Forbidden Structures in the Boolean Lattice

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Extensions and Copies

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![Diagram of a lattice with nodes labeled a, b, c, d, and edge connections]

$P$
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An extension of $P$ in $2^{[3]}$
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An extension of $P$ in $2^3$

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The Turán Problem

- When $\mathcal{F} \subseteq 2^n$, we view $\mathcal{F}$ as a poset ordered by inclusion.
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- If \( \mathcal{F} \) does not contain a copy of \( P \), then \( \mathcal{F} \) is \( P \)-free.
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- If $\mathcal{F}$ does not contain a copy of $P$, then $\mathcal{F}$ is $P$-free. If $\mathcal{F}$ does not contain an extension of $P$, then $\mathcal{F}$ is $E(P)$-free.
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\text{La}^*(n, P) = \max\{|\mathcal{F}| : \mathcal{F} \subseteq 2^{[n]} \text{ and } \mathcal{F} \text{ is } P\text{-free}\}
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- Clearly, $\text{La}(n, P) \leq \text{La}^*(n, P)$.
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Theorem (Sperner (1928); Erdős (1945))

$\text{La}^*(n, K_r)$ equals the sum of the $r - 1$ largest binomial coefficients in $\{\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\}$. For fixed $r$ and $n \to \infty$,

\[
\text{La}^*(n, K_r) = (r - 1 + o(1))\left(\binom{n}{\lfloor n/2 \rfloor}\right).
\]
Growth of $L_a(n, P)$ vs. $L_a^*(n, P)$

- If $P$ has $r$ elements, then $K_r$ is an extension of $P$. 

Question

For fixed $P$, how does $L_a^*(n, P)$ grow?

Main Conjecture

For fixed $P$, we have $L_a^*(n, P) = O\left(2^{n/\sqrt{n}}\right)$. 
Growth of $\text{La}(n, P)$ vs. $\text{La}^*(n, P)$

- If $P$ has $r$ elements, then $K_r$ is an extension of $P$.
- $\text{La}(n, P) \leq \text{La}(n, K_r)$.

The challenge is to determine the asymptotics of $\text{La}(n, P)$.
Growth of $\text{La}(n, P)$ vs. $\text{La}^*(n, P)$

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- $\text{La}(n, P) \leq \text{La}(n, K_r) = (r - 1 + o(1))\binom{n}{\lfloor n/2 \rfloor} = O(2^n/\sqrt{n})$. 
If $P$ has $r$ elements, then $K_r$ is an extension of $P$.

If $P$ is not an antichain, then $La(n, P) = \Theta(2^n/\sqrt{n})$. 

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Growth of $\text{La}(n, P)$ vs. $\text{La}^*(n, P)$

- If $P$ has $r$ elements, then $K_r$ is an extension of $P$.
- $\text{La}(n, P) \leq \text{La}(n, K_r) = (r - 1 + o(1))\left(\left\lfloor \frac{n}{2} \right\rfloor \right) = O\left(2^n / \sqrt{n}\right)$.
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**Main Conjecture**

For fixed $P$, we have $\text{La}^*(n, P) = O(2^n/\sqrt{n})$. 
The Turán Threshold

The Turán threshold of $P$, denoted $\pi^*(P)$, is given by

$$\pi^*(P) = \limsup_{n \to \infty} \frac{\text{La}^*(n, P)}{\binom{n}{\lfloor n/2 \rfloor}}.$$
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Posets with finite Turán thresholds

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- [Lu–Milans]: If $P$ is series-parallel, then $\pi^*(P) = O(|P|)$.
- [Lu–Milans]: If $P$ has height 2, then $\pi^*(P) = O(|P|)$.
- Cor: $\pi^*(2[3]) \leq 24$. 

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- Cor: $\pi^*(2^3) \leq 24$. 
Given $\mathcal{F} \subseteq 2^{[n]}$, let $X$ be the number of times a random full chain meets $\mathcal{F}$. 

The Lubell function of $\mathcal{F}$, denoted $\ell_n(\mathcal{F})$ or $\ell(\mathcal{F})$, is $E[X]$. 

Think of $\ell_n(\mathcal{F})$ as a measure of the size of $\mathcal{F}$, with $0 \leq \ell_n(\mathcal{F}) \leq n + 1$. 

For $A \subseteq B$, we define $\ell(F; [A, B])$ to be the expected number of times a random, full chain from $A$ to $B$ meets $\mathcal{F}$. 
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Since $\ell(F) = \sum_{A \in F} \frac{1}{|A|} \geq \frac{|F|}{\binom{n}{\lfloor n/2 \rfloor}}$, we have $\lambda^*(P) \geq \pi^*(P)$. 
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Since $\ell(\mathcal{F}) = \sum_{A \in \mathcal{F}} \frac{1}{|A|} \geq \frac{|\mathcal{F}|}{n \lceil n/2 \rceil}$, we have $\lambda^*(P) \geq \pi^*(P)$.

Conjecture

Always $\lambda^*(P)$ is finite.
The Standard Example: Proof Sketch

Theorem

\[ r - 2 \leq \lambda^*(S_r) \leq 2r + O(\sqrt{r}) \]
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Proof (sketch).
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\[ r - 2 \leq \lambda^* (S_r) \leq 2r + O(\sqrt{r}) \]

Proof (sketch).

Let \( F_0 \subseteq 2^{[n]} \) with \( \ell(F_0) \geq 2r + \Theta(\sqrt{r}) \).

1. Let \( F_1 = \{ A \in F_0 : |A| \leq n/2 \} \). By self-duality, we may assume \( \ell(F_1) \geq \frac{1}{2} \ell(F_0) \geq r + \Theta(\sqrt{r}) \).
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2. When \( t \in A \in F \), we say that \( t \) is a pivot of \( A \) if

   \[ t \in A \subseteq F \]
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2. When \( t \in A \in F \), we say that \( t \) is a **pivot** of \( A \) if there exists an element \( t' \notin A \) such that \( A - \{ t \} + \{ t' \} \in F \).
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2. When \( t \in A \in F \), we say that \( t \) is a pivot of \( A \) if there exists an element \( t' \notin A \) such that \( A - \{t\} + \{t'\} \in F \).

A set \( A \in F \) is \( \gamma \)-flexible if it has at least \( \gamma |A| \) pivots.
The Standard Example: Proof Sketch

Theorem

\[ r - 2 \leq \lambda^*(S_r) \leq 2r + O(\sqrt{r}) \]

Proof (sketch).

3. Form \( F_2 \) from \( F_1 \) by throwing away all sets that are not \((1 - \frac{1}{\sqrt{r}})\)-flexible. We have \( \ell(F_2) \geq \ell(F_1) - O(\sqrt{r}) \).
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4. Pick a set \( B \in F_2 \) with \( \ell(F_2; [\emptyset, B]) \geq \ell(F_2) \).
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Proof (sketch).

3. Form \( \mathcal{F}_2 \) from \( \mathcal{F}_1 \) by throwing away all sets that are not \((1 - \frac{1}{\sqrt{r}})\)-flexible. We have \( \ell(\mathcal{F}_2) \geq \ell(\mathcal{F}_1) - O(\sqrt{r}) \).

4. Pick a set \( B \in \mathcal{F}_2 \) with \( \ell(\mathcal{F}_2; [\emptyset, B]) \geq \ell(\mathcal{F}_2) \).

Let \( T \) be the set of pivots in \( B \).
The Standard Example: Proof Sketch

**Theorem**

\[ r - 2 \leq \lambda^*(S_r) \leq 2r + O(\sqrt{r}) \]

**Proof (sketch).**

3. Form \( \mathcal{F}_2 \) from \( \mathcal{F}_1 \) by throwing away all sets that are not \((1 - \frac{1}{\sqrt{r}})\)-flexible. We have \( \ell(\mathcal{F}_2) \geq \ell(\mathcal{F}_1) - O(\sqrt{r}) \).

4. Pick a set \( B \in \mathcal{F}_2 \) with \( \ell(\mathcal{F}_2; [\emptyset, B]) \geq \ell(\mathcal{F}_2) \).

Let \( T \) be the set of pivots in \( B \). Note that \( |T| \geq \gamma |B| \).
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5. Since \( \ell(\mathcal{F}_2; [\emptyset, B]) \) is large and \(|T| \geq \gamma|B|\), we find sets \( A_1, \ldots, A_r \in \mathcal{F}_2 \) and elements \( t_1, \ldots, t_r \in T \) such that
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   \begin{enumerate}[(a)]
     \item \( A_i \subseteq B \)
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   (a) \( A_i \subseteq B \)
   (b) \( t_i \in A_j \) if and only if \( i = j \).
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6. Since \( t_i \in T \), we can pivot \( B \) away from \( t_i \) to obtain \( B_i \in \mathcal{F}_2 \).
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6. Since \( t_i \in T \), we can pivot \( B \) away from \( t_i \) to obtain \( B_i \in \mathcal{F}_2 \).

7. \( t_i \in A_i \) but \( t_i \not\in B_i \), so \( A_i \not\subset B_i \).
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8. If \( j \neq i \), then \( A_j \subseteq B - \{t_i\} \subseteq B_i \).
9. \( A_1, \ldots, A_r \) and \( B_1, \ldots, B_r \) form a copy of \( S_r \).
Toward Height 3

Definition
The generalized standard example of width $r$ and height $h$, denoted $S_{r,h}$, has $h$ disjoint $r$-antichains $A_1, \ldots, A_h$ where $A_i \cup A_{i+1}$ is a copy of $S_r$. 

$S_{5,3}$

Theorem
$\pi^*(S_{r,h}) = O(r)$. 

Proof involves about 10 cleaning steps. 

Not clear how to extend to other posets of height 3 or $S_{r,h}$. 

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A natural attack

- Each poset is contained in a Boolean lattice.

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\lambda^* \left(2^3\right) \leq 2 + \lambda^* \left(S^3\right) \leq 24.
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Open problem: show that \(\lambda^* \left(2^4\right)\) or \(\pi^* \left(2^4\right)\) is finite.
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- Given disjoint sets $X_0, X_1, \ldots, X_d$, with $X_i \neq \emptyset$ for $i \geq 1$, the generated $d$-dimensional Boolean algebra is the family of all sets formed by the union of $X_0$ with 0 or more members of $\{X_1, \ldots, X_d\}$.
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Fact
There exists $\mathcal{F} \subseteq 2^n$ such that $\mathcal{F}$ is $B_2$-free and $|\mathcal{F}| \geq \Omega(n^{1/4} \cdot \frac{2^n}{\sqrt{n}})$. 
The Turán Problem for Boolean Algebras

- What is the largest size of a $B_d$-free subfamily of $2^{[n]}$?
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- [Erdős–Kleitman 1971] For some constants $c_1, c_2$ and $n$ sufficiently large

\[ c_1 \cdot n^{-1/4} \cdot 2^n \leq b(n, 2) \leq c_2 \cdot n^{-1/4} \cdot 2^n. \]
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[Gunderson–Rödl–Sidorenko 1999] For each $d$, there exists $c_d$ such that for $n$ sufficiently large

$$n^{-\frac{d}{2d+1-2 \cdot (1-o(1))}} \cdot 2^n \leq b(n, d) \leq c_d \cdot n^{-\frac{1}{2d}} \cdot 2^n.$$
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- Here, $c_d = (10d)^d(1 + o(1))$. 
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Theorem

\[ b(n, d) \leq 50 \cdot n^{-\frac{1}{2d}} \cdot 2^n. \]
Summary & Open Problems

Theorems

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If \( P \) has height 2, then \( \lambda^*(P) \leq O(|P|) \).

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- Conjecture: Always \( \lambda^*(P) \) is finite.
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Thank You.