Largest union-intersecting families

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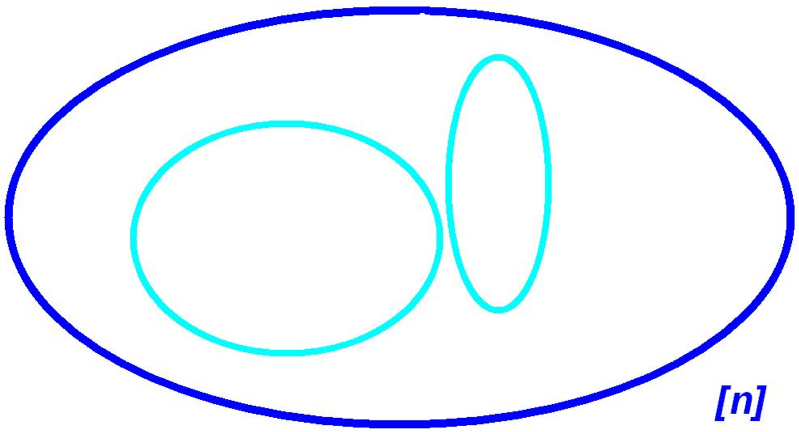
Notation: $[n] = \{1, 2, \ldots, n\}$.

A family $\mathcal{F} \subset 2^{[n]}$ is **intersecting** if $F \cap G \neq \emptyset$ holds for every pair $F, G \in \mathcal{F}$.



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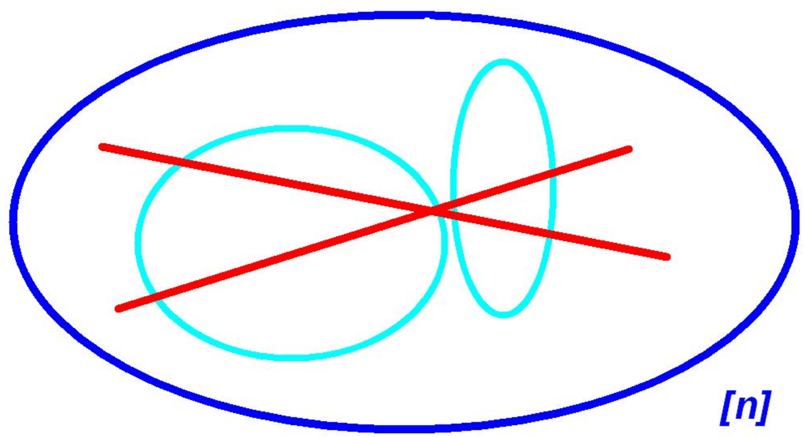

Observation **(Erdős – Ko – Rado, 1961)** If $\mathcal{F} \subset 2^{[n]}$ is intersecting then

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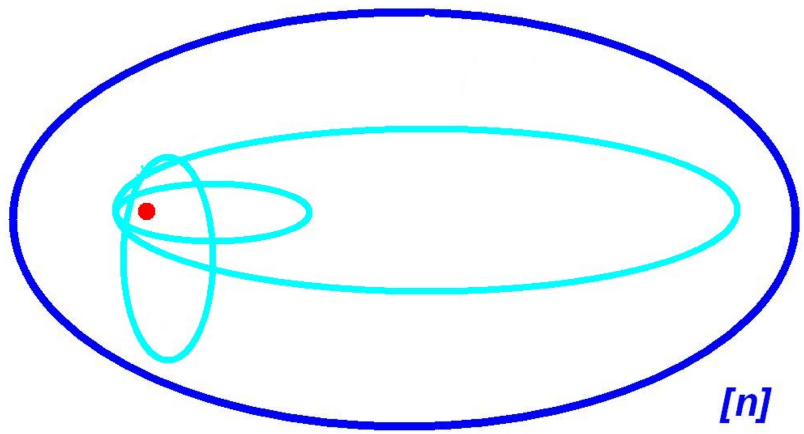

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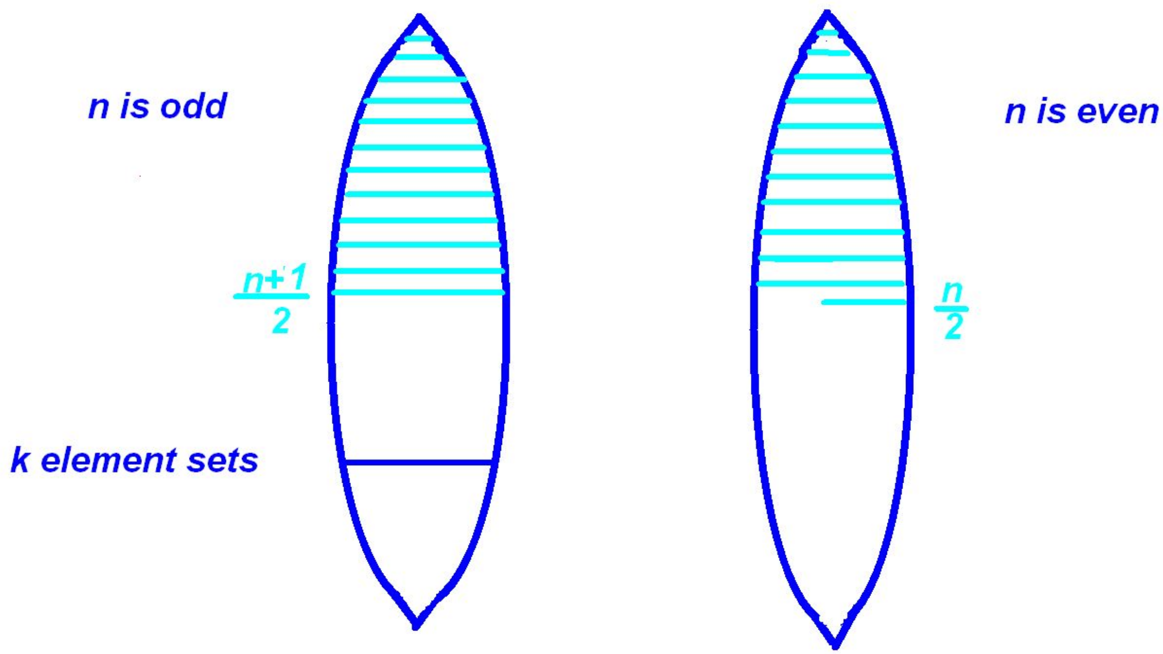
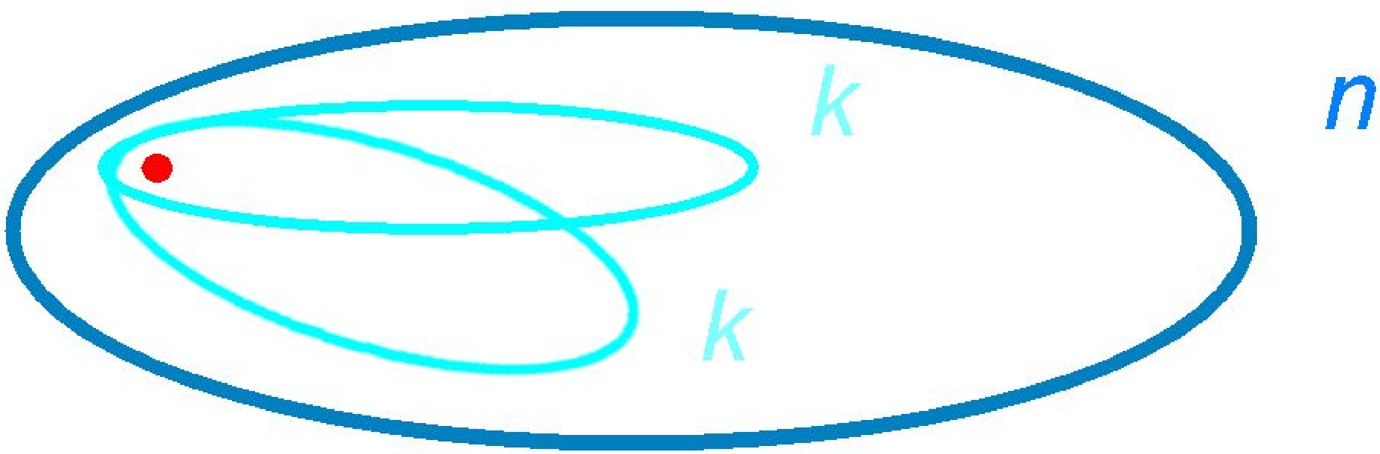
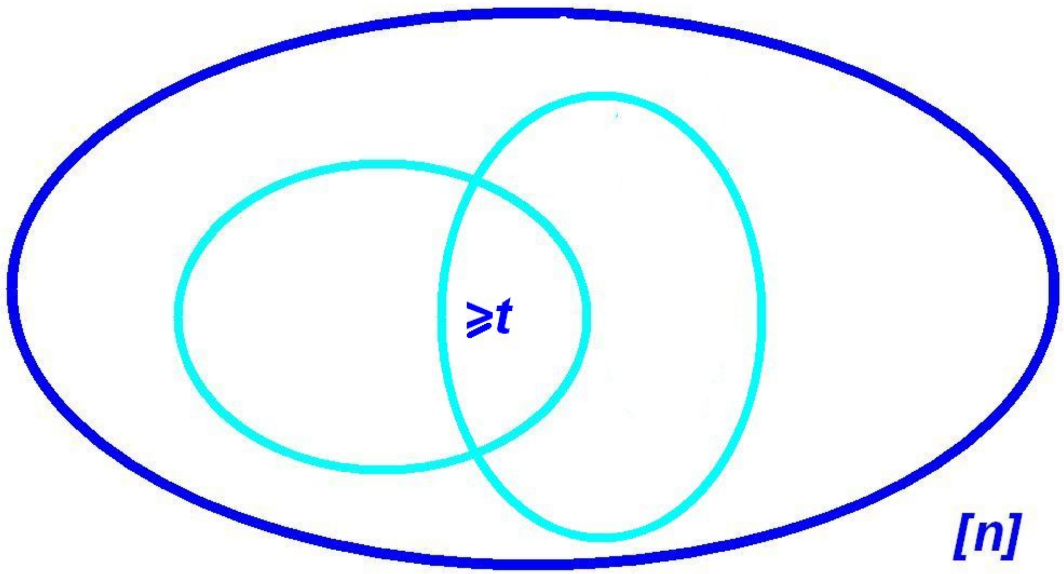


Theorem (Erdős – Ko – Rado, 1961) If $\mathcal{F} \subset \binom{[n]}{k}$ is intersecting where $k \leq \frac{n}{2}$ then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}.$$



A family $\mathcal{F} \subset 2^{[n]}$ is **t-intersecting** if $|F \cap G| \geq t$ holds for every pair $F, G \in \mathcal{F}$.



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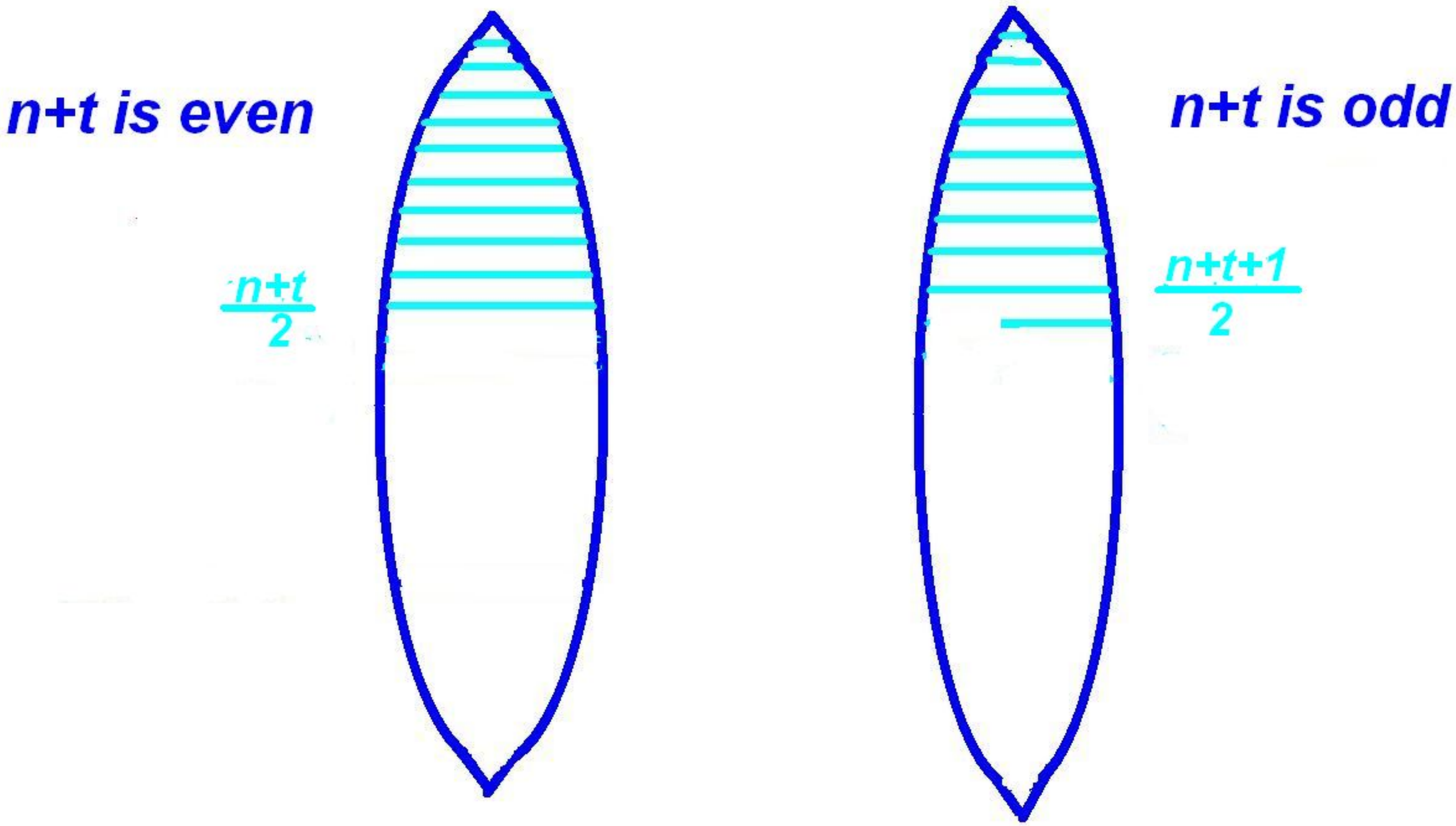
Theorem (K, 1964) If $\mathcal{F} \subset 2^{[n]}$ is t-intersecting then

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=\frac{n+t}{2}}^{n} \binom{n}{i} & \text{if } n+t \text{ is even} \\ \sum_{i=\frac{n+t+1}{2}}^{n} \binom{n}{i} + \binom{n-1}{\frac{n+t-1}{2}} & \text{if } n+t \text{ is odd .} \end{cases}$$


n+t is even
$\frac{n+t}{2}$
n+t is odd
$\frac{n+t+1}{2}$

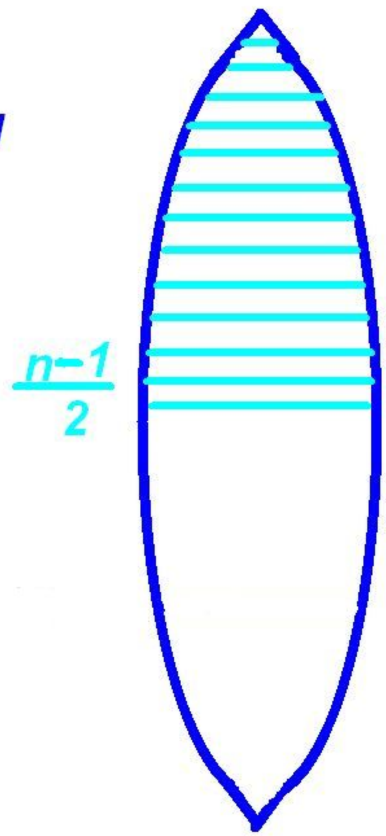
A **problem** of **Körner**.

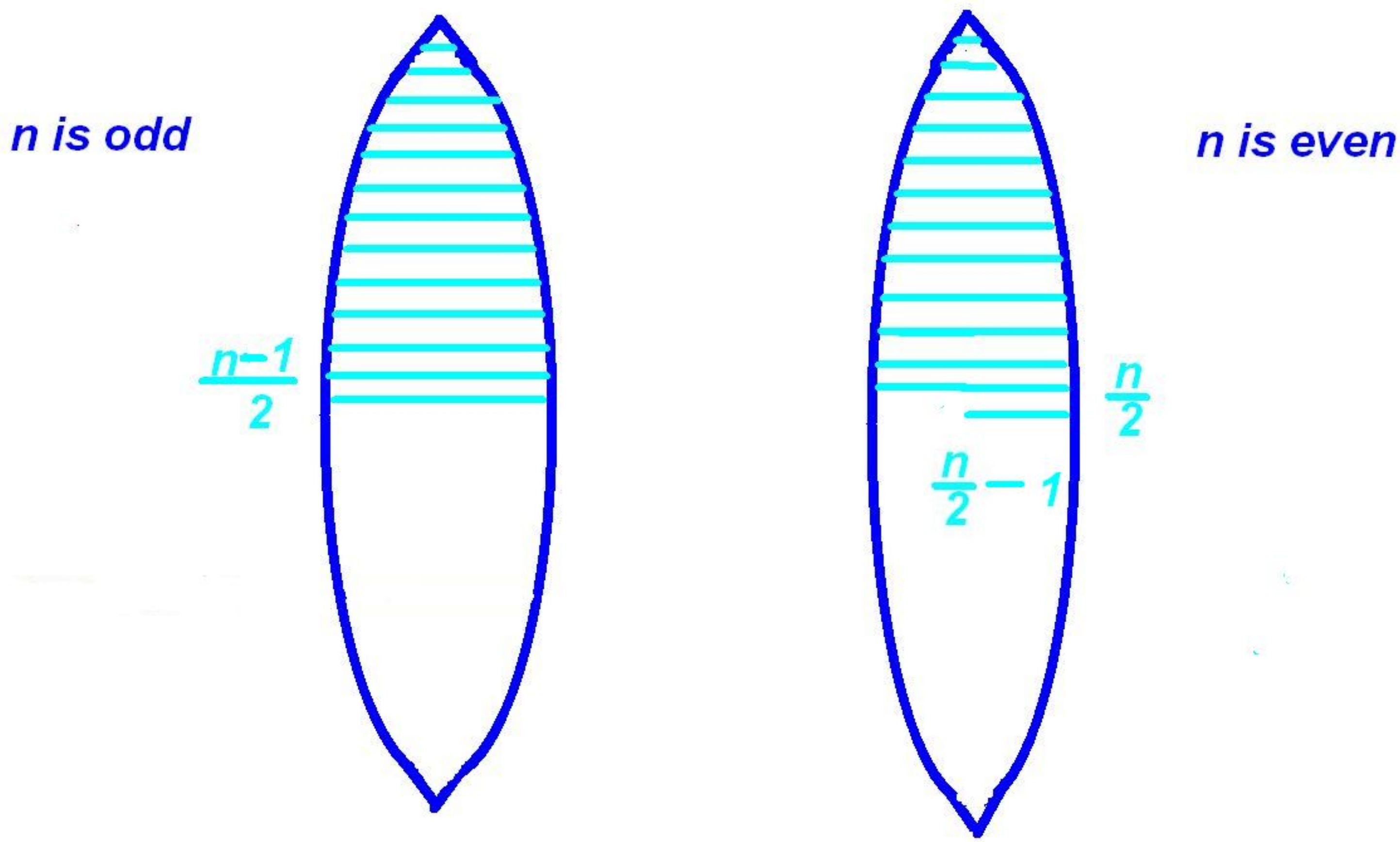
Let $\mathcal{F} \subset 2^{[n]}$ and suppose that if $F_1, F_2, G_1, G_2 \in \mathcal{F}, F_1 \neq F_2, G_1 \neq G_2$ holds then

$$(F_1 \cup F_2) \cap (G_1 \cup G_2) \neq \emptyset.$$

What is the maximum size of such a **union-intersecting** family?

n is odd




n is odd
n−1
2
n is even
n
2
n
2 − 1

Theorem (Katona-D.T. Nagy 2014+) Suppose that the family $\mathcal{F} \subset 2^{[n]}$ is a union–intersecting family then

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=\frac{n-1}{2}}^{n} \binom{n}{i} & \text{if } n+1 \text{ is odd} \\ \sum_{i=\frac{n}{2}}^{n} \binom{n}{i} + \binom{n-1}{\frac{n}{2}-1} & \text{if } n+1 \text{ is even .} \end{cases}$$

holds.

A family $\mathcal{F} \subset 2^{[n]}$ is called a (u, v)-**union-intersecting** if for different members $F_1, \ldots, F_u, G_1, \ldots, G_v$ the following holds:

$$(\cup_{i=1}^{u} F_i) \cap \left(\cup_{j=1}^{v} G_j\right) \neq \emptyset.$$

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$f(n, 2, 2)$ is the previous theorem.

Theorem (Katona-D.T. Nagy 2014+)

$$f(n,1,2) = \begin{cases} \sum_{i=\frac{n}{2}}^{n} \binom{n}{i} & \text{if } n \text{ is even} \\ \sum_{i=\frac{n+1}{2}}^{n} \binom{n}{i} + \binom{n-1}{\frac{n-3}{2}} & \text{if } n \text{ is odd .} \end{cases}$$

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Proof

Standardization of mathematical lectures

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§345, ¶59. Every lecture should contain one proof and one joke but they must not be the same.

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$$2|\mathcal{F}|\le 2^n+|\mathcal{G}|\to|\mathcal{F}|\le 2^{n-1}+\frac{1}{2}|\mathcal{G}|$$

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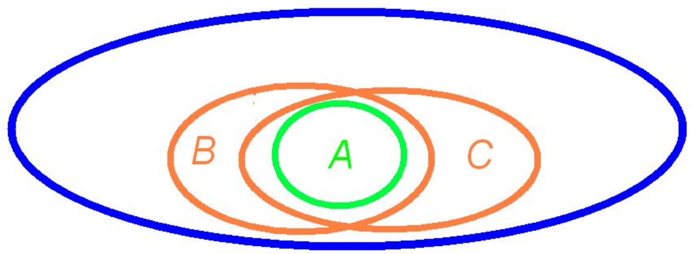

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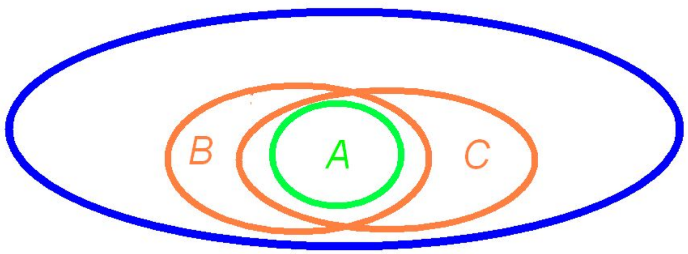

$$A \cap (\overline{A} \cup \overline{B} \cup \overline{C}) = A \cap \overline{A} = \emptyset$$

a contradiction.

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Theorem (K-Tarján, 1981)

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Hence

$$|\mathcal{F}| \leq 2^{n-1} + \frac{1}{2}|\mathcal{G}| \leq 2^{n-1} + \binom{n-1}{\left\lfloor \frac{n-1}{2} \right\rfloor}$$

Theorem (Katona-D.T. Nagy 2014+) If $v \geq 4$ then

$$2^{n-1} + \frac{1}{2}\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq f(n,1,v) \leq 2^{n-1} + \frac{1}{2}\binom{n}{\lfloor \frac{n}{2} \rfloor} + \frac{v-2}{n} + O\left(\frac{1}{n^2}\right)$$

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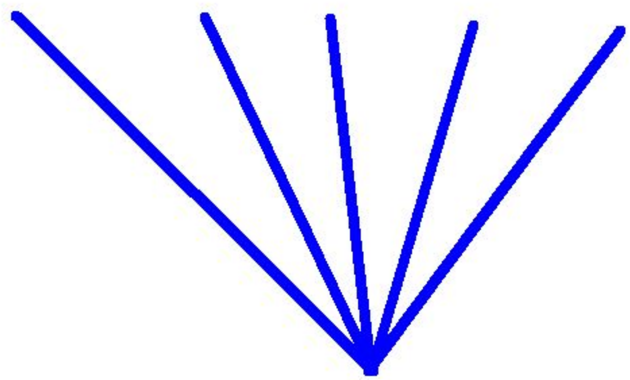

Theorem (Katona-D.T. Nagy 2014+) If $v \geq u \geq 2, v \geq 3$ then

$$2^{n-1} + \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 - \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right) \leq f(n,u,v) \leq$$

$$2^{n-1} + \binom{n}{\lfloor \frac{n}{2} \rfloor}\left(1 + \frac{u+v-3}{n} + O\left(\frac{1}{n^2}\right)\right)$$

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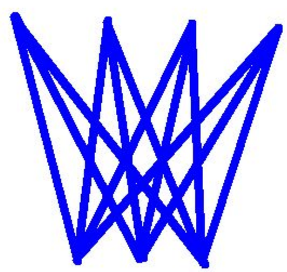

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holds if $n > n(k, v)$.

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Is there an **Ahlswede-Khachatrian** theorem also here?

Շնորհակալություն

[redacted]

Thank you