Symmetric Chains in Quotients of Boolean Lattices

Dwight Duffus and Kyle Thayer

Mathematics & Computer Science, Emory University, Atlanta GA USA
dwight@mathcs.emory.edu
kyle.thayer@gmail.com

SIAM DM Meeting
Minneapolis / 16 - 19 June 2014
Symmetry in ranked partially ordered sets

A partially ordered set $P$ of length $n$ is ranked if all maximal chains between fixed endpoints have the same length; ranked orders admit a partition $P = \bigsqcup_{i=0}^{n} P_i$ where $P_i = \min(P - \bigsqcup_{j=0}^{i-1} P_j)$.

Let $r_i = |P_i|, i = 0, 1, \ldots, n$, denote the rank numbers of $P$.

$P$ is rank-symmetric if $r_i = r_{n-i}$ for all $i$.

$P$ is rank-unimodal if $r_0 \leq r_1 \leq \cdots \leq r_j \geq r_{j+1} \geq \cdots \geq r_n$.

$P$ is $k$-Sperner if no union of $k$ antichains is larger than the union of the $k$ largest ranks, and $P$ is strongly Sperner if it is $k$-Sperner for $k = 1, 2, \ldots, n+1$.

$P$ has the LYM property if for all antichains $A \subseteq P$, \[ \sum_{i=0}^{n} |A \cap P_i| / r_i \leq 1. \]

$P$ has a symmetric chain decomposition if $P = \bigsqcup_{i=0}^{n} C_i$ with each $C_i$ a symmetric, saturated chain in $P$. 
A partially ordered set $P$ of length $n$ is *ranked* if all maximal chains between fixed endpoints have the same length; ranked orders admit a partition $P = \bigsqcup_{i=0}^{n} P_i$ where $P_i = \min(P - \bigcup_{j=0}^{i-1} P_j)$.

Let $r_i = |P_i|$, $i = 0, 1, \ldots, n$, denote the *rank numbers of* $P$. 

$P$ is *rank-symmetric* if $r_i = r_{n-i}$ for all $i$.

$P$ is *rank-unimodal* if $r_0 \leq r_1 \leq \cdots \leq r_j \geq r_{j+1} \geq \cdots \geq r_n$.

$P$ is *$k$-Sperner* if no union of $k$ antichains is larger than the union of the $k$ largest ranks, and $P$ is *strongly Sperner* if it is $k$-Sperner for $k = 1, 2, \ldots, n+1$.

$P$ has the *LYM property* if for all antichains $A \subseteq P$, $\sum_{i=0}^{n} |A \cap P_i| / r_i \leq 1$.

$P$ has a *symmetric chain decomposition* if $P = \bigsqcup_{i=0}^{n} C_i$ with each $C_i$ a symmetric, saturated chain in $P$. 


A partially ordered set $P$ of length $n$ is \emph{ranked} if all maximal chains between fixed endpoints have the same length; ranked orders admit a partition $P = \bigsqcup_{i=0}^{n} P_i$ where $P_i = \min(P - \bigcup_{j=0}^{i-1} P_j)$.

Let $r_i = |P_i|$, $i = 0, 1, \ldots, n$, denote the \emph{rank numbers} of $P$.

$P$ is \emph{rank-symmetric} if $r_i = r_{n-i}$ for all $i$.
A partially ordered set $P$ of length $n$ is *ranked* if all maximal chains between fixed endpoints have the same length; ranked orders admit a partition $P = \bigsqcup_{i=0}^{n} P_i$ where $P_i = \min(P - \bigcup_{j=0}^{i-1} P_j)$.

Let $r_i = |P_i|$, $i = 0, 1, \ldots, n$, denote the *rank numbers* of $P$.

- $P$ is *rank-symmetric* if $r_i = r_{n-i}$ for all $i$.
- $P$ is *rank-unimodal* if $r_0 \leq r_1 \leq \cdots \leq r_j \geq r_{j+1} \geq \cdots \geq r_n$. 

Symmetry in ranked partially ordered sets

A partially ordered set $P$ of length $n$ is *ranked* if all maximal chains between fixed endpoints have the same length; ranked orders admit a partition $P = \bigsqcup_{i=0}^{n} P_i$ where $P_i = \min(P - \bigcup_{j=0}^{i-1} P_j)$.

Let $r_i = |P_i|$, $i = 0, 1, \ldots, n$, denote the *rank numbers* of $P$.

- $P$ is *rank-symmetric* if $r_i = r_{n-i}$ for all $i$
- $P$ is *rank-unimodal* if $r_0 \leq r_1 \leq \cdots \leq r_j \geq r_{j+1} \geq \cdots \geq r_n$
- $P$ is *$k$-Sperner* if no union of $k$ antichains is larger than the union of the $k$ largest ranks, and $P$ is *strongly Sperner* if it is $k$-Sperner for $k = 1, 2, \ldots, n + 1$
A partially ordered set $P$ of length $n$ is \textit{ranked} if all maximal chains between fixed endpoints have the same length; ranked orders admit a partition $P = \bigsqcup_{i=0}^{n} P_i$ where $P_i = \min(P - \bigcup_{j=0}^{i-1} P_j)$.

Let $r_i = |P_i|$, $i = 0, 1, \ldots, n$, denote the \textit{rank numbers} of $P$.

\begin{itemize}
  \item $P$ is \textit{rank-symmetric} if $r_i = r_{n-i}$ for all $i$
  \item $P$ is \textit{rank-unimodal} if $r_0 \leq r_1 \leq \cdots \leq r_j \geq r_{j+1} \geq \cdots \geq r_n$
  \item $P$ is \textit{k-Sperner} if no union of $k$ antichains is larger than the union of the $k$ largest ranks, and $P$ is \textit{strongly Sperner} if it is $k$-Sperner for $k = 1, 2, \ldots, n + 1$
  \item $P$ has the \textit{LYM property} if for all antichains $A \subseteq P$,
    \[ \sum_{i=0}^{n} \frac{|A \cap P_i|}{r_i} \leq 1 \]
\end{itemize}
A partially ordered set $P$ of length $n$ is \textit{ranked} if all maximal chains between fixed endpoints have the same length; ranked orders admit a partition $P = \bigsqcup_{i=0}^{n} P_i$ where $P_i = \min(P - \bigcup_{j=0}^{i-1} P_j)$.

Let $r_i = |P_i|$, $i = 0, 1, \ldots, n$, denote the \textit{rank numbers} of $P$.

\begin{itemize}
    \item $P$ is \textit{rank-symmetric} if $r_i = r_{n-i}$ for all $i$.
    \item $P$ is \textit{rank-unimodal} if $r_0 \leq r_1 \leq \cdots \leq r_j \geq r_{j+1} \geq \cdots \geq r_n$.
    \item $P$ is \textit{$k$-Sperner} if no union of $k$ antichains is larger than the union of the $k$ largest ranks, and $P$ is \textit{strongly Sperner} if it is $k$-Sperner for $k = 1, 2, \ldots, n + 1$.
    \item $P$ has the \textit{LYM property} if for all antichains $A \subseteq P$,
        $$\sum_{i=0}^{n} \frac{|A \cap P_i|}{r_i} \leq 1$$
    \item $P$ has a \textit{symmetric chain decomposition} if $P = \bigsqcup_{i=0}^{n} C_i$ with each $C_i$ a symmetric, saturated chain in $P$.
\end{itemize}
Quotients and automorphisms of partially ordered sets

For a partially ordered set $P$ and $G \leq \text{Aut}(P)$, the quotient of $P$ by $G$, $P/G$, is the set of orbits of $P$ under $G$, ordered by $[x] \leq [y] \iff \exists x' \in [x], y' \in [y]$ with $x' \leq y'$ in $P$.

The Boolean lattice $2^n$ of all subsets of $\{1, 2, \ldots, n\}$ ordered by $\subseteq$ has $\text{Aut}(2^n) \cong S_n$ the symmetric group on $\{1, 2, \ldots, n\}$.

More generally, for any finite chain $C$, $\text{Aut}(C_n) \cong S_n$ and for chains $C_i$ of distinct lengths and $n_i \in \mathbb{N}$ ($i = 1, 2, \ldots, m$), $\text{Aut}(C_{n_1} \times C_{n_2} \times \cdots \times C_{n_m}) \cong S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}$. 
For a partially ordered set $P$ and $G \leq \text{Aut}(P)$, the *quotient of $P$ by $G$, $P/G$, is the set of orbits of $P$ under $G$, ordered by*

$$[x] \leq [y] \iff \exists x' \in [x], \ y' \in [y] \text{ with } x' \leq y' \text{ in } P.$$
For a partially ordered set $P$ and $G \leq \text{Aut}(P)$, the quotient of $P$ by $G$, $P/G$, is the set of orbits of $P$ under $G$, ordered by

$$[x] \leq [y] \iff \exists x' \in [x], y' \in [y] \text{ with } x' \leq y' \text{ in } P.$$ 

The Boolean lattice $2^n$ of all subsets of $[n] = \{1, 2, \ldots, n\}$ ordered by $\subseteq$ has

$$\text{Aut}(2^n) \cong S_n$$

the symmetric group on $[n]$. 
For a partially ordered set $P$ and $G \leq \text{Aut}(P)$, the \textit{quotient of $P$ by $G$}, $P/G$, is the set of orbits of $P$ under $G$, ordered by

$$[x] \leq [y] \iff \exists x' \in [x], \ y' \in [y] \text{ with } x' \leq y' \text{ in } P.$$ 

The Boolean lattice $2^n$ of all subsets of $[n] = \{1, 2, \ldots, n\}$ ordered by $\subseteq$ has

$$\text{Aut}(2^n) \cong S_n$$

the symmetric group on $[n]$.

More generally, for any finite chain $C$,

$$\text{Aut}(C^n) \cong S_n$$
For a partially ordered set $P$ and $G \leq \text{Aut}(P)$, the \textit{quotient of $P$ by $G$}, $P/G$, is the set of orbits of $P$ under $G$, ordered by

$$[x] \leq [y] \iff \exists x' \in [x], y' \in [y] \text{ with } x' \leq y' \text{ in } P.$$ 

The Boolean lattice $2^n$ of all subsets of $[n] = \{1, 2, \ldots, n\}$ ordered by $\subseteq$ has

$$\text{Aut}(2^n) \cong S_n$$

the symmetric group on $[n]$.

More generally, for any finite chain $C$,

$$\text{Aut}(C^n) \cong S_n$$

and for chains $C_i$ of distinct lengths and $n_i \in \mathbb{N}$ ($i = 1, 2, \ldots, m$)

$$\text{Aut}(C_1^{n_1} \times C_2^{n_2} \times \cdots \times C_m^{n_m}) \cong S_{n_1} \times S_{n_2} \times \cdots \times S_{n_m}.$$
Fig. 5. The Hasse diagram for $N_6$.

Figure credit: K.K. Jordan (2010)
Example

\[ B_6 / \langle (1 2 3 4 5 6) \rangle \]

Figure credit: K K Jordan (2010)
Symmetry and quotients

Stanley [1980] proved that some interesting quotients are rank-symmetric, rank-unimodal, strongly Sperner. Stanley [1984], Pouzet [1976], Pouzet and Rosenberg [1986], and Harper [1984] proved results from which these follow:

- for any $G \leq S_n$, $2^n/G$ is rank-symmetric, rank-unimodal and strongly Sperner;
- for any $P$ that is a product of chains and $G \leq \text{Aut}(P)$, $P/G$ is rank-symmetric, rank-unimodal and strongly Sperner.

Question: Do the quotients $2^n/G$ always have SCDs? Special cases of this were posed by Stanley but, in this generality, the question was not asked until 20 years after these papers.
Stanley [1980] proved that some interesting quotients are

- rank-symmetric, rank-unimodal, strongly Sperner
Symmetry and quotients

Stanley [1980] proved that some interesting quotients are

- rank-symmetric, rank-unimodal, strongly Sperner

Stanley [1984], Pouzet [1976], Pouzet and Rosenberg [1986], and Harper [1984] proved results from which these follow:

For any $G \leq S_n$, $2^n/G$ is rank-symmetric, rank-unimodal and strongly Sperner;

For any $P$ that is a product of chains and $G \leq \text{Aut}(P)$, $P/G$ is rank-symmetric, rank-unimodal and strongly Sperner.

Question: Do the quotients $2^n/G$ always have SCDs?

Special cases of this were posed by Stanley but, in this generality, the question was not asked until 20 years after these papers.
Stanley [1980] proved that some interesting quotients are

- rank-symmetric, rank-unimodal, strongly Sperner

Stanley [1984], Pouzet [1976], Pouzet and Rosenberg [1986], and Harper [1984] proved results from which these follow:

- for any $G \leq S_n$, $2^n/G$ is rank-symmetric, rank-unimodal and strongly Sperner.
Stanley [1980] proved that some interesting quotients are rank-symmetric, rank-unimodal, strongly Sperner.

Stanley [1984], Pouzet [1976], Pouzet and Rosenberg [1986], and Harper [1984] proved results from which these follow:

- for any $G \leq S_n$, $2^n/G$ is rank-symmetric, rank-unimodal and strongly Sperner;
- for any $P$ that is a product of chains and $G \leq \text{Aut}(P)$, $P/G$ is rank-symmetric, rank-unimodal and strongly Sperner.
Stanley [1980] proved that some interesting quotients are rank-symmetric, rank-unimodal, strongly Sperner.

Stanley [1984], Pouzet [1976], Pouzet and Rosenberg [1986], and Harper [1984] proved results from which these follow:

- For any $G \leq S_n$, $2^n/G$ is rank-symmetric, rank-unimodal, and strongly Sperner;
- For any $P$ that is a product of chains and $G \leq \text{Aut}(P)$, $P/G$ is rank-symmetric, rank-unimodal, and strongly Sperner.

**Question**: Do the quotients $2^n/G$ always have SCDs?
Stanley [1980] proved that some interesting quotients are

- rank-symmetric, rank-unimodal, strongly Sperner

Stanley [1984], Pouzet [1976], Pouzet and Rosenberg [1986], and Harper [1984] proved results from which these follow:

- for any $G \leq S_n$, $2^n/G$ is rank-symmetric, rank-unimodal and strongly Sperner;
- for any $P$ that is a product of chains and $G \leq \text{Aut}(P)$, $P/G$ is rank-symmetric, rank-unimodal and strongly Sperner.

**Question:** Do the quotients $2^n/G$ always have SCDs?

Special cases of this were posed by Stanley but, in this generality, the question was not asked until 20 years after these papers.
Symmetric chains

Let $P$ be a rank-symmetric partially ordered set of length $n$:

$$P = \bigsqcup_{i=0}^{n} P_i$$

and $r_i = |P_i|$ ($i = 0, 1, \ldots, n$).

$P$ is rank-unimodal and strongly Sperner iff for $i = 0, 1, \ldots, \lfloor n/2 \rfloor$ there exist $r$-saturated chains $x_i < x_i + 1 < \cdots < x_{n-i}$, $x_j \in P_j$:
Let $P$ be a rank-symmetric partially ordered set of length $n$:

$$P = \bigsqcup_{i=0}^{n} P_i \text{ and } r_i = |P_i| \ (i = 0, 1, \ldots, n).$$
Let $P$ be a rank-symmetric partially ordered set of length $n$:

$$P = \bigsqcup_{i=0}^{n} P_i \text{ and } r_i = |P_i| \ (i = 0, 1, \ldots, n).$$

$P$ is rank-unimodal and strongly Sperner iff for $i = 0, 1, \ldots, \lfloor n/2 \rfloor$

$$\exists r_i \text{ pwd saturated chains } x_i < x_{i+1} < \cdots < x_{n-i}, \ x_j \in P_j :$$
Symmetric chains

Let $P$ be a rank-symmetric partially ordered set of length $n$: 

$$P = \bigsqcup_{i=0}^{n} P_i \text{ and } r_i = |P_i| \ (i = 0, 1, \ldots, n).$$

$P$ is rank-unimodal and strongly Sperner iff for $i = 0, 1, \ldots, \left\lfloor n/2 \right\rfloor$

$$\exists r_i \text{ pwd saturated chains } x_i < x_{i+1} < \cdots < x_{n-i}, \ x_j \in P_j :$$
Rank-symmetry, rank-unimodality and strongly Sperner guarantee matchings

\[ \phi_i : P_i \rightarrow P_{i+1}, \quad \psi_i : P_{n-i} \rightarrow P_{n-(i+1)}, \quad i = 0, 1, \ldots, \lfloor n/2 \rfloor \]

so there is a partition of \( P \) into chains, but not necessarily symmetric ones.
Rank-symmetry, rank-unimodality and strongly Sperner guarantee matchings

\[ \phi_i : P_i \to P_{i+1}, \quad \psi_i : P_{n-i} \to P_{n-(i+1)}, \quad i = 0, 1, \ldots, \left\lfloor n/2 \right\rfloor \]

so there is a partition of \( P \) into chains, but not necessarily symmetric ones.

**Example:** [Griggs]
Questions and conjecture

1980 Is the lattice \( L(m, n) \) of all partitions of an integer into at most \( m \) parts of size at most \( n \) a symmetric chain order (SCO) [Stanley]?

2004 Is \( 2^n/Z_n \) an SCO? [Griggs, Killian, Savage] (\( Z_n \) is generated by an \( n \)-cycle.)

Conjecture:

2006 For all \( G \leq S_n \), \( 2^n/G \) is an SCO. [Canfield & Mason]
Questions:

1980 Is the lattice $L(m, n)$ of all partitions of an integer into at most $m$ parts of size at most $n$ a symmetric chain order (SCO) [Stanley]?
Questions and conjecture

Questions:

1980 Is the lattice $L(m, n)$ of all partitions of an integer into at most $m$ parts of size at most $n$ a symmetric chain order (SCO) [Stanley]?

2004 Is $2^n/\mathbb{Z}_n$ an SCO? [Griggs, Killian, Savage] ($\mathbb{Z}_n$ is generated by an $n$-cycle.)
Questions:  

1980  Is the lattice $L(m, n)$ of all partitions of an integer into at most $m$ parts of size at most $n$ a symmetric chain order (SCO) [Stanley]?

2004  Is $2^n/\mathbb{Z}_n$ an SCO? [Griggs, Killian, Savage]  
($\mathbb{Z}_n$ is generated by an $n$-cycle.)

Conjecture:  

2006  For all $G \leq S_n$, $2^n/G$ is an SCO. [Canfield & Mason]
Stanley’s tableau example

$L(m, n)$ is the collection of all downsets of an $m \times n$ grid, ordered by $\subseteq$.

Some Background and Motivation:
Given an $m \times n$ grid, $L(m, n)$ is the collection of all downsets, ordered by containment.

Let $G \leq S_{mn}$ be the group of all permutations constructed from $n$ independent permutations within the columns, followed by a permutation of the columns: $G \sim = S_m \wr S_n$. Each orbit under $G$ has a unique downset representative: thus, $L(m, n) \sim = \frac{2^{mn}}{G}$.

Question: [Stanley 1980] Is $L(m, n)$ an SCO?
\[ L(m, n) \] is the collection of all downsets of an \( m \times n \) grid, ordered by \( \subseteq \).
$L(m, n)$ is the collection of all downsets of an $m \times n$ grid, ordered by $\subseteq$. 

\[ m \]

\[ n \]
Stanley’s tableau example

$L(m, n)$ is the collection of all downsets of an $m \times n$ grid, ordered by $\subseteq$.

Let $G \leq S_{mn}$ be the group of all permutations constructed from $n$ independent permutations within the columns, followed by a permutation of the columns:
Stanley’s tableau example

\[ L(m, n) \] is the collection of all downsets of an \( m \times n \) grid, ordered by \( \subseteq \).

Let \( G \leq S_{mn} \) be the group of all permutations constructed from \( n \) independent permutations within the columns, followed by a permutation of the columns: \( G \cong S_m \wr S_n \). Each orbit under \( G \) has a unique downset representative: thus, \( L(m, n) \cong 2^{mn}/G \).
Stanley’s tableau example

$L(m, n)$ is the collection of all downsets of an $m \times n$ grid, ordered by $\subseteq$.

Let $G \leq S_{mn}$ be the group of all permutations constructed from $n$ independent permutations within the columns, followed by a permutation of the columns: $G \cong S_m \wr S_n$. Each orbit under $G$ has a unique downset representative: thus, $L(m, n) \cong 2^{mn} / G$.

**Question:** [Stanley 1980] Is $L(m, n)$ an SCO?
What is known?

1. For $n$ prime, $\mathbb{Z}_n \leq S_n$ is an SCO [Griggs, Killian, Savage 2004];
2. For all $n$, $\mathbb{Z}_n$ is an SCO [Jordan 2010; Hersh and Schilling 2011];
3. For $P$ a product of chains and $\mathcal{K} \leq \text{Aut}(P)$ generated by powers of disjoint cycles, $P/\mathcal{K}$ is an SCO [Duffus, McKibben-Sanders and Thayer 2011];
4. For all $n$ and all SCOs $P$, $P_n/\mathbb{Z}_n$ is an SCO [Dhand 2011];
What is known?

Let $\mathbb{Z}_n \leq S_n$ to be generated by the shift or $n$-cycle $(1 \ 2 \ \cdots \ n)$. 

1. for $n$ prime, $2^n / \mathbb{Z}_n$ is an SCO [Griggs, Killian, Savage 2004];
2. for all $n$, $2^n / \mathbb{Z}_n$ is an SCO [Jordan 2010; Hersh and Schilling 2011];
3. for $P$ a product of chains and $K \leq \operatorname{Aut}(P)$ generated by powers of disjoint cycles, $P / K$ is an SCO [Duffus, McKibben-Sanders and Thayer 2011];
4. for all $n$ and all SCOs $P$, $P^n / \mathbb{Z}_n$ is an SCO [Dhand 2011].
Let $\mathbb{Z}_n \leq S_n$ to be generated by the shift or $n$-cycle $(1 \ 2 \ \cdots \ n)$.

1. for $n$ prime, $2^n/\mathbb{Z}_n$ is an SCO [Griggs, Killian, Savage 2004];
What is known?

Let $\mathbb{Z}_n \leq S_n$ to be generated by the shift or $n$-cycle $(1 \ 2 \ \cdots \ n)$.

1. for $n$ prime, $2^n/\mathbb{Z}_n$ is an SCO [Griggs, Killian, Savage 2004];

2. for all $n$, $2^n/\mathbb{Z}_n$ is an SCO [Jordan 2010; Hersh and Schilling 2011];
What is known?

Let $\mathbb{Z}_n \leq S_n$ to be generated by the shift or $n$-cycle $(1 \ 2 \ \cdots \ n)$.

1. for $n$ prime, $2^n/\mathbb{Z}_n$ is an SCO [Griggs, Killian, Savage 2004];

2. for all $n$, $2^n/\mathbb{Z}_n$ is an SCO [Jordan 2010; Hersh and Schilling 2011];

3. for $P$ a product of chains and $K \leq \text{Aut}(P)$ generated by powers of disjoint cycles, $P/K$ is an SCO [Duffus, McKibben-Sanders and Thayer 2011];
Let $\mathbb{Z}_n \leq S_n$ to be generated by the shift or $n$-cycle $(1\ 2\ \cdots\ n)$.

1. for $n$ prime, $2^n/\mathbb{Z}_n$ is an SCO [Griggs, Killian, Savage 2004];

2. for all $n$, $2^n/\mathbb{Z}_n$ is an SCO [Jordan 2010; Hersh and Schilling 2011];

3. for $P$ a product of chains and $K \leq \text{Aut}(P)$ generated by powers of disjoint cycles, $P/K$ is an SCO [Duffus, McKibben-Sanders and Thayer 2011];

4. for all $n$ and all SCOs $P$, $P^n/\mathbb{Z}_n$ is an SCO [Dhand 2011];
5. Let $n = kt$, $G \leq S_n$, $K \leq S_k$, $T \leq S_t$, $G = K \wr T$ via the natural action of $T$ on $K^t$. If

(a) $2^k/K$ is an SCO, and

(b) $T$ is generated by powers of disjoint cycles

then $2^n/G$ is an SCO.
5. Let $n = kt$, $G \leq S_n$, $K \leq S_k$, $T \leq S_t$, $G = K \wr T$ via the natural action of $T$ on $K^t$. If

(a) $2^k/K$ is an SCO, and

(b) $T$ is generated by powers of disjoint cycles

then $2^n/G$ is an SCO.

Base case for (a): $K$ is generated by powers of disjoint cycles.

[Duffus and Thayer(2014\textsuperscript{+})]
Open Problems

Problem 1:
For all $n \geq 1$, let $D_{2n}$ denote the dihedral group of symmetries of a regular $n$-gon. Show that $\mathbb{Z}_2 \times \mathbb{Z}_{2n}$ is an SCO.

[Griggs, Killian, Savage (2004)]

Problem 2:
Show that for all $k, t$, $L(k, t)$ is an SCO. Equivalently, show that this quotient is an SCO:

$$\mathbb{Z}_{2kt} / (\mathbb{Z}_k \rtimes \mathbb{Z}_t) \cong \mathbb{Z}_{(k+1)t} / \mathbb{Z}_t.$$

[Stanley 1980]

Problem 3:
Determine if for every embedding $\phi$ of a finite abelian group $A$ in $S_n$, $\mathbb{Z}_n / \phi(A)$ is an SCO.
Problem 1: For all \( n \geq 1 \), let \( D_{2n} \) denote the dihedral group of symmetries of a regular \( n \)-gon. Show that \( 2^n/D_{2n} \) is an SCO.
[Griggs, Killian, Savage (2004)]
Problem 1: For all $n \geq 1$, let $D_{2n}$ denote the dihedral group of symmetries of a regular $n$-gon. Show that $2^n/D_{2n}$ is an SCO.
[Griggs, Killian, Savage (2004)]

Problem 2: Show that for all $k, t$, $L(k, t)$ is an SCO. Equivalently, show that this quotient is an SCO:

$$2^{kt}/(S_k \wr S_t) \cong (k + 1)^t/S_t.$$  
[Stanley 1980]
Problem 1: For all \( n \geq 1 \), let \( D_{2n} \) denote the dihedral group of symmetries of a regular \( n \)-gon. Show that \( 2^n / D_{2n} \) is an SCO.
[Griggs, Killian, Savage (2004)]

Problem 2: Show that for all \( k, t \), \( L(k, t) \) is an SCO. Equivalently, show that this quotient is an SCO:

\[
2^{kt} / (S_k \wr S_t) \cong (k + 1)^t / S_t.
\]
[Stanley 1980]

Problem 3: Determine if for every embedding \( \phi \) of a finite abelian group \( A \) in \( S_n \), \( 2^n / \phi(A) \) is an SCO.
Let $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$. 
Let $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$. Then with

$$A_1 = \langle (1 \ 2), (3 \ 4), (5 \ 6 \ 7 \ 8 \ 9) \rangle \leq S_9$$

$$2^9/A_1 \text{ is an SCO}$$
Let $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$. Then with

$$A_1 = \langle (1 \ 2), (3 \ 4), (5 \ 6 \ 7 \ 8 \ 9) \rangle \leq S_9$$

$2^9/A_1$ is an SCO

$$A_2 = \langle (1 \ 2), (3 \ 4 \ \cdots \ 12) \rangle \leq S_{12}$$

$2^{12}/A_2$ is an SCO
Let $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$. Then with

$$A_1 = \langle (1 \ 2), (3 \ 4), (5 \ 6 \ 7 \ 8 \ 9) \rangle \leq S_9$$

$2^9/A_1$ is an SCO

$$A_2 = \langle (1 \ 2), (3 \ 4 \ \cdots \ 12) \rangle \leq S_{12}$$

$2^{12}/A_2$ is an SCO

Take $A_3$ to the regular representation of $A$ in $S_{20}$. Is $2^{20}/A_3$ an SCO?
Let $A = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5$. Then with

\[ A_1 = \langle (1 2), (3 4), (5 6 7 8 9) \rangle \leq S_9 \]

\[ 2^9/A_1 \text{ is an SCO} \]

\[ A_2 = \langle (1 2), (3 4 \cdots 12) \rangle \leq S_{12} \]

\[ 2^{12}/A_2 \text{ is an SCO} \]

Take $A_3$ to the regular representation of $A$ in $S_{20}$. Is $2^{20}/A_3$ an SCO?

**Problem 4:** Determine if the regular representation of an abelian group $A$ produces a quotient of $2^{|A|}$ with an SCD.
Each embedding of an abelian group $A$ in a symmetric group can be obtained as a product of actions $A$ on factor groups $A/H$. Here is a test case.

**Example:** Let $A$ be an elementary abelian $p$-group, say $A \cong \mathbb{Z}_p^t$, and let $H_i$ ($i = 1, 2, \ldots, k$) be the maximal subgroups of $A$. 
Each embedding of an abelian group $A$ in a symmetric group can be obtained as a product of actions $A$ on factor groups $A/H$. Here is a test case.

**Example:** Let $A$ be an elementary abelian $p$-group, say $A \cong \mathbb{Z}_p^t$, and let $H_i$ ($i = 1, 2, \ldots, k$) be the maximal subgroups of $A$.

- the $H_i$’s are the hyperplanes of $A$, $k = (p^t - 1)/(p - 1)$
Each embedding of an abelian group $A$ in a symmetric group can be obtained as a product of actions $A$ on factor groups $A/H$. Here is a test case.

**Example:** Let $A$ be an elementary abelian $p$-group, say $A \cong \mathbb{Z}_p^t$, and let $H_i$ ($i = 1, 2, \ldots, k$) be the maximal subgroups of $A$.

- the $H_i$’s are the hyperplanes of $A$, $k = (p^t - 1)/(p - 1)$
- each quotient $A/H_i$ is a cyclic group of order $p$
Each embedding of an abelian group $A$ in a symmetric group can be obtained as a product of actions $A$ on factor groups $A/H$. Here is a test case.

**Example:** Let $A$ be an elementary abelian $p$-group, say $A \cong \mathbb{Z}^t_p$, and let $H_i$ ($i = 1, 2, \ldots, k$) be the maximal subgroups of $A$.

- the $H_i$’s are the hyperplanes of $A$, $k = (p^t - 1)/(p - 1)$
- each quotient $A/H_i$ is a cyclic group of order $p$
- for each $a \in A$, $x + H_i \rightarrow a + x + H_i$ is in $S_{A/H_i}$
Each embedding of an abelian group $A$ in a symmetric group can be obtained as a product of actions $A$ on factor groups $A/H$. Here is a test case.

**Example:** Let $A$ be an elementary abelian $p$-group, say $A \cong \mathbb{Z}_p^t$, and let $H_i$ ($i = 1, 2, \ldots, k$) be the maximal subgroups of $A$.

- the $H_i$'s are the hyperplanes of $A$, $k = (p^t - 1)/(p - 1)$
- each quotient $A/H_i$ is a cyclic group of order $p$
- for each $a \in A$, $x + H_i \mapsto \widehat{a}_i$ is in $S_{A/H_i}$
- for each $i$, $\widehat{A}_i := \{\widehat{a}_i \mid a \in A\} \cong \mathbb{Z}_p$
Each embedding of an abelian group $A$ in a symmetric group can be obtained as a product of actions $A$ on factor groups $A/H$. Here is a test case.

**Example:** Let $A$ be an elementary abelian $p$-group, say $A \cong \mathbb{Z}_p^t$, and let $H_i$ ($i = 1, 2, \ldots, k$) be the maximal subgroups of $A$.

- the $H_i$’s are the hyperplanes of $A$, $k = (p^t - 1)/(p - 1)$
- each quotient $A/H_i$ is a cyclic group of order $p$
- for each $a \in A$, $x + H_i \rightarrow \hat{a}_i a + x + H_i$ is in $S_{A/H_i}$
- for each $i$, $\hat{A}_i := \{\hat{a}_i \mid a \in A\} \cong \mathbb{Z}_p$
- $a \rightarrow (\hat{a}_1, \hat{a}_2, \ldots \hat{a}_k)$ is an embedding of $A$ in $S_N$ where $N = \bigcup N_i$, with $N_i := A/H_i$, and so $|N| = k \cdot p$
Each embedding of an abelian group $A$ in a symmetric group can be obtained as a product of actions $A$ on factor groups $A/H$. Here is a test case.

**Example:** Let $A$ be an elementary abelian $p$-group, say $A \cong \mathbb{Z}_p^t$, and let $H_i \ (i = 1, 2, \ldots, k)$ be the maximal subgroups of $A$.

- the $H_i$'s are the hyperplanes of $A$, $k = (p^t - 1)/(p - 1)$
- each quotient $A/H_i$ is a cyclic group of order $p$
- for each $a \in A$, $x + H_i \overset{\hat{a}_i}{\rightarrow} a + x + H_i$ is in $S_{A/H_i}$
- for each $i$, $\hat{A}_i := \{\hat{a}_i \mid a \in A\} \cong \mathbb{Z}_p$
- $a \mapsto (\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_k)$ is an embedding of $A$ in $S_N$ where $N = \bigcup N_i$, with $N_i := A/H_i$, and so $|N| = k \cdot p$
- $\hat{A}$ is the diagonal subgroup of $\hat{A}_1 \times \hat{A}_2 \times \cdots \times \hat{A}_k$
Proposition: $2^N / \hat{A}$ has an SCD.
Proposition: $2^N/\hat{A}$ has an SCD.

We know that

- each $2^{N_i}/\hat{A}_i$ has an SCD, since each is isomorphic to $2^p/\mathbb{Z}_p$.
**Proposition:** \( 2^N / \hat{A} \) has an SCD.

We know that

\[ \begin{align*}
\diamond & \text{ each } 2^{N_i} / \hat{A}_i \text{ has an SCD, since each is isomorphic to } 2^p / \mathbb{Z}_p \\
\diamond & 2^N / (\hat{A}_1 \times \cdots \times \hat{A}_k) = 2^{N_1} / \hat{A}_1 \times \cdots \times 2^{N_k} / \hat{A}_k
\end{align*} \]
Proposition: $2^N/\hat{A}$ has an SCD.

We know that

- each $2^{N_i}/\hat{A}_i$ has an SCD, since each is isomorphic to $2^p/\mathbb{Z}_p$
- $2^N/(\hat{A}_1 \times \cdots \times \hat{A}_k) = 2^{N_1}/\hat{A}_1 \times \cdots \times 2^{N_k}/\hat{A}_k$
- a product of SCOs is an SCO
Proposition: $2^N / \hat{A}$ has an SCD.

We know that

- each $2^N_i / \hat{A}_i$ has an SCD, since each is isomorphic to $2^P / \mathbb{Z}_p$

- $2^N / (\hat{A}_1 \times \cdots \times \hat{A}_k) = 2^{N_1} / \hat{A}_1 \times \cdots \times 2^{N_k} / \hat{A}_k$

- a product of SCOs is an SCO

- $\hat{A}$ is a subgroup of $\hat{A}_1 \times \cdots \times \hat{A}_k$ so the orbits of $2^N$ under $\hat{A}$ refine those under $\hat{A}_1 \times \cdots \times \hat{A}_k$
Proposition: \( 2^N / \hat{A} \) has an SCD.

We know that

\( \diamond \) each \( 2^{N_i} / \hat{A}_i \) has an SCD, since each is isomorphic to \( 2^p / \mathbb{Z}_p \)

\( \diamond \) \( 2^N / (\hat{A}_1 \times \cdots \times \hat{A}_k) = 2^{N_1} / \hat{A}_1 \times \cdots \times 2^{N_k} / \hat{A}_k \)

\( \diamond \) a product of SCOs is an SCO

\( \diamond \) \( \hat{A} \) is a subgroup of \( \hat{A}_1 \times \cdots \times \hat{A}_k \) so the orbits of \( 2^N \) under \( \hat{A} \) refine those under \( \hat{A}_1 \times \cdots \times \hat{A}_k \)

Let \( C \) be a member of an SCD of \( 2^{N_1} / \hat{A}_1 \times \cdots \times 2^{N_k} / \hat{A}_k \) with its rank \( j \) element \( \mathcal{X}_j = ([X_{1,j}], [X_{2,j}], \ldots, [X_{k,j}]), j = r, r+1, \ldots, n-r \).

We may assume that representatives are chosen such that

\[ X_{i,j} \subseteq X_{i,j+1}, j = r, r+1, \ldots, n-r-1. \]
Each class $\mathcal{X}_j$ refines into $p^{k-1}$ members of $2^N/\hat{A}$ as follows: let

$$\bar{\pi} = (1, \pi_2, \pi_3, \ldots, \pi_k), \quad \pi_i \in \mathbb{Z}_p,$$

and

$$\bar{\pi}(\mathcal{X}_j) = X_{1,j} \cup \pi_2(X_{2,j}) \cup \pi_3(X_{3,j}) \cup \cdots \cup \pi_k(X_{k,j}).$$
Each class \( \mathcal{X}_j \) refines into \( p^{k-1} \) members of \( 2^N/\hat{A} \) as follows: let

\[
\bar{\pi} = (1, \pi_2, \pi_3, \ldots, \pi_k), \quad \pi_i \in \mathbb{Z}_p, \quad \text{and}
\]

\[
\bar{\pi}(\mathcal{X}_j) = X_{1,j} \cup \pi_2(X_{2,j}) \cup \pi_3(X_{3,j}) \cup \cdots \cup \pi_k(X_{k,j}).
\]

Then the element \([\bar{\pi}(\mathcal{X}_j)]\) in \( 2^N/\hat{A} \) is comprised of the \( p \) images of \( \bar{\pi}(\mathcal{X}_j) \) under \( \hat{A} \).
Each class $\mathcal{X}_j$ refines into $p^{k-1}$ members of $2^N/\hat{A}$ as follows: let

$$\bar{\pi} = (1, \pi_2, \pi_3, \ldots, \pi_k), \quad \pi_i \in \mathbb{Z}_p,$$ and

$$\bar{\pi}(\mathcal{X}_j) = X_{1,j} \cup \pi_2(X_{2,j}) \cup \pi_3(X_{3,j}) \cup \cdots \cup \pi_k(X_{k,j}).$$

Then the element $[\bar{\pi}(\mathcal{X}_j)]$ in $2^N/\hat{A}$ is comprised of the $p$ images of $\bar{\pi}(\mathcal{X}_j)$ under $\hat{A}$.

Since each $X_{i,j} \subseteq X_{i,j+1}$, $j = r, r + 1, \ldots, n - r - 1$,

$$\bar{\pi}(\mathcal{X}_r) \subseteq \bar{\pi}(\mathcal{X}_{r+1}) \subseteq \cdots \subseteq \bar{\pi}(\mathcal{X}_{n-r}),$$ so

$$[\bar{\pi}(\mathcal{X}_r)] < [\bar{\pi}(\mathcal{X}_{r+1})] < \cdots < [\bar{\pi}(\mathcal{X}_{n-r})] \text{ in } 2^N/\hat{A}.$$ 

Hence, there exist $p^{k-1}$ symmetric chains partitioning all refined classes of $C$ in $2^N/\hat{A}$. 
Let $S_n$ have its induced action on the $k$-element subsets of $[n]$ and let $S_n^{(k)}$ denote the resulting subgroup of $S_{[n]}^{(k)}$. Then $2^{([n]/S_n^{(k)})}$ is the set of isomorphism types of uniform $k$-hypergraphs ordered by embedding. This is known to be rank-symmetric, rank-unimodal, strongly Sperner . . .
Let $S_n$ have its induced action on the $k$-element subsets of $[n]$ and let $S_n^{(k)}$ denote the resulting subgroup of $S^{([n])}$. Then

$$2^{([n])}/S_n^{(k)}$$

is the set of isomorphism types of uniform $k$-hypergraphs ordered by embedding.
Let $S_n$ have its induced action on the $k$-element subsets of $[n]$ and let $S_n^{(k)}$ denote the resulting subgroup of $S^{([n])}$. Then

$$2^{([n])}/S_n^{(k)}$$

is the set of isomorphism types of uniform $k$-hypergraphs ordered by embedding.

This is known to be rank-symmetric, rank-unimodal, strongly Sperner . . .
Let $S_n$ have its induced action on the $k$-element subsets of $[n]$ and let $S_n^{(k)}$ denote the resulting subgroup of $S^{([n])}$. Then

$$2^{([n]/k}) / S_n^{(k)}$$

is the set of isomorphism types of uniform $k$-hypergraphs ordered by embedding.

This is known to be rank-symmetric, rank-unimodal, strongly Sperner . . .

**Problem 5:** Determine if the set of unlabelled $k$-graphs on $[n]$, ordered by subgraph, has an SCD.