Recent Progress on Diamond-Free Families

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Joint work with Lucas Kramer and Michael Young

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Joint Work

This talk is based on joint work with:

Lucas Kramer
Iowa State

Michael Young
Iowa State

The Boolean lattice

Definition

For any (finite) set S, the POWER SET OF S *is the set of all subsets of S and is denoted 2^S.*

Definition

The BOOLEAN LATTICE OF DIMENSION n *is a partially-ordered set with ground set $2^{\{1,\ldots,n\}}$ so that $X \preceq Y$ in the poset whenever $X \subseteq Y$ as sets. We denote it as $\mathcal{B}_n$.*

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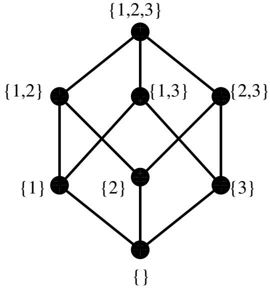

Figure: $\mathcal{B}_3$

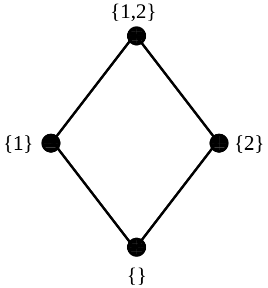

Figure: $\mathcal{B}_2$, the "DIAMOND"

Formal definitions

Definition

Let $\mathcal{P}$ be a finite poset. Another poset $\mathcal{Q}$ is said to CONTAIN $\mathcal{P}$ AS A (WEAK) SUBPOSET *if there exists an injective map $\varphi : \mathcal{P} \to \mathcal{Q}$ such that if $p_1 \preceq_{\mathcal{P}} p_2$, then $\phi(p_1) \preceq_{\mathcal{Q}} \phi(p_2)$.*

If $\mathcal{Q}$ does not contain $\mathcal{P}$ as a subposet, then $\mathcal{Q}$ is said to be $\mathcal{P}$-FREE.

A family of sets $\mathcal{F} \subseteq \mathcal{B}_n$ is said to be $\mathcal{P}$-free if the induced order on $\mathcal{F}$ is $\mathcal{P}$-free.

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Example

A family $\mathcal{F}$ is $\mathcal{B}_1$-free if and only if it is an ANTICHAIN.

Figure: $\mathcal{B}_1$

(Just a few) prior results

Theorem (Sperner, 1928)

Let $\mathrm{La}(n, \mathcal{C}_2)$ *denote the size of the largest family in* $\mathcal{B}_n$*, which has no chain of height 2. Then,*

$$\mathrm{La}\left(n, \mathcal{C}_2\right) = \binom{n}{\lfloor n/2 \rfloor}.$$

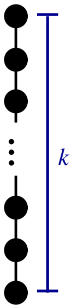
(Just a few) prior results

Theorem (Erdős, 1945)

Let $\mathrm{La}(n, \mathcal{C}_k)$ *denote the size of the largest family in* $\mathcal{B}_n$*, which has no chain of height* k *as a (weak) subposet. Then,*

$$\mathrm{La}\,(n, \mathcal{C}_k) = \Sigma(n, k-1) \sim (k-1)\binom{n}{\lfloor n/2 \rfloor},$$

where $\Sigma(n, k-1)$ *denotes the sum of the* $k-1$ *largest binomial coefficients* $\binom{n}{\ell}$*.*

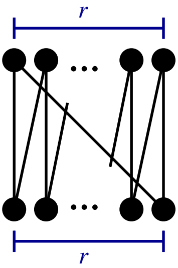


(Just a few) prior results

Theorem (Griggs-Lu, 2009 ($r \geq 4$ even); Lu, 2012+ ($r \geq 7$ odd))

Let $r \in \{4, 6, 7, 8, \ldots\}$ and $\mathrm{La}(n, \mathcal{CR}_r)$ denote the size of the largest family in $\mathcal{B}_n$, which has no "crown" of length $2r$ as a (weak) subposet. Then,

$$\mathrm{La}\left(n, \mathcal{CR}_r\right) \sim \binom{n}{\lfloor n/2 \rfloor}.$$



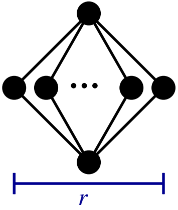
(Just a few) prior results

Theorem (Griggs-Li-Lu, 2011)

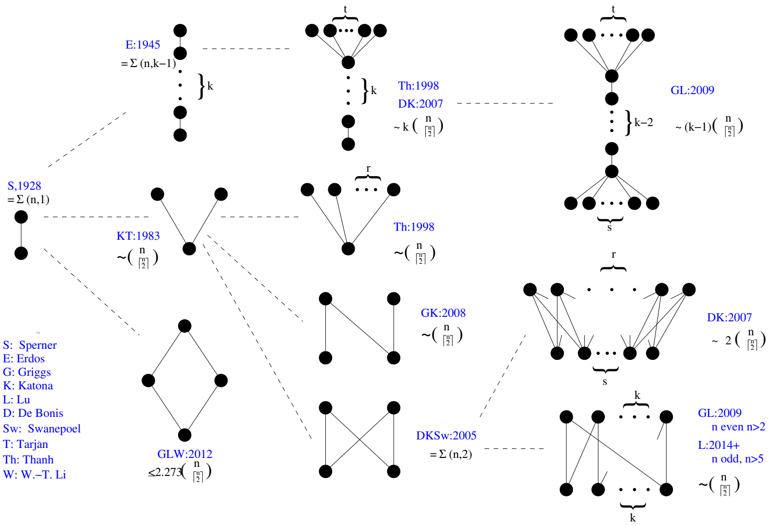
Let $r \in \{3, 4, 7, 8, 9, 15, \ldots\}$ *(∞ly many) and* $\mathrm{La}(n, \mathcal{D}_r)$ *denote the size of the largest family in* $\mathcal{B}_n$*, which has no "generalized diamond" of order* $r + 2$ *as a (weak) subposet. Then,*

$$\begin{aligned}\mathrm{La}\left(n, \mathcal{D}_r\right) &= \Sigma\left(n, \lceil \log_2(r+2) \rceil\right) \\ &\sim \lceil \log_2(r+2) \rceil \binom{n}{\lfloor n/2 \rfloor},\end{aligned}$$

where $\Sigma(n, k)$ *denotes the sum of the* k *largest binomial coefficients* $\binom{n}{\ell}$.



(Just a few) prior results



Thanks to Wei-Tian Li for this summary and Lucas Kramer for the figures.

(Just a few) prior results

Theorem (Bukh, 2009)

Let $\mathcal{P}$ be a poset whose Hasse diagram is a tree and let $e(\mathcal{P})$ be the least number such that $\mathcal{P}$ is not a subposet of $e(\mathcal{P})$ layers in $\mathcal{B}_n$, for n sufficiently large. Then,

$$\mathrm{La}(n,\mathcal{P}) \sim e(\mathcal{P})\binom{n}{\lfloor n/2 \rfloor}.$$

(Just a few) prior results

Conjecture (Griggs-Lu, 2009)

Let $\mathcal{P}$ be any fixed poset. Then,

$$\pi(\mathcal{P}) = \lim_{n\to\infty} \frac{\mathrm{La}(n,\mathcal{P})}{\binom{n}{\lfloor n/2\rfloor}}$$

exists and is an integer.

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Conjecture (Saks-Winkler, 2009)

In addition, if $e(\mathcal{P})$ is the least number such that $\mathcal{P}$ is not a subposet of any $e(\mathcal{P})$ layers in $\mathcal{B}_n$, for n sufficiently large then,

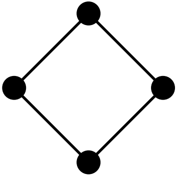
$$\pi(\mathcal{P}) = e(\mathcal{P}).$$

(Just a few) prior results

Theorem (Griggs-Lu, 2009)

Let $\mathrm{La}(n,\Diamond)$ *denote the size of the largest family in* $\mathcal{B}_n$*, which has no "diamond" as a (weak) subposet. Then,*

$$\mathrm{La}\,(n,\Diamond) \leq (2.3 \quad + o(1)) \binom{n}{\lfloor n/2 \rfloor}.$$

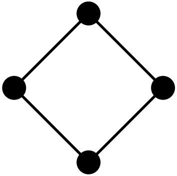


(Just a few) prior results

Theorem (Axenovich-Manske-M., 2012)

Let $\mathrm{La}(n,\Diamond)$ *denote the size of the largest family in* $\mathcal{B}_n$*, which has no "diamond" as a (weak) subposet. Then,*

$$\mathrm{La}\left(n,\Diamond\right) \leq (2.284 + o(1)) \binom{n}{\lfloor n/2 \rfloor}.$$



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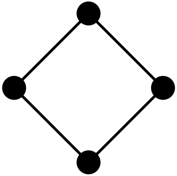

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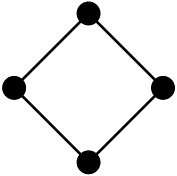
Theorem (Kramer-M.-Young, 2012)

Let $\mathrm{La}(n,\Diamond)$ *denote the size of the largest family in* $\mathcal{B}_n$*, which has no "diamond" as a (weak) subposet. Then,*

$$\mathrm{La}\left(n,\Diamond\right) \leq (2.25 \ + o(1))\binom{n}{\lfloor n/2 \rfloor}.$$



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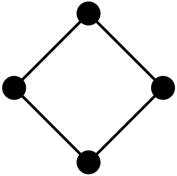

Proposition

$$\mathrm{La}\,(n, \Diamond) \geq \Sigma(n, 2) \sim 2\tbinom{n}{\lfloor n/2 \rfloor}.$$

The Lubell function

Definition

Let $\mathcal{F}$ be a family of sets in the n-dimensional Boolean lattice, $\mathcal{B}_n$. The LUBELL FUNCTION OF $\mathcal{F}$ *is*

$$\mathrm{Lu}(n, \mathcal{F}) = \mathrm{Lu}(\mathcal{F}) = \sum_{F \in \mathcal{F}} \binom{n}{|F|}^{-1}.$$

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This function features prominently in some proofs of Sperner's theorem. In particular, we have the LYM (also called YBLM) inequality:

Theorem (Yamamoto, 1954; Meshalkin, 1963; Bollobás, 1965; Lubell, 1966)

If $\mathcal{F}$ is an antichain, then

$$\mathrm{Lu}(\mathcal{F}) \leq 1.$$

Key Lemma 1

Definition

Let $\mathcal{P}$ be a nontrivial poset and let

$$\mathrm{Lu}^*(\mathcal{P}) \stackrel{\mathrm{def}}{=} \limsup_{n\to\infty} \left\{\mathrm{Lu}(n,\mathcal{F}) : \mathcal{F} \textit{ is } \mathcal{P}\textit{-free and } \emptyset \in \mathcal{F}\right\}.$$

The quantity $\mathrm{Lu}^*(\mathcal{P})$ is integral to our Key Lemma 1:

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Lemma (Griggs-Li-Lu, 2012)

Let $\mathcal{P}$ be a nontrivial poset and let $\mathcal{F}$ be a $\mathcal{P}$-free family in the n-dimensional Boolean lattice, $\mathcal{B}_n$. Then,

$$|\mathcal{F}| \le \left(\mathrm{Lu}^*(\mathcal{P}) + o(1)\right) \binom{n}{\lfloor n/2 \rfloor}.$$

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So, it suffices to upper bound Lubell functions for $\mathcal{P}$-free families that CONTAIN THE EMPTYSET.

The Lubell bound cannot match lower bound

Recall that for the diamond poset, there is a trivial lower bound.

Proposition

$$\mathrm{La}\,(n, \Diamond) \geq \Sigma(n, 2) \sim 2\binom{n}{\lfloor n/2 \rfloor}.$$

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Key Lemma 1 reduces to the following for $\mathcal{P} = \Diamond$:

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Unfortunately, we cannot use this to tighten the bound further.

Proposition (Griggs-Li-Lu, 2012)

For every $n \ge 4$, there are at least two nonisomorphic $\Diamond$-free families, $\mathcal{F} \subseteq \mathcal{B}_n$ with

$$\mathrm{Lu}(\mathcal{F}) = 2 + \frac{1}{n(n-1)} \left\lfloor \frac{n^2}{4} \right\rfloor \sim 2.25.$$

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Our goal is to show that $\mathrm{Lu}^*(\Diamond) = 2.25$.

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There is a function $f(n, v, G)$ such that, for every $\Diamond$-free family $\mathcal{F}$, in $\mathcal{B}_n$, with $\emptyset \in \mathcal{F}$, there is a graph G on $v \leq n$ vertices such that

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$$\begin{aligned} \mathrm{Lu}(\mathcal{F}) &\leq 2 + f(n, v, G) \\ &:= 2 + \tfrac{2\alpha_1(G) - 2\alpha_2(G)}{n(n-1)(n-2)} + \tfrac{6\beta_0(G)}{n(n-1)(n-2)(n-3)} \end{aligned}$$

- $\alpha_i(G)$ *is the # of triples that induce exactly i edges, $i = 0, 1, 2, 3$,*
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&\quad + \textstyle\sum_{w \in W} \left[\tfrac{|X_w| - |Y_w|}{n(n-1)} + \tfrac{4\overline{e}(Y_w) - 2\overline{e}(X_w)}{n(n-1)(n-2)} \right],
\end{aligned}$$

- $\alpha_i(G)$ *is the # of triples that induce exactly i edges, $i = 0, 1, 2, 3$,*
- $\beta_0(G)$ *is the # of quadruples that induce exactly 0 edges,*
- $W = [n] - V(G)$,
- (X_w, Y_w) *is a partition of $V(G)$, $\forall w$ (X_w, Y_w could be empty)*
- $\overline{e}(S)$ *is the number of nonedges induced by $S \subseteq V(G)$.*

Bounding f

Definition

$$f(n,v,G) := \frac{2\alpha_1(G)-2\alpha_2(G)}{n(n-1)(n-2)} + \frac{6\beta_0(G)}{n(n-1)(n-2)(n-3)}$$

$$+ \sum_{w\in W}\left[\frac{|X_w|-|Y_w|}{n(n-1)} + \frac{4\overline{e}(Y_w)-2\overline{e}(X_w)}{n(n-1)(n-2)}\right].$$

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Our goal is to show that, for any $v \le n$ and any G on v vertices,

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It turns out to be relatively easy to verify that

Proposition

$$\max\{f(n,v,G) : v < 2n/3\} = \frac{1}{n(n-1)}\left\lfloor\frac{n^2}{4}\right\rfloor \le \frac{1}{4} + \frac{1}{4(n-1)}.$$

The maximum occurs only if $v = \{\lfloor n/2\rfloor, \lceil n/2\rceil\}$ and $G = K_v$.

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The maximum occurs only if $v = \{\lfloor n/2\rfloor, \lceil n/2\rceil\}$ *and* $G = K_v$.

So, we have to consider $v \geq 2n/3$. For purposes of illustration, let's eliminate the summation term by assuming $v = n$.

The simplified problem

Definition

$$g(n, G) := \frac{2\alpha_1(G) - 2\alpha_2(G)}{n(n-1)(n-2)} + \frac{6\beta_0(G)}{n(n-1)(n-2)(n-3)}$$

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&= \frac{1}{3}\frac{\alpha_1(G) - \alpha_2(G)}{\binom{n}{3}} + \frac{1}{4}\frac{\beta_0(G)}{\binom{n}{4}}
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&= \frac{1}{\binom{n}{4}} \sum_{H:|V(H)|=4} d(H),
\end{aligned}
$$

where

$$
d(H) = \frac{1}{3}\frac{\alpha_1(H)}{4} - \frac{1}{3}\frac{\alpha_2(H)}{4} + \frac{1}{4}\frac{\beta_0(H)}{1}.
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Clearly, $g(n,G) \leq \max\{d(H) : |V(H)| = 4\}$.

Unfortunately, $d\Big($  $\Big) = \frac{1}{3}$.

The correction factor

Suppose we can find a function $c(H)$ such that $\frac{1}{\binom{n}{4}} \sum_H c(H) + o(1) \geq 0$.

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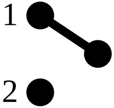
\end{aligned}
$$

How do we find such a c?

First we set up flags

Given two nonadjacent vertices "1" and "2", the probability that a random vertex is adjacent to "1" and nonadjacent to "2" is denoted:



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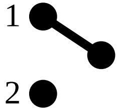
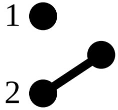


Given two nonadjacent vertices "1" and "2", the probability that a random vertex is adjacent to "2" and nonadjacent to "1" is denoted:



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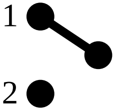

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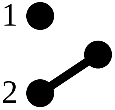
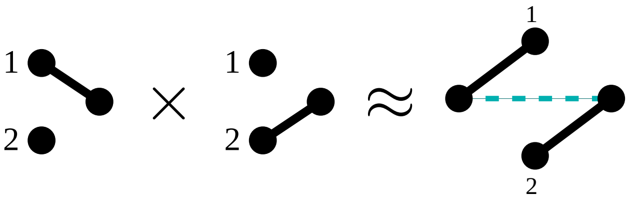
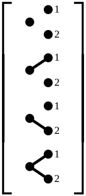


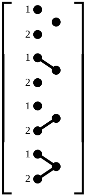


Then we guess

$$\frac{1}{\binom{n}{4}}\sum_H c(H) \;=\; \frac{1}{4}\sum_{\substack{(1,2)\\ 1\not\sim 2}}$$



$$^{T}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$



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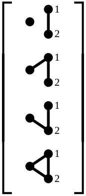

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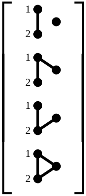

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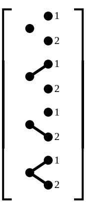

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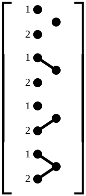

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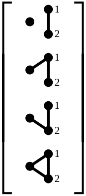

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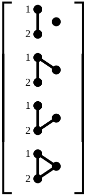

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The matrices are positive semidefinite.

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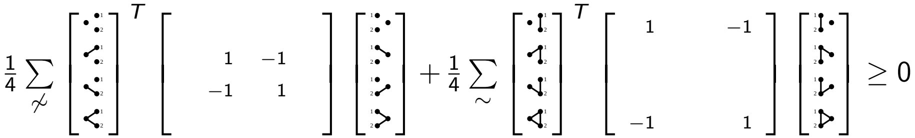

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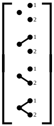
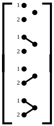
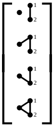
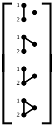
$\frac{1}{4}\sum_{\not\sim}$  ^{T} $\begin{bmatrix} & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & \end{bmatrix}$  $+\frac{1}{4}\sum_{\sim}$  ^{T} $\begin{bmatrix} 1 & & & -1 \\ & & & \\ & & & \\ -1 & & & 1 \end{bmatrix}$  ≥ 0

So,

$$\mathrm{Lu}(\mathcal{F}) \leq 2 + \frac{1}{\binom{n}{4}} \sum_{H} (d(H) + c(H)) + o(1)$$

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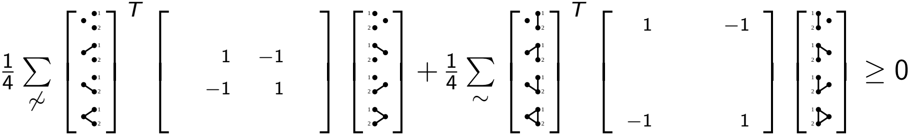

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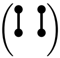
Recall d  $= \frac{1}{3}$.

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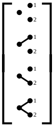
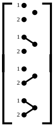
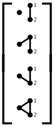
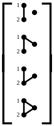
$\frac{1}{4}\sum_{\not\sim}$  $^{T} \begin{bmatrix} & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & \end{bmatrix}$  $+\frac{1}{4}\sum_{\sim}$  $^{T} \begin{bmatrix} 1 & & & -1 \\ & & & \\ & & & \\ -1 & & & 1 \end{bmatrix}$  ≥ 0

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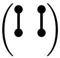
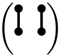
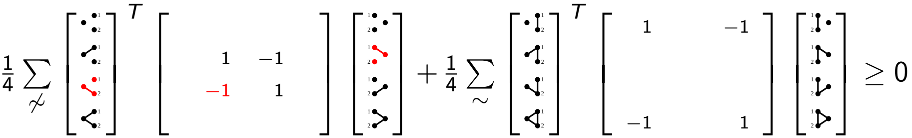
Recall d  $= \frac{1}{3}$.

Since c  $= -\frac{1}{4}\left(\frac{1}{3}\right) - \frac{1}{4}\left(\frac{1}{3}\right) + \frac{1}{4}\left(\frac{1}{3}\right) = -\frac{1}{12}$, The matrices are positive semidefinite, hence



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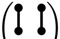
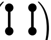
Recall d () $= \frac{1}{3}$.

Since c () $= -\frac{1}{4}\left(\frac{1}{3}\right) - \frac{1}{4}\left(\frac{1}{3}\right) + \frac{1}{4}\left(\frac{1}{3}\right) = -\frac{1}{12}$,

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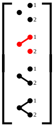
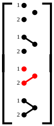
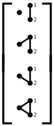
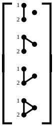
$\frac{1}{4} \sum_{\not\sim}$  $^T \begin{bmatrix} & & & \\ & 1 & -1 & \\ & -1 & 1 & \\ & & & \end{bmatrix}$  $+ \frac{1}{4} \sum_{\sim}$  $^T \begin{bmatrix} 1 & & & -1 \\ & & & \\ & & & \\ -1 & & & 1 \end{bmatrix}$  ≥ 0

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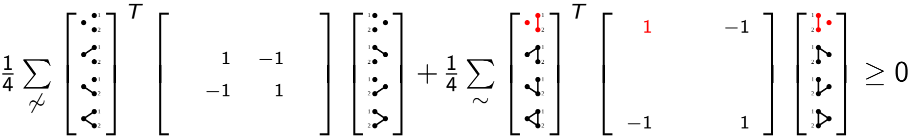

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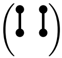
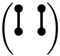
Recall d  $= \frac{1}{3}$.

Since c  $= -\frac{1}{4}\left(\frac{1}{3}\right) - \frac{1}{4}\left(\frac{1}{3}\right) + \frac{1}{4}\left(\frac{1}{3}\right) = -\frac{1}{12}$,

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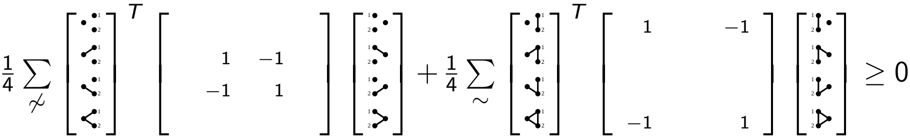

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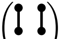
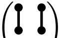
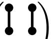
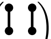
Recall d () $= \frac{1}{3}$.

Since c () $= -\frac{1}{4}\left(\frac{1}{3}\right) - \frac{1}{4}\left(\frac{1}{3}\right) + \frac{1}{4}\left(\frac{1}{3}\right) = -\frac{1}{12}$,

d () $+\ c$ () $= \frac{1}{4}$.

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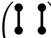
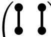
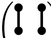
Recall d () $= \frac{1}{3}$.

Since c () $= -\frac{1}{4}\left(\frac{1}{3}\right) - \frac{1}{4}\left(\frac{1}{3}\right) + \frac{1}{4}\left(\frac{1}{3}\right) = -\frac{1}{12}$,

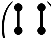
d () $+\, c$ () $= \dfrac{1}{4}$.

This choice of matrices makes $d(H) + c(H) \leq \frac{1}{4}$ for every H and equality if H is a subgraph of 2 disjoint cliques.

Then we guess

So,

$$\mathrm{Lu}(\mathcal{F}) \leq 2 + \frac{1}{\binom{n}{4}} \sum_{H} (d(H) + c(H)) + o(1) \leq 2 + \frac{1}{4} + o(1).$$

d () $+\, c$ () $= \frac{1}{4}.$

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Hence $g(n, G) \leq \frac{1}{4} + o(1)$, the objective.

Results on three layers

Theorem (Kramer-M.-Young, 2012)

Let $\mathrm{La}(n,\Diamond)$ *denote the size of the largest family in* $\mathcal{B}_n$*, which has no "diamond" as a (weak) subposet. Then,*

$$\mathrm{La}\,(n,\Diamond) \leq (2.25 + o(1)) \binom{n}{\lfloor n/2 \rfloor}.$$

Suppose we restrict ourselves to $\Diamond$-free families $\mathcal{F}$ that exist in only three layers. That is, the subsets of $[n]$ in it only have three different sizes.

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Theorem

If $\mathcal{F}$ *is a* $\Diamond$*-free family in three layers of* $\mathcal{B}_n$*, then*

- $|\mathcal{F}| \leq (2.20711 + o(1))\binom{n}{\lfloor n/2\rfloor}$ *[Axenovich-Manske-M., 2012]*
- $|\mathcal{F}| \leq (2.15471 + o(1))\binom{n}{\lfloor n/2\rfloor}$ *[Manske-Shen, 2012]*
- $|\mathcal{F}| \leq (2.15121 + o(1))\binom{n}{\lfloor n/2\rfloor}$ *[Balogh-Hu-Lidický-Liu, 2012]*

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- The last one uses flag algebras more-or-less directly.

Results on three layers

Theorem

If $\mathcal{F}$ is a $\Diamond$-free family in three layers of $\mathcal{B}_n$, then

- $|\mathcal{F}| \leq \left(1 + \frac{2\sqrt{3}}{3} + o(1)\right) \binom{n}{\lfloor n/2 \rfloor}$ *[Manske-Shen, 2012]*

- The first two use a method that involves something similar to the chain counting arguments that gave the general bound of 2.25.
- The last one uses flag algebras more-or-less directly.
- Let us focus on the slightly higher Manske-Shen bound.

Manske-Shen method

Let $\mathcal{F}$ be a $\Diamond$-free family in three layers of $\mathcal{B}_n$.

We observe that we can ensure that the three layers are consecutive and the sizes of the sets in those layers are $\approx n/2$.

Therefore, $|\mathcal{F}| \approx \mathrm{Lu}(\mathcal{F})\binom{n}{\lfloor n/2 \rfloor}$.

Manske-Shen method

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Therefore, $|\mathcal{F}| \approx \mathrm{Lu}(\mathcal{F})\binom{n}{\lfloor n/2 \rfloor}$.

Moreover, $\mathcal{F} = \mathcal{S} \mathbin{\dot\cup} \mathcal{T} \mathbin{\dot\cup} \mathcal{U}$, where

- $\mathcal{U}$ is in the top layer,
- $\mathcal{T}$ is in the middle layer, and
- $\mathcal{S}$ is in the bottom layer.

f and g functions

$$\mathcal{F} = \mathcal{S} \mathbin{\dot\cup} \mathcal{T} \mathbin{\dot\cup} \mathcal{U}$$

- For $Y \in \mathcal{T}$, let $\breve{f}(Y)$ denote the proportion of chains from $\emptyset$ to Y that have a member of $\mathcal{S}$.
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These are duals of other functions f and g which we don't need to introduce, but we'll keep the notation.

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Cauchy-Schwarz via Manske-Shen

Let $\mathcal{T}_1 = \left\{ Y \in \mathcal{T} : \check{R}(Y) \geq 0 \right\}$.

The proof begins with a quick application of Cauchy-Schwarz.

$$\left(\sum_{Y \in \mathcal{T}} 2\right)^2 \leq \left(\sum_{Y \in \mathcal{T}} (1 - \check{R}(Y))\right) \qquad \times \left(\sum_{Y \in \mathcal{T}} \frac{4}{1 - \check{R}(Y)}\right)$$

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By the quadratic formula,

$$\mathrm{Lu}(\mathcal{F}) \leq \frac{3 + 2\sqrt{3}}{2} \approx 2.15471.$$

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Can we exploit this technique to the general case?

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Preliminary work suggests we can get an upper bound strictly less than 2.25, however. Work (with Lucas Kramer) is ongoing.

Future work

- Improve the bounds $2\binom{n}{\lfloor n/2 \rfloor} \leq |\mathcal{F}| \leq (2.25 + o(1))\binom{n}{\lfloor n/2 \rfloor}$ if $\mathcal{F}$ is $\Diamond$-free.

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- Improve the Manske-Shen approach using Flag Algebras (a generalization of Cauchy-Schwarz).
- Develop a general theory for poset Turán problems.