On the dimension of posets with cover graphs of treewidth 2

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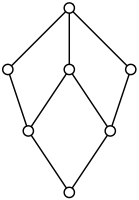
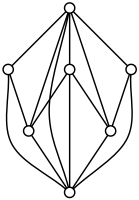
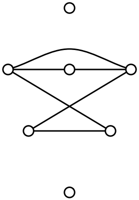
Cover graphs



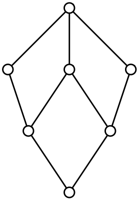
Order diagram



Comparability graph



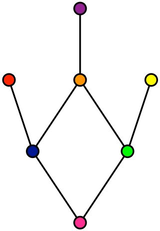
Incomparability graph



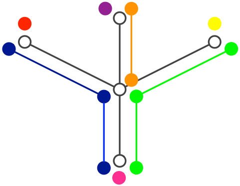
Cover graph

Treewidth

Tree decompositions:



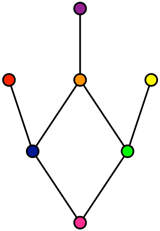
G



width = 2

Treewidth

Tree decompositions:



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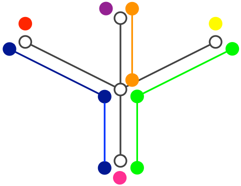

width = 2

Width $=$ max. number of subtrees seen by a node -1

Treewidth $=$ min. width of a tree decomposition

N.B. If tree required to be a path $\rightarrow$ path decomposition / pathwidth

Theorem (Trotter & Moore, 1977)

Cover graph of P is a forest $\Rightarrow \dim(P) \leqslant 3$

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Restatement:

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Cover graph of P has treewith $\leqslant 1$ $\Rightarrow$ $\dim(P) \leqslant 3$

Can this be extended to graphs of bounded treewidth?

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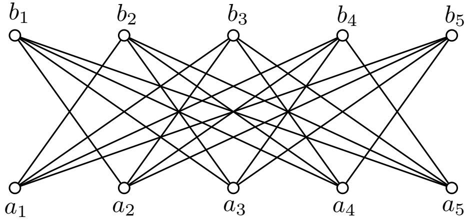
Restatement:

Theorem (Trotter & Moore, 1977)

Cover graph of P has treewith $\leqslant 1$ $\Rightarrow$ $\dim(P) \leqslant 3$

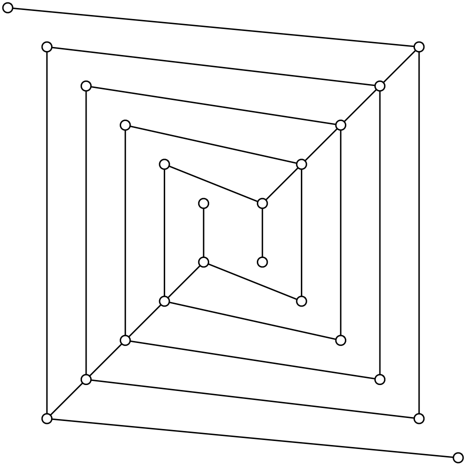
Can this be extended to graphs of bounded treewidth? **No!**

Standard examples



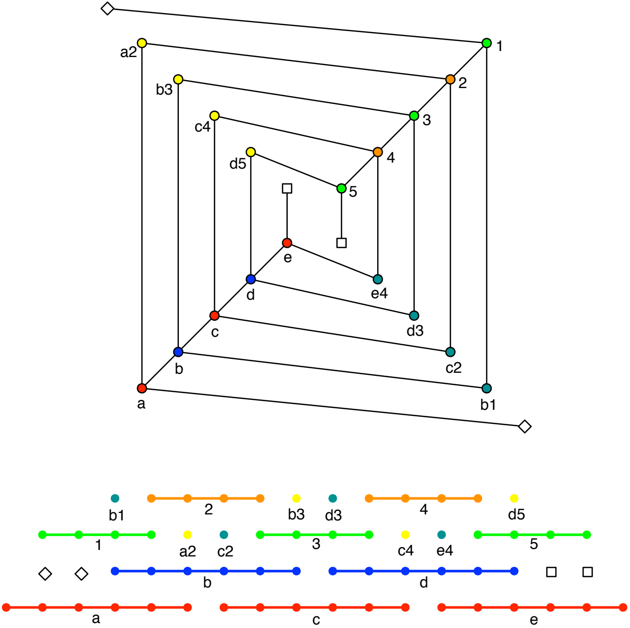
Standard example S_n has dimension n

Kelly's construction (illustration for $n = 6$)



- contains standard example S_n $\quad\Rightarrow\quad$ $\dim(P) \geqslant n$
- treewidth $=$ pathwidth $= 3$

Kelly's construction – path decomposition

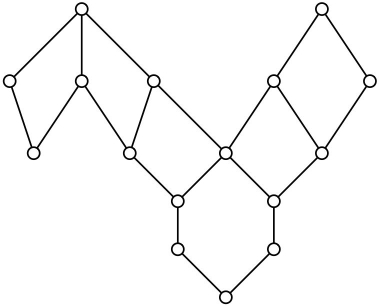


What about treewidth 2?

Graphs of treewidth $\leqslant 2$ have a simple structure:

- treewidth $\leqslant 2 \quad \Leftrightarrow \quad$ no K_4-minor
- treewidth $\leqslant 2 \quad \Leftrightarrow \quad$ series-parallel
- ...

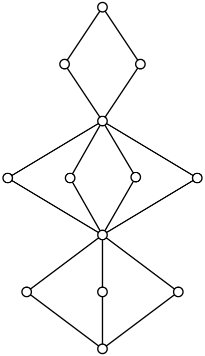
Two special cases: 1. Outerplanar cover graphs



Theorem (Felsner, Trotter, Wiechert, 2014)

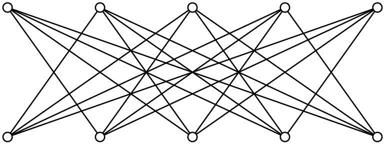
Cover graph of P outerplanar $\Rightarrow$ $\dim(P) \leqslant 4$

Two special cases: 2. Cover graphs of pathwidth $\leqslant 2$



Theorem (Biró, Keller, Young, 2014)

Cover graph of P has pathwidth $\leqslant 2 \;\Rightarrow\; \dim(P) \leqslant 17$



Theorem (Biró, Keller, Young, 2014)

Cover graph of P has treewidth $\leqslant 2 \;\Rightarrow\; P$ does not contain S_5

$\rightarrow$ no hope of adapting Kelly's construction so that treewidth $\leqslant 2$

A natural conjecture

"Every poset whose cover graph has treewidth $\leqslant 2$ *has bounded dimension"*

A natural conjecture

"Every poset whose cover graph has treewidth $\leqslant 2$ has bounded dimension"

Theorem (J., Micek, Trotter, Wang, Wiechert, 2014)

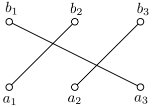
Cover graph of P has treewidth $\leqslant 2$ $\Rightarrow \dim(P) \leqslant 2554$

About the proof

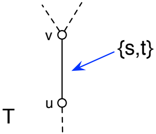
- Each incomparable pair (a, b) receives a signature $\sigma(a, b)$ from a finite set Σ of signatures, encoding various properties of the pair
- Goal: Show that, for each $\sigma \in \Sigma$, the set of pairs receiving signature σ is reversible
- Proof uses lots of "congestion strategies"

Toy example:

- say set under consideration not reversible because three pairs $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ form a strict alternating cycle:



- say $\exists$ edge uv of T corresponding to a cutset $\{s, t\}$ separating $\{a_1, a_2, a_3\}$ from $\{b_1, b_2, b_3\}$:



- path witnessing $a_i \leqslant b_{i+1}$ meets $\{s, t\}$ for $i = 1, 2, 3$

  $\Rightarrow$ two meet the same element, say s

  $\Rightarrow$ $a_i \leqslant b_j$ for some i, j with $j \neq i + 1$, contradiction

What next?

Forbidding large standard examples

$\mathrm{se}(P) :=$ largest n s.t. P contains S_n

Fix a class of graphs	Fix a class of posets
$\chi(G) \geqslant \omega(G) \quad \forall G$	$\dim(P) \geqslant \mathrm{se}(P) \quad \forall P$
class is χ-bounded if	class is dim-bounded if
$\chi(G) \leqslant f(\omega(G)) \quad \forall G$	$\dim(P) \leqslant f(\mathrm{se}(P)) \quad \forall P$
for some function f	for some function f

- Are planar posets dim-bounded? (Trotter)
- Same question for posets with cover graphs of treewidth $\leqslant k$
  $\rightarrow$ would generalize result for treewidth 2