_2 \in (0, 1/2]$. \hspace{1cm} \left( \frac{1}{2} = \frac{1}{t+1} \right)

If $A, B \subset 2^n$ are cross 1-intersecting, then

$$\mu_{p_1}(A)\mu_{p_2}(B) \leq p_1p_2.$$ 

The proof is done by solving a corresponding SDP problem. In fact we got a refined bipartite ratio bound based on SDP.
Our setup

Let $G$ be a bi-regular bipartite graph with $V(G) = V_1 \cup V_2$ and $\tilde{\mu}_i : V_i \rightarrow [0, 1]$ ($i = 1, 2$). Let $A$ be a bipartite adjacency matrix of $G$. Suppose $U_1 \subset V_1$ and $U_2 \subset V_2$ are cross indep.

Easy ratio bound (reprise)

Let $\sigma_1 \geq \sigma_2 \geq \cdots$ be the singular values of $A$. Then

$$\sqrt{\tilde{\mu}_1(U_1)\tilde{\mu}_2(U_2)} \leq \frac{\sigma_2}{\sigma_1 + \sigma_2}.$$
New ratio bound (idea)

If $A$ has singular values $\sqrt{\alpha_1 \beta_1} \geq \sqrt{\alpha_2 \beta_2} \geq \cdots$ with some extra properties, then

$$\tilde{\mu}_1(U_1)\tilde{\mu}_2(U_2) \leq \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{\beta_1}{\beta_1 + \beta_2}.$$
New ratio bound (still oversimplified)

If there are nonsingular matrices $P_1, P_2$ and a nonnegative symmetric matrix $A_1$ such that

- $P_1^T A P_2 = \bigoplus (-1)^s \sqrt{\alpha_s \beta_s} I_{m_s}$,
- $P_1^T A_1 P_1 = \bigoplus (-1)^s \alpha_s I_{m_s}$,
- $\alpha_s$ and $\beta_s$ satisfy some inequalities.

Then

$$\tilde{\mu}_1(U_1) \tilde{\mu}_2(U_2) \leq \frac{\alpha_1}{\alpha_1 + \alpha_2} \frac{\beta_1}{\beta_1 + \beta_2}.$$
This new ratio bound can be applied to the following type of cross 1-intersecting EKR:

- weighted subsets ($p_{i} \leq 1/2$)
  
  $$\mu_{p_{1}}(\mathcal{A})\mu_{p_{2}}(\mathcal{B}) \leq p_{1}p_{2},$$

- uniform subsets ($n \geq 2k_{i}$)
  
  $$|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{k_{1}-1}\binom{n-1}{k_{2}-1},$$

- subspaces ($n \geq 2k_{i}$) (Suda–Tanaka 2013)
  
  $$|\mathcal{A}||\mathcal{B}| \leq \begin{bmatrix} n-1 \\ k_{1}-1 \end{bmatrix} \begin{bmatrix} n-1 \\ k_{2}-1 \end{bmatrix}.$$
Conjectures

Conjecture 1

Let $p_1, p_2 \leq \frac{1}{t+1}$. If $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ are cross $t$-intersecting, then

$$\mu_{p_1}(\mathcal{A}) \mu_{p_2}(\mathcal{B}) \leq (p_1p_2)^t.$$ 

True if

- $t = 1$,
- $t \geq 14$ and $p_1 = p_2$,
- $p_1 = p_2 \leq \frac{\log 2}{t+1}$.
\( A, B, C \subset 2^{[n]} \) are 3-cross intersecting if

\[
A \cap B \cap C \neq \emptyset
\]

for all \( A \in \mathcal{A}, B \in \mathcal{B}, C \in \mathcal{C} \).

**Conjecture 2**

Let \( \mathcal{A} \subset \binom{[n]}{k_1}, \mathcal{B} \subset \binom{[n]}{k_2}, \mathcal{C} \subset \binom{[n]}{k_3} \) be 3-cross intersecting, and \( 2n \geq 3k_i \). Then

\[
|\mathcal{A}||\mathcal{B}||\mathcal{C}| \leq \binom{n-1}{k_1-1}\binom{n-1}{k_2-1}\binom{n-1}{k_3-1}.
\]

True if \( k_1 = k_2 = k_3 \).
Conjecture 2 would imply

**Conjecture 3**

Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \subset 2^{[n]}$ be 3-cross intersecting, and $p_1, p_2, p_3 \leq 2/3$. Then

$$\mu_{p_1}(\mathcal{A})\mu_{p_2}(\mathcal{B})\mu_{p_3}(\mathcal{C}) \leq p_1p_2p_3.$$  

Not known even if $p_1 = p_2 = p_3$. 