

Poset-free Families and Lubell-boundedness

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Abstract

Given a finite poset P , we consider the largest size $\text{La}(n, P)$ of a family \mathcal{F} of subsets of $[n] := \{1, \dots, n\}$ that contains no subposet P . This continues the study of the asymptotic growth of $\text{La}(n, P)$; it has been conjectured that for all P , $\pi(P) := \lim_{n \rightarrow \infty} \text{La}(n, P) / \binom{n}{\lfloor \frac{n}{2} \rfloor}$ exists and equals a certain integer, $e(P)$. While this is known to be true for paths, and several more general families of posets, for the simple diamond poset \mathcal{D}_2 , the existence of π frustratingly remains open. Here we develop theory to show that $\pi(P)$ exists and equals the conjectured value $e(P)$ for many new posets P . We introduce a hierarchy of properties for posets, each of which implies $\pi = e$, and some implying more precise information about $\text{La}(n, P)$. The properties relate to the Lubell function of a family \mathcal{F} of subsets, which is the average number of times a random full chain meets \mathcal{F} . We present an array of examples and constructions that possess the properties.

1 Introduction

Let the Boolean lattice \mathcal{B}_n denote the partially ordered set (poset, for short) $(2^{[n]}, \subseteq)$ of all subsets of the n -set $\{1, \dots, n\}$. For finite posets $P = (P, \leq)$ and $P' = (P', \leq')$, we say P *contains* P' , or we say P' is a (*weak*) *subposet of* P , if there exists an injection $f: P' \rightarrow P$ that preserves the partial ordering [16]. This means that whenever $u \leq' v$ in P' , we have $f(u) \leq f(v)$ in P .

We consider collections $\mathcal{F} \subseteq 2^{[n]}$ that contain no subposet P . We say \mathcal{F} is *P -free*. We are interested in determining the largest size of a P -free family of subsets of $[n]$, denoted $\text{La}(n, P)$.

The foundational result of this sort, Sperner's Theorem from 1928 [15], solves this problem for families that contain no two-element chain (that is, for antichains). Erdős [6] generalized this to give the largest size of a family that does not contain a chain of size

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k , the path poset \mathcal{P}_k consisting of k totally ordered elements $a_1 < \dots < a_k$. In recent years Katona [11, 5, 4, 7] brought the attention of researchers to the generalization of this problem, which is to investigate $\text{La}(n, P)$ for many posets P . The problem turns out to be far more challenging for most posets P . A comprehensive survey of prior work can be found in [9].

From the early results, Griggs and Lu [10] found that, while $\text{La}(n, P)$ may not be as simple as the sum of middle binomial coefficients in n , in the cases P that had been solved, it is at least true that $\text{La}(n, P)$ is asymptotic to an integer multiple of $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. (This observation may be implicit in the earlier work of Katona *et al.*) Moreover, when Saks and Winkler pointed out a pattern of the known values of $\pi(P)$, Griggs and Lu introduced the parameter $e(P)$, defined below, and proposed:

Conjecture 1.1 [9] *For any poset P , the limit $\pi(P) := \lim_{n \rightarrow \infty} \frac{\text{La}(n, P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ exists and it is an integer, $e(P)$.*

For a set S , the collection of all k -subsets of S is denoted by $\binom{S}{k}$. Following [9], by $\mathcal{B}(n, k)$ and $\Sigma(n, k)$ we mean the families of subsets of $[n]$ of the k middle sizes and the size of the families. More precisely, $\mathcal{B}(n, k) = \binom{[n]}{\lfloor (n-k+1)/2 \rfloor} \cup \dots \cup \binom{[n]}{\lfloor (n+k-1)/2 \rfloor}$ or $\binom{[n]}{\lceil (n-k+1)/2 \rceil} \cup \dots \cup \binom{[n]}{\lceil (n+k-1)/2 \rceil}$. So it has one or two possible families, depending on the parity of $n+k$.

For a poset P , we let $e(P)$ denote the maximum k such that for any integers n and s , the family $\mathcal{F} = \binom{[n]}{s} \cup \dots \cup \binom{[n]}{s+k-1}$ is P -free. In particular, the union $\mathcal{B}(n, k)$ of k middle levels in \mathcal{B}_n does not contain P as a subposet.

For instance, the *butterfly poset* \mathcal{B} of elements $A_1, A_2, B_1,$ and B_2 with $A_i < B_j$ for $i, j = 1, 2$ is not contained in the union of two consecutive levels in the Boolean lattice, while of course the union of three middle levels does contain \mathcal{B} for $n \geq 3$. One gets that $e(\mathcal{B}) = 2$.

Since the family $\mathcal{B}(n, e)$, where $e = e(P)$, contains no P , it is clear that when $\pi(P)$ exists, it must be at least $e(P)$. Griggs and Lu then conjectured that in fact $\pi(P) = e(P)$ for general P . They presented some new families of posets for which the conjecture holds [10], and they improved the known bounds on $\pi(P)$ for some families for which the existence of $\pi(P)$ is still not settled. It remains a daunting problem to obtain $\pi(P)$, even for some particular small posets P .

The most-studied case is the *diamond poset* \mathcal{D}_2 , consisting of four elements A, B, C, D with $A < B, C$ and $B, C < D$. While the conjectured value of $\pi(\mathcal{D}_2)$ is $e(\mathcal{D}_2) = 2$, a series of studies has so far only brought down the upper bound to 2.25 [12], and the existence of $\pi(\mathcal{D}_2)$ remains unproven. It appears that additional tools must be developed.

In their subsequent work with Lu [9], the authors discovered that for a P -free family \mathcal{F} , it can be valuable to consider the average number of times a random full chain of subsets of $[n]$ intersects \mathcal{F} , which they called the Lubell function of \mathcal{F} , denoted $\bar{h}(\mathcal{F})$. With this approach, they were able to obtain new families of posets P that satisfy the conjecture. Griggs and Li [8] subsequently described a “partition method” for using

the Lubell function to derive simple new proofs of several fundamental poset examples satisfying the conjecture.

In this paper, we extensively expand this approach of bounding the Lubell function, introducing a series of new properties of posets P for which the conjecture above is satisfied. For posets P satisfying the most restrictive of these properties, which we name here uniform L-boundedness, it was shown in [9, 13] that P not only satisfies the asymptotic conjecture, but $\text{La}(n, P)$ is exactly determined, for general n . Moreover, we know all extremal families, just as we saw for the path posets \mathcal{P}_k . We introduce a series of successively weaker properties, m -L-boundedness, for integers $m \geq 0$, where $m = 0$ is uniform L-boundedness. Posets that are 1-L-bounded are said to be centrally L-bounded. Here m is a restriction on the sizes of subsets in the P -free families, which will help us to study $\text{La}(n, P)$ in some circumstances. If a poset P is m -L-bounded for some m , we say it is L-bounded. All L-bounded posets behave well in that they satisfy the conjecture. Uniform L-boundedness is extended in a different way, with properties called lower L-bounded and upper L-bounded, that also imply the conjecture.

In Section 2 we review the required poset terminology and concepts related to the Lubell function. The new properties and their connections to the conjectures are developed in Section 3.

In Section 4 we introduce a notion connected with $e(P)$, called a large interval of a poset. This plays an important role in the constructions discussed later.

A new class of posets is introduced in Section 5, called fan posets, which are simply wedges of paths. These include the posets \mathcal{V}_r previously investigated by Katona *et al.*, for integer $r \geq 1$, with elements $A < B_i$ for $1 \leq i \leq r$. We determine which fans are L-bounded, and give examples for all m of a fan that is m -L-bounded but not $(m + 1)$ -L-bounded. Since all fan posets are trees, and since trees satisfy the $\pi(P) = e(P)$ conjecture by a theorem of Bukh [2], it follows that all fans satisfy the conjecture. We give a simpler direct proof of this using the Lubell properties.

We describe constructive methods to generate a surprising variety of m -L-bounded and lower-L-bounded posets from old ones in Sections 6 and 7. All posets generated in this way satisfy the $\pi = e$ conjecture. Some of the many interesting problems for further research are discussed in the last section.

2 The Lubell Function and Three Poset Parameters

Let us recall some standard poset notions. For elements $a \leq b$ in poset P , a (closed) *interval* $[a, b] \subset P$ is the subposet of P consisting of elements c such that $a \leq c \leq b$. Note that if P is the Boolean lattice \mathcal{B}_n , then an interval $[A, B]$ has the same structure as $\mathcal{B}_{|B|-|A|}$.

The *dual* of a poset $P = (P, \leq)$ is the poset $d(P) = (P, \leq_d)$ such that $x \leq_d y$ in $d(P)$ if and only if $y \leq x$ in P .

An element x of a poset P is $\hat{0}$ (resp., $\hat{1}$), if for every element $p \in P$, $x \leq p$ ($x \geq p$, resp.).

We introduce notation for the *filter* or *up-set* (resp., *ideal* or *down-set*) generated by an element $p \in P$: Let $\{p\}^+$ (resp., $\{p\}^-$) denote the sets $\{q \in P \mid q \geq p\}$ and $\{q \in P \mid q \leq p\}$, resp.

The *ordinal sum* of disjoint posets P_1, P_2 , denoted $P_1 \oplus P_2$, is the poset $P_1 \cup P_2$, ordered by $x \leq y$ if $x \in P_1$ and $y \in P_2$, or if x, y are in the same P_i with $x \leq y$. The boldface number \mathbf{k} is the k -element antichain.

Fix a family $\mathcal{F} \subseteq 2^{[n]}$. Let $\mathcal{C} := \mathcal{C}_n$ denote the collection of all $n!$ full (maximal) chains $\emptyset \subset \{i_1\} \subset \{i_1, i_2\} \subset \cdots \subset [n]$ in the Boolean lattice \mathcal{B}_n . We collect information about the average number of times chains $\mathcal{C} \in \mathcal{C}$ meet \mathcal{F} to give an upper bound on $|\mathcal{F}|$.

The *height* of \mathcal{F} , viewed as a poset, is

$$h(\mathcal{F}) := \max_{\mathcal{C} \in \mathcal{C}} |\mathcal{F} \cap \mathcal{C}|.$$

Following [9] we consider the *Lubell function* of \mathcal{F} , which is defined to be

$$\bar{h}(\mathcal{F}) = \bar{h}_n(\mathcal{F}) := \text{ave}_{\mathcal{C} \in \mathcal{C}} |\mathcal{F} \cap \mathcal{C}|.$$

The Lubell function bounds the size of a family:

Lemma 2.1 [9] *Let \mathcal{F} be a collection of subsets of $[n]$. Then $\bar{h}(\mathcal{F}) = \sum_{F \in \mathcal{F}} 1/\binom{n}{|F|} \geq |\mathcal{F}|/\binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

This lemma is an extension of the heart of Lubell's elegant proof of Sperner's Theorem [14]. We see that $\bar{h}(\mathcal{F})$ can be viewed as a weighted sum, where each set F has weight $1/\binom{n}{|F|}$. To maximize $|\mathcal{F}|$ over families \mathcal{F} of given weight, the sets in the family must have weights as small as possible.

By computing $\bar{h}(\mathcal{F})$ for all P -free \mathcal{F} , we obtain an upper bound on $\text{La}(n, P)$, so also on $\pi(P)$, if it exists. Let $\lambda_n(P)$ be the maximum value of $\bar{h}(\mathcal{F})$ over all P -free families $\mathcal{F} \subset 2^{[n]}$. Then, $\text{La}(n, P)/\binom{n}{\lfloor \frac{n}{2} \rfloor} \leq \lambda_n(P)$. We define $\lambda(P) = \lim_{n \rightarrow \infty} \lambda_n(P)$, if this limit exists. Thus,

$$\pi(P) \leq \lambda(P), \tag{1}$$

if both limits exist. In the following sections we will see many posets for which $\pi(P) = \lambda(P)$. On the other hand, there exist posets with $\pi(P) < \lambda(P)$, with the smallest example being \mathcal{V}_2 , the poset on three elements $A < B$ and $A < C$, for which $1 = \pi(\mathcal{V}_2) < \lambda(\mathcal{V}_2) = 2$.

Combining this inequality above with the lower bound on $\pi(P)$, we have

$$e(P) \leq \pi(P) \leq \lambda(P), \tag{2}$$

if both limits exist. Unlike the earlier inequality for $\lambda(P)$, we have not found any poset P with $e(P) < \pi(P)$, and we continue to believe that $\pi(P) = e(P)$ for general P .

While $e(P)$ exists for any poset P , we do not know how to determine it in general. We do know that the height $h(P)$ alone is not sufficient to determine $e(P)$, since Jiang and Lu independently observed [9] that $e(P)$ can be made arbitrarily large by considering the *k-diamond* poset \mathcal{D}_k , $k \geq 2$, which has elements $\{A, B_1, \dots, B_k, C\}$ ordered by $A < B_i < C$ for $1 \leq i \leq k$. We can also express \mathcal{D}_k as an ordinal sum of antichains, $\mathbf{1} \oplus \mathbf{k} \oplus \mathbf{1}$. While $e(\mathcal{D}_k) \rightarrow \infty$ with growing k , the height stays at just three.

3 Lubell-bounded Posets

We next introduce and investigate properties based on the Lubell function that are useful for obtaining posets P that satisfy the $\pi(P) = e(P)$ conjecture. With Lu in [9], we considered posets P for which for all n ,

$$\bar{h}_n(\mathcal{F}) \leq e(P)$$

for any P -free family \mathcal{F} of subsets of $[n]$. Here we shall say such posets are *uniformly L-bounded*, with L for Lubell, since the Lubell functions of P -free families are bounded by $e(P)$, for every n . On the other hand, since $\mathcal{B}(n, e)$ is P -free for $e = e(P)$ and $n \geq e - 1$, it follows that uniformly L-bounded posets P satisfy $\lambda_n(P) = e(P)$ for all $n \geq e - 1$, and so,

$$\lambda(P) = \lim_{n \rightarrow \infty} \lambda_n(P) = e(P).$$

For the diamond poset \mathcal{D}_2 the existence of the limit $\pi(\mathcal{D}_2)$ remains elusive. The diamond \mathcal{D}_2 is certainly not uniformly L-bounded (as there are \mathcal{D}_2 -free families \mathcal{F} for which $\bar{h}_n(\mathcal{F}) > 2.25$ for large n , as compared to $e(\mathcal{D}_2) = 2$). However, for “most” values $k > 2$ the diamond \mathcal{D}_k is uniformly L-bounded. Another poset class we introduced [9] was the *harp* $\mathcal{H}(\ell_1, \dots, \ell_k)$, consisting of k chains $\mathcal{P}_{\ell_1}, \dots, \mathcal{P}_{\ell_k}$ with their minimum elements identified and their maximum elements identified. Provided the chains have distinct lengths ($\ell_1 > \dots > \ell_k \geq 3$) we showed the harp is uniformly L-bounded. Here is what we showed concerning largest P -free families for uniformly L-bounded P . Note that beyond determining $\pi(P)$, we know $\text{La}(n, P)$, and we even know the extremal P -free families.

Theorem 3.1 [9] *Let P be a poset that is uniformly L-bounded. Let $e = e(P)$. Then for all n , $\text{La}(n, P) = \Sigma(n, e)$, and so $\pi(P) = e$. Moreover, if \mathcal{F} is a P -free family of subsets of $[n]$ of maximum size, \mathcal{F} must be $\mathcal{B}(n, e)$.*

A problem with the Lubell function is the large contribution to it (one each) from the empty set or the full set $[n]$. For example, consider the butterfly poset \mathcal{B} : For general n the family \mathcal{F} consisting of $\emptyset, [n]$, and the singletons is a (small) \mathcal{B} -free family with $\bar{h}(\mathcal{F}) = 3 > e(P) = 2$, so the butterfly is not uniformly L-bounded.

Then to investigate asymptotically the maximum size $\text{La}(n, P)$ of P -free families, it makes sense to restrict attention to families that do not contain $\emptyset, [n]$. If the Lubell function of such families stays small, then we still get good bounds on $\text{La}(n, P)$, at least asymptotically. Let us say P is *centrally L-bounded*, if for all n , $\bar{h}_n(\mathcal{F}) \leq e(P)$ for all P -free families \mathcal{F} of proper subsets of $[n]$, that is, excluding $\emptyset, [n]$. As an example of such a poset, in [8] we showed that the butterfly \mathcal{B} is centrally L-bounded.

Theorem 3.2 *Let P be a centrally L-bounded poset, and let $e = e(P)$. For $n \geq e + 3$,*

$$\text{La}(n, P) = \Sigma(n, e).$$

Hence, $\pi(P) = e$.

Proof. Given a centrally L-bounded poset P , suppose that \mathcal{F} is a P -free family of subsets of $[n]$ of maximum size, where $n \geq e+3$. If \mathcal{F} contains neither \emptyset nor $[n]$, then by definition, $\bar{h}(\mathcal{F}) \leq e$, and easily by Lemma 2.1 we get $|\mathcal{F}| \leq \Sigma(n, e)$.

Next consider if \mathcal{F} contains \emptyset , but not $[n]$. We know that $\bar{h}(\mathcal{F}) \leq e+1$, since P is centrally L-bounded and \mathcal{F} with \emptyset removed remains P -free. Then \mathcal{F} cannot contain all singleton subsets of $[n]$: Otherwise, we could form $\mathcal{F}' = \mathcal{F} \setminus (\{\emptyset\} \cup \binom{[n]}{1})$, and have $\bar{h}(\mathcal{F}') \leq e-1$. But then, $|\mathcal{F}| = 1 + n + |\mathcal{F}'| \leq 1 + n + \Sigma(n, e-1) < \Sigma(n, e)$. This contradicts that $|\mathcal{F}| = \text{La}(n, P) \geq \Sigma(n, e)$. So some singleton subset $\{i\}$ is missing from \mathcal{F} , and we may obtain a new family \mathcal{F}'' from \mathcal{F} by replacing \emptyset by $\{i\}$.

We claim \mathcal{F}'' is P -free. Otherwise, \mathcal{F}'' contains P , and $\{i\}$ must be a minimal element of P . All other elements in P are in $\mathcal{F} \setminus \{\emptyset\}$. Then replacing $\{i\}$ by \emptyset , we see that \mathcal{F} itself contains P , a contradiction. By central L-boundedness of P , $\bar{h}(\mathcal{F}'') \leq e$. We deduce the desired bound from $|\mathcal{F}| = |\mathcal{F}''| \leq \Sigma(n, e)$.

Finally consider \mathcal{F} containing both \emptyset and $[n]$. Suppose $n \neq 4$ or $e \neq 1$. Then if $\binom{[n]}{1} \subset \mathcal{F}$, we obtain $|\mathcal{F}| \leq 2 + n + \Sigma(n, e-1) < \Sigma(n, e)$, which contradicts \mathcal{F} having maximum size. Similarly, \mathcal{F} cannot contain all $(n-1)$ -subsets of $[n]$. We can do replacements as before to obtain a new P -free family that contains neither \emptyset nor $[n]$ and has size as large as \mathcal{F} . Applying the central L-boundedness of P to this new family, $|\mathcal{F}| = \Sigma(n, e)$.

The last case is $n = 4$, $e = 1$, and \mathcal{F} contains both \emptyset and $[4]$. If \mathcal{F} contains all singleton subsets of $[4]$, then $\mathcal{F} = \{\emptyset, [4]\} \cup \binom{[4]}{1}$. Let $\mathcal{F}' = (\mathcal{F} \setminus \{[4]\}) \cup \{S\}$ for some 3-subset S . On the one hand, \mathcal{F}' is P -free since if S represents some element of P , then $[4]$ could be the same element of P . With other elements of P in $\mathcal{F} \setminus \{[4]\}$, we conclude that \mathcal{F} contains P , which contradicts our assumption. On the other hand, \mathcal{F}' must contain P since $\bar{h}(\mathcal{F}' \setminus \{\emptyset\}) = 1\frac{1}{4} > e(P)$. The dilemma is caused by the assumption $\binom{[4]}{1} \subset \mathcal{F}$. Hence \mathcal{F} cannot contain all singleton subsets. By the same reasoning, \mathcal{F} cannot contain all 3-subsets of $[4]$. Therefore we can just replace \emptyset and $[4]$ by a singleton subset and a 3-subset, resp. Again, we conclude that $|\mathcal{F}| = \Sigma(n, e)$. \square

This property of a poset being centrally L-bounded generalizes how we showed in [8] that when P is the butterfly, $\text{La}(n, P) = \Sigma(n, e)$ for all sufficiently large n . Note that unlike the more restrictive class of uniformly L-bounded posets, we do not get that $\text{La}(n, P) = \Sigma(n, e)$ for *all* n —it can fail for small n . Indeed, we see this for the butterfly, where $e = 2$: For $n = 2$, \mathcal{B}_2 is itself a \mathcal{B} -free family of size 4, which is more than $\Sigma(2, 2) = 3$.

A further distinction is that extremal families for centrally L-bounded P are not restricted to the middle level families $\mathcal{B}(n, e)$, as they are when P is uniformly L-bounded. For instance, when P is the butterfly, a construction in [5] for \mathcal{B}_4 is

$$\mathcal{F} = \{\{1\}, \{2\}, \{1, 3, 4\}, \{2, 3, 4\}\} \cup \binom{[4]}{2},$$

which is a different butterfly-free family of maximum size ($e = 2$ and $|\mathcal{F}| = \Sigma(4, 2)$).

Another such example is for the poset \mathcal{J} on four elements $A_1 < A_2 < A_3$ and $A_1 < B$ studied in [13]. This is one of the fan posets introduced in the next section, and

proven there to be centrally L-bounded. Again we have $e = 2$, and in \mathcal{B}_2 the family $\mathcal{F} = \{\emptyset, \{1\}, \{1, 2\}\}$ is a largest \mathcal{J} -free family other than $\mathcal{B}(2, 2)$.

We shall be seeing new examples of centrally L-bounded posets momentarily. It is possible that the requirement $n \geq e + 3$ in the theorem above can be strengthened to $n \geq e + 1$.

Let us consider weakening the property on posets P even more, by further limiting the families for which we require the Lubell function to be bounded, while still verifying the asymptotic conjecture $\pi(P) = e(P)$. For integer $m \geq 0$, we ignore the subsets in both the bottom and top m levels of the Boolean lattice: We say poset P is m -L-bounded, if for all n $\bar{h}_n(\mathcal{F}) \leq e(P)$ for all P -free families \mathcal{F} of subsets of sizes in $[m, n - m]$. So, a poset is 0-L-bounded when it is uniformly L-bounded, and 1-L-bounded means it is centrally L-bounded. As m increases, there is more restriction on the families \mathcal{F} that must satisfy the Lubell function condition, and so more posets potentially have the property. We develop the theory for these properties, and derive the asymptotic conjecture $\pi = e$, so that we can handle such posets when we find them. In our section on fan posets, we shall give examples for all m of posets that are m -L-bounded but not $(m + 1)$ -bounded.

Proposition 3.3 *Let integer $m \geq 0$. Let P be an m -L-bounded poset, and let $e = e(P)$. Then for all n ,*

$$\text{La}(n, P) \leq \Sigma(n, e) + 2 \sum_{i=0}^{m-1} \binom{n}{i}.$$

Hence, $\pi(P) = e$.

Proof. For an m -L-bounded poset P , the bound on $\text{La}(n, P)$ follows by removing the tails (bottom and top m levels) of a P -free family \mathcal{F} that achieves $\text{La}(n, P)$, since the family that remains is uniformly L-bounded. Asymptotically, $\Sigma(n, e) \sim e \binom{n}{\lfloor \frac{n}{2} \rfloor}$, while the number of sets at the tails (the m smallest and largest sizes) is only $O(n^{m-1})$. \square

Is every poset m -L-bounded for some integer m ? Definitely not. Consider the poset \mathcal{V}_r for integer $r \geq 2$. The family $\mathcal{F} = \binom{[n]}{n-r+2} \cup \binom{[n]}{n-r+1}$ is \mathcal{V}_r -free and has $\bar{h}(\mathcal{F}) = 2 > e(\mathcal{V}_r)$. Thus no matter how large the constant m we choose for the size condition $[m, n - m]$ on the subsets, there always exists an r such that the Lubell function of some \mathcal{V}_r -free family satisfying the size condition can be arbitrarily larger than $e(\mathcal{V}_r)$. However, we have proved that if we delete “sufficiently many” subsets from \mathcal{B}_n , then the gap between the Lubell function of \mathcal{V}_r -free families and $e(\mathcal{V}_r)$ will be very small for large enough n [8].

To capture m -L-boundedness for general m , we introduce another class of posets, saying that P is L -bounded if it is m -L-bounded for some m . This class is then the union of the classes of m -L-bounded posets over all m . In view of the proposition above, we record

Corollary 3.4 *If poset P is L -bounded, then $\pi(P) = e(P)$.*

We introduce a different type of weakening of uniform L-boundedness that is satisfied by additional posets that are not L-bounded: We say poset P is *lower-L-bounded* if, for any numbers $\beta \in (\frac{1}{2}, 1)$ and $\varepsilon > 0$, there exists $N := N(\beta, \varepsilon)$ such that for all $n \geq N$,

$$\bar{h}(\mathcal{F}) \leq e(P) + \varepsilon,$$

for all P -free families of subsets of $[n]$ of sizes less than βn . We are interested as well in the dual property: We say poset P is *upper-L-bounded* if, for any numbers $\alpha \in (0, \frac{1}{2})$ and $\varepsilon > 0$, there exists $N := N(\alpha, \varepsilon)$ such that for all $n \geq N$,

$$\bar{h}(\mathcal{F}) \leq e(P) + \varepsilon,$$

for all P -free families of subsets of $[n]$ of sizes greater than αn .

Proposition 3.5 *Let P be a lower-L-bounded or upper-L-bounded poset. Then $\pi(P) = e(P)$.*

Proof. Since the properties are dual, it suffices to prove this for any lower-L-bounded poset P . For each n , let \mathcal{F}_n be a largest P -free family of subsets of $[n]$. Fix some $\beta \in (\frac{1}{2}, 1)$. Partition \mathcal{F}_n into \mathcal{F}'_n and \mathcal{F}''_n such that \mathcal{F}'_n has sets of \mathcal{F}_n of sizes at most βn and $\mathcal{F}''_n = \mathcal{F}_n \setminus \mathcal{F}'_n$. It is well-known that $\sum_{i=0}^{\alpha n} \binom{n}{i} = O(\frac{1}{n^2}) \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for any constant $0 < \alpha < \frac{1}{2}$ [1, page 256]. Given any $\varepsilon > 0$, for $n \geq N$, we have

$$\begin{aligned} \frac{|\mathcal{F}_n|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} &\leq \sum_{F \in \mathcal{F}'_n} \frac{1}{\binom{n}{|F|}} + \frac{|\mathcal{F}''_n|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \\ &\leq \bar{h}_n(\mathcal{F}'_n) + O\left(\frac{1}{n^2}\right) \\ &\leq e(P) + \varepsilon + O\left(\frac{1}{n^2}\right). \end{aligned}$$

Since ε is arbitrary, this implies $\pi(P) = e(P)$. □

4 Large Intervals

The structure of a poset P sometimes helps to estimate the three parameters of P compared in Section 2. In this section we introduce and investigate the notion of a large interval in a poset, which we need in connection with our theory of L-boundedness. We say an interval $I = [a, b]$ in P is a *large interval* if it is a maximal interval with $e(I) = e(P)$.

Lemma 4.1 *Suppose P is a poset with element p , such that $P = P_1 \cup P_2$, where $P_1 \cap P_2 = \{p\}$ and $P_2 = \{p\}^+$. Then*

$$(i) \lambda_n(P) \leq \lambda_n(P_1) + \lambda_n(P_2),$$

(ii) $\text{La}(n, P) \leq \text{La}(n, P_1) + \text{La}(n, P_2)$, and

(iii) $\pi(P) \leq \pi(P_1) + \pi(P_2)$, if they exist.

In particular, if p is the maximal element of a large interval I of P_1 , or $P_1 = \{p\}^-$ (that is, p is the maximal element of P_1), then

(iv) $e(P) = e(P_1) + e(P_2)$.

Proof.

Suppose \mathcal{F} is a P -free family. Define $\mathcal{F}_1 := \{S \mid S \in \mathcal{F}, \mathcal{F} \cap [S, [n]] \text{ contains } P_2\}$ and $\mathcal{F}_2 := \mathcal{F} \setminus \mathcal{F}_1$.

Claim: \mathcal{F}_1 is P_1 -free, and \mathcal{F}_2 is P_2 -free.

If \mathcal{F}_1 contains P_1 , then there is a set S in \mathcal{F}_1 that represents the element p of P_1 . Let T be a maximal set in \mathcal{F}_1 containing S . Since it is maximal in \mathcal{F}_1 , $(\mathcal{F} \setminus \{T\}) \cap [T, [n]]$ is contained in \mathcal{F}_2 . Furthermore, $\mathcal{F} \cap [T, [n]]$ contains P_2 as a subposet. Thus, \mathcal{F} contains P , which contradicts our assumption. For the family \mathcal{F}_2 , note that there does not exist a set in \mathcal{F}_2 that is the minimal element (the element p) of P_2 . Hence \mathcal{F}_2 is P_2 -free.

Since every family can be partitioned as above, we have $\bar{h}(\mathcal{F}) \leq \bar{h}(\mathcal{F}_1) + \bar{h}(\mathcal{F}_2)$. This implies (i). Furthermore, $|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2|$ implies $\text{La}(n, P) \leq \text{La}(n, P_1) + \text{La}(n, P_2)$, which is (ii). Finally, (iii) is a consequence of (ii).

For (iv), let $e(P_1) = e_1$ and $e(P_2) = e_2$. First we show $e(P) \geq e_1 + e_2$. Suppose for some n, s the family $\bigcup_{i=0}^{e_1+e_2-1} \binom{[n]}{s+i}$ contains P , where $0 \leq s \leq n - e_1 - e_2 + 1$. Let S be the set in the family corresponding to element $p \in P$. Its size $|S|$ cannot be less than $s + e_1$ nor greater than $s + e_1 - 1$ otherwise it would imply $e(P_1) < e_1$ or $e(P_2) < e_2$. Hence, the family is P -free and $e(P) \geq e_1 + e_2$.

Then we must show $e(P) \leq e_1 + e_2$. By the definition of e_i , we can find an integer n_i large enough so that there exists P_i in the family of consecutive $e_i + 1$ levels $\bigcup_{j=0}^{e_i} \binom{[n_i]}{s_i+j}$ of \mathcal{B}_{n_i} for $i = 1, 2$. Let $n_0 = n_1 + n_2 + e_1 + e_2$ and $s_0 = s_1 + s_2 + e_1 + e_2$. Consider the family $\bigcup_{i=0}^{e_1+e_2} \binom{[n_0]}{s_0+i}$. An interval $[S_1, T_1]$ in $[n_0]$, with $|S_1| = s_1 + e_1 + e_2$ and $|T_1| = s_1 + e_1 + e_2 + n_2$, is a poset that is the same as the Boolean lattice \mathcal{B}_{n_2} . Hence, $[S_1, T_1] \cap \left(\bigcup_{i=0}^{e_1+e_2} \binom{[n_0]}{s_0+i}\right)$ contains P_2 . Furthermore, the element p in P_2 could be chosen as a set of size at least $s_1 + s_2 + e_1 + e_2$. On the other hand, an interval $[S_2, T_2]$ with $|S_2| = s_2 + e_2$ and $|T_2| = s_2 + e_2 + n_1$ is the same as the Boolean lattice \mathcal{B}_{n_1} . Thus, $[S_2, T_2] \cap \left(\bigcup_{i=0}^{e_1+e_2} \binom{[n_0]}{s_0+i}\right)$ contains P_1 , and all elements in P_1 could be some sets of sizes at most $s_1 + s_2 + e_1 + e_2$ as well. By relabelling the elements if needed, one sees that the family $\bigcup_{i=0}^{e_1+e_2} \binom{[n_0]}{s_0+i}$ contains P as a subposet. This proves (iv). \square

Note that the second part of Lemma 4.1 (iv) has recently been discovered independently by Burcsi and Nagy [3].

Our purpose in studying large intervals is to find when will the parameter $e(P_1 \cup P_2)$ have the ‘‘additive property’’, as presented in Lemma 4.1 (iv). However, not every poset contains a large interval. For example, every maximal interval of the butterfly poset \mathcal{B} is a \mathcal{P}_2 , but $e(\mathcal{P}_2) < e(\mathcal{B})$. So poset \mathcal{B} , which is centrally L-bounded (hence, L-bounded),

has no large interval. Nevertheless, if an L -bounded poset contains a large interval, then it is unique.

Proposition 4.2 *Let P be an L -bounded poset. Then P contains at most one large interval. Furthermore, if an m - L -bounded P contains a large interval I , then I is also m - L -bounded.*

Proof. Let P be an L -bounded poset, say it is m - L -bounded. Assume that P contains two large intervals $I_1 = [a_1, b_1]$ and $I_2 = [a_2, b_2]$ such that $e := e(P) = e(I_1) = e(I_2)$. Without loss of generality, we assume $b_1 \neq b_2$ (or we may instead consider the dual of P). Consider any Boolean lattice \mathcal{B}_n with $n \geq 2m + e$.

Claim: The family $\mathcal{G} := (\bigcup_{i=1}^e \binom{[n]}{m+i-1}) \cup \{S\}$, where S is any set of size $m + e$, is P -free.

If \mathcal{G} contains I_1 then S must be b_1 since the family of any e consecutive levels does not contain I_1 . Similar reasoning shows that S is b_2 . However, $b_1 \neq b_2$ in P . We conclude that \mathcal{G} does not contain P .

The P -free family \mathcal{G} satisfies $\bar{h}(\mathcal{G}) = e + 1/\binom{n}{m+e}$ and every set in \mathcal{G} has sizes in $[m, n - m]$. This violates the assumption of the m - L -boundedness for P . Therefore, P cannot contain two large intervals.

Now suppose that P contains a large interval I . Let \mathcal{F} be any I -free family such that every set in \mathcal{F} has size in $[m, n - m]$. Since \mathcal{F} is I -free, it is also P -free. By the m - L -boundedness of P , we have $\bar{h}(\mathcal{F}) \leq e = e(I)$. So I is m - L -bounded. \square

Remark. The proof above also implies that if P has two large intervals, then $\text{La}(n, P)$ is strictly greater than $\Sigma(n, e)$ for all large enough n : One can take $\mathcal{B}(n, e)$ and one extra set, to form a P -free family.

5 Fans

For our theory of Lubell boundedness, it turns out to be very interesting to study the natural common generalization of the posets \mathcal{V}_r (studied by Katona *et al.*) and the poset \mathcal{J} . We say a *fan* $\mathcal{V}(\ell_1, \dots, \ell_k)$, $\ell_1 \geq \dots \geq \ell_k \geq 2$ is the poset obtained by identifying the minimum elements of k chains $\mathcal{P}_{\ell_1}, \dots, \mathcal{P}_{\ell_k}$. That is, a fan is simply a wedge of paths. We then have $\mathcal{V}_r = \mathcal{V}(2, \dots, 2)$, where there are r 2's, while $\mathcal{J} = \mathcal{V}(3, 2)$.

Harps introduced in [9] are similar, except the maximum elements of the chains are also identified. So a harp can be viewed as a suspension of paths.

By the theorem of Bukh [2] $\pi = e$ for posets P that are trees (the Hasse diagram is a tree). Specifically, for trees $e = h - 1$, where h is the height (cf. [9]). Since fans are trees, it follows that for the fan $P = \mathcal{V}(\ell_1, \dots, \ell_k)$, $\ell_1 \geq \dots \geq \ell_k \geq 2$, $\pi = e = \ell_1 - 1$, so the asymptotic conjecture is true for fans. Here we give a simple direct proof of this fan result for fans by showing their L -boundedness. By comparison, Bukh's result requires a rather elaborate proof by probabilistic arguments.

Theorem 5.1 *Let P be the fan poset $\mathcal{V}(\ell_1, \dots, \ell_k)$, with $\ell_1 \geq \dots \geq \ell_k \geq 2$. The L -boundedness of P can be classified according to the ℓ_i 's as follows.*

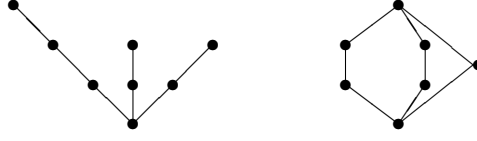


Figure 1: The fan $\mathcal{V}(4, 3, 3)$ and the harp $\mathcal{H}(4, 4, 3)$.

(i) If $k = 1$ or if $\ell_1 - 1 > \ell_2 > \dots > \ell_k$, then P is uniformly L -bounded.

(ii) If $\ell_1 > \ell_2 > \dots > \ell_k$, then P is centrally L -bounded.

(iii) If $\ell_1 > \ell_2$, then P is L -bounded.

(iv) If $\ell_1 = \ell_2$, then P is not L -bounded, but it is lower- L -bounded.

Proof. Let P be the fan $\mathcal{V}(\ell_1, \dots, \ell_k)$ with $\ell_1 \geq \dots \geq \ell_k \geq 2$. Note that we have $e(P) = h(P) - 1 = \ell_1 - 1$.

For (i), if $k = 1$, the result follows from Erdős's result on chains of any given length [6]. If $k \geq 1$, then P is a subposet of the harp $\mathcal{H}(\ell_1, \ell_2 + 1, \dots, \ell_k + 1)$, which has distinct lengths since $\ell_1 > \ell_2 + 1 > \dots > \ell_k + 1$. It means any family \mathcal{F} that is P -free avoids this harp. Applying the results on harp-free families [9],

$$\bar{h}(\mathcal{F}) \leq e(\mathcal{H}(\ell_1, \ell_2 + 1, \dots, \ell_k + 1)) = \ell_1 - 1 = e(P),$$

proving P is uniformly L -bounded.

For (ii), consider any P -free family \mathcal{F} of subsets of $[n]$ and $\emptyset, [n] \notin \mathcal{F}$. Following [9] apply the *min partition* on the set of full chains to get blocks \mathcal{C}_A containing full chains \mathcal{C} with $\min(\mathcal{F} \cap \mathcal{C}) = A$ for distinct A 's. One block contains chains that avoid \mathcal{F} . Suppose there is a block of chains having $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}| > \ell_1 - 1$ for some A . Let \mathcal{F}_1 be $(\mathcal{F} \cap [A, [n]]) \setminus \{A\}$ and Z_1 be a chain of size $\ell_1 - 1$ in \mathcal{F}_1 . Such a chain exists since $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F}_1 \cap \mathcal{C}| = \text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}| > (\ell_1 - 1) - 1 = \ell_1 - 2$. For $i > 1$, let $\mathcal{F}_i = \mathcal{F}_{i-1} \setminus Z_{i-1}$ and Z_i be a chain of size $\ell_i - 1$ in \mathcal{F}_i . We will show such Z_i exists for $1 \leq i \leq k$. Then these chains together with A contains P as a subposet. By induction, for $i > 1$,

$$\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F}_i \cap \mathcal{C}| = \text{ave}_{\mathcal{C} \in \mathcal{C}_A} |(\mathcal{F}_{i-1} \setminus Z_{i-1}) \cap \mathcal{C}| > \ell_{i-1} - 2 - \frac{|Z_{i-1}|}{n - |A|} > \ell_i - 2.$$

This suffices to show there exists Z_i of size $\ell_i - 1$ in \mathcal{F}_i . Thus, \mathcal{F} contains P , contradiction to the P -freeness of \mathcal{F} . Hence no block \mathcal{C}_A has $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}| > \ell_1 - 1$. Then $\bar{h}(\mathcal{F}) \leq \ell_1 - 1$.

To show (iii), we show P is m - L -bounded for some integer m . Let m be an integer satisfying $m > \sum_{i=1}^{k-1} (\ell_i - 1)$. Let \mathcal{F} be a P -free family of subsets of $[n]$ and each set in \mathcal{F} has size in $[m, n - m]$. Again, apply the min partition on \mathcal{F} and suppose there is a block \mathcal{C}_A with $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}| > \ell_1 - 1$. Define \mathcal{F}_i and Z_i for $i \geq 1$ as before. The existence of Z_1 of size $\ell_1 - 1$ is clear. For other Z_i with $i > 1$, Note $n - |A| \geq m$. We have

$$\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F}_i \cap \mathcal{C}| = \text{ave}_{\mathcal{C} \in \mathcal{C}_A} |(\mathcal{F}_1 \setminus \cup_{j=1}^{i-1} Z_j) \cap \mathcal{C}| > \ell_1 - 2 - \frac{\sum_{j=1}^{i-1} |Z_j|}{n - |A|} > \ell_2 - 2 \geq \ell_i - 2,$$

for $2 \leq i \leq k$. So there exists a chain Z_i of size $\ell_i - 1$ in \mathcal{F}_i for $1 \leq i \leq k$. This again contradicts the P -freeness of \mathcal{F} . We conclude that $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}| \leq \ell_1 - 1$ and $\bar{h}(\mathcal{F}) \leq \ell_1 - 1$. Hence P is m -L-bounded.

The last case is (iv), $\ell_1 = \ell_2$. Since $e(P) = \ell_1 - 1$, the chains of size ℓ_1 are large intervals of P , so by Proposition 4.2, P is not L-bounded. Suppose it is not lower-L-bounded either. Then for some β and ε with $\frac{1}{2} < \beta < 1$ and $\varepsilon > 0$, for infinitely many n there is a P -free family \mathcal{F} of subsets of $[n]$ with $\bar{h}(\mathcal{F}) > e(P) + \varepsilon$, where every set in \mathcal{F} has size less than βn . Apply the min partition on \mathcal{F} and let \mathcal{C}_A be a block with $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}| > e(P) + \varepsilon$. As before, we claim $[A, [n]]$ contains P . By the size condition, removing a chain of size $\ell_1 - 1$ from $(\mathcal{F} \cap [A, [n]]) \setminus \{A\}$ reduces $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}|$ by at most $\frac{\ell_1 - 1}{(1 - \beta)n}$. When $n > \frac{k(\ell_1 - 1)}{(1 - \beta)\varepsilon}$, we can find k disjoint chains of size $\ell_1 - 1$ in $(\mathcal{F} \cap [A, [n]]) \setminus \{A\}$. Thus, \mathcal{F} contains P , a contradiction. Therefore, P is lower-L-bounded. \square

We shall mention that all fan posets $\mathcal{V}(\ell_1, \dots, \ell_k)$ are actually lower-L-bounded. It is also clear that a uniformly L-bounded poset is a lower-L-bounded poset. However, there are m -L-bounded posets that are not lower-L-bounded since the size condition for lower-L-boundedness does not exclude the small-sized subsets in the family. We will provide such examples in next section.

Since L-boundedness and lower-L-boundedness imply $\pi = e$, we have the following special case of Bukh's Tree Theorem [2].

Corollary 5.2 *For any fan poset $\mathcal{V}(\ell_1, \dots, \ell_k)$ with $\ell_1 \geq \dots \geq \ell_k \geq 2$, we have $\pi = e = \ell_1 - 1$.*

Although we defined m -L-boundedness, so far we have not presented any poset that is m -L-bounded but not $(m - 1)$ -L-bounded, except when $m = 1$. In the following, we offer examples of posets to show that the m -L-boundedness property is not vacuous.

Theorem 5.3 *For $m \geq 1$, the poset $P = \mathcal{V}(3, 2, \dots, 2)$, where there are $m + 1$ 2's, is m -L-bounded but not $(m - 1)$ -L-bounded.*

Proof. Consider the family $\mathcal{F} = \binom{[n]}{n-m-1} \cup \binom{[n]}{n-m} \cup \mathcal{F}_0$, where \mathcal{F}_0 consists of a set in $\binom{[n]}{n-m+1}$. For any set $F \in \mathcal{F}$, at most $m + 2$ sets strictly contain it, hence \mathcal{F} is P -free. We have $\bar{h}(\mathcal{F}) > 2 = e(P)$, and so P is not $(m - 1)$ -L-bounded.

Next consider any P -free family \mathcal{F} of subsets of $[n]$ with sizes in $[m, n - m]$. We apply the min-partition on the set of full chains. Let \mathcal{C}_A be any block. If $\mathcal{F} \cap [A, [n]]$ does not contain a chain of size 3, then the average of $|\mathcal{F} \cap \mathcal{C}| \leq 2$ for full chains $\mathcal{C} \in \mathcal{C}_A$. Suppose there is a chain of size 3 in $\mathcal{F} \cap [A, [n]]$, say $A \subset B \subset C$. There are at most m sets in $\mathcal{F} \cap [A, [n]]$ besides A , B , and C . Recall that $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}|$ is equal to the Lubell function of the P -free family, obtained by removing the elements of A from each set in $\mathcal{F} \cap [A, [n]]$, in the smaller Boolean lattice $\mathcal{B}_{n-|A|}$. Here, $m + 2 \leq n - |A| \leq n - m$. By the size condition, $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}|$ is at most $(1 + \frac{1}{m+2} + \frac{2}{(m+1)(m+2)}) + m(\frac{1}{m+2}) \leq 2$, as $m \geq 1$. We conclude that $\bar{h}(\mathcal{F}) \leq 2$, which gives the m -L-boundedness of P . \square

6 Constructing L-bounded Posets

We have seen that L-bounded posets (including those that are uniformly L-bounded or centrally L-bounded) have nice properties. We now describe methods to construct more L-bounded posets. We begin with a construction using ordinal sums.

Theorem 6.1 *For any centrally L-bounded poset P , both $\mathbf{1} \oplus P$ and $P \oplus \mathbf{1}$ are centrally L-bounded. Furthermore, $\mathbf{1} \oplus P \oplus \mathbf{1}$ is uniformly L-bounded.*

Proof. Let P be a 1-L-bounded poset. Let \mathcal{F} be a $(\mathbf{1} \oplus P)$ -free family of subsets of $[n]$ not containing \emptyset nor $[n]$. We again apply the min partition on the set of full chains. For any block \mathcal{C}_A , the subfamily $(\mathcal{F} \setminus \{A\}) \cap [A, [n]]$ is P -free. Hence it contributes at most $e(P)$ to $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}|$. Therefore, each block has $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}| \leq e(P) + 1$. Then $\bar{h}_n(\mathcal{F}) \leq e(P) + 1$. On the other hand, it is clear that the union any consecutive $e(P) + 1$ levels is $(\mathbf{1} \oplus P)$ -free. For $P \oplus \mathbf{1}$, note that the dual of a 1-L-bounded poset is also 1-L-bounded. So $P \oplus \mathbf{1} = d(\mathbf{1} \oplus d(P))$ is 1-L-bounded. The first part is proved.

To show the second part, on the one hand, we take the union of any $e(P)+2$ consecutive levels. There are at most $e(P)$ consecutive levels strictly contained in any interval $[A, B]$ in the union of $e(P) + 2$ consecutive levels. So the family of sets in $[A, B] \setminus \{A, B\}$ is P -free and the union of any $e(P) + 2$ consecutive levels is $(\mathbf{1} \oplus P \oplus \mathbf{1})$ -free. Hence $e(P) + 2 \leq e(\mathbf{1} \oplus P \oplus \mathbf{1})$.

On the other hand, let \mathcal{F} be a $(\mathbf{1} \oplus P \oplus \mathbf{1})$ -free family. Apply the min-max partition [9] on \mathcal{C}_n to get blocks $\mathcal{C}_{A,B}$ containing full chains \mathcal{C} with $\min(\mathcal{F} \cap \mathcal{C}) = A$ and $\max(\mathcal{F} \cap \mathcal{C}) = B$ for pairs $A \subseteq B$. One block contains chains that avoid \mathcal{F} . If $\mathcal{C}_{A,B}$ is a block in the partition, then $(\mathcal{F} \setminus \{A, B\}) \cap [A, B]$ is P -free. Since $(\mathcal{F} \setminus \{A, B\}) \cap [A, B]$ could be viewed as a P -free family in $\mathcal{B}_{|B|-|A|}$, which contributes no more than $e(P)$ to the average, hence $\text{ave}_{\mathcal{C} \in \mathcal{C}_{A,B}} |\mathcal{F} \cap \mathcal{C}| \leq e(P) + 2$. Thus, $\bar{h}_n(\mathcal{F}) \leq e(P) + 2 \leq e(\mathbf{1} \oplus P \oplus \mathbf{1})$ for any $(\mathbf{1} \oplus P \oplus \mathbf{1})$ -free family. So the poset $\mathbf{1} \oplus P \oplus \mathbf{1}$ is uniformly L-bounded. \square

Figure 2 illustrates how to obtain a uniformly L-bounded poset from the butterfly poset using Theorem 6.1. Recall that the butterfly poset \mathcal{B} is centrally L-bounded.

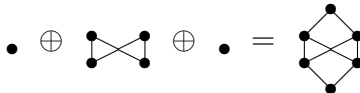


Figure 2: The poset $\mathbf{1} \oplus \mathcal{B} \oplus \mathbf{1}$ is uniformly L-bounded.

We next introduce an operation that involves large intervals of posets. For $1 \leq i \leq k$ let P_i be a poset having a large interval I_i (which may be not be unique). We denote by $P_1 \oplus_I P_2 \oplus_I \cdots \oplus_I P_k$ the sum of the posets P_i where we identify the maximal element of I_i to the minimal element of I_{i+1} for $1 \leq i \leq k - 1$. This operation depends on the choice

of the large intervals. However, if all P_i 's are L-bounded, then this poset is unique, since no L-bounded poset has more than one large interval.

Theorem 6.2 For $k, \ell \geq 0$ let P_i ($1 \leq i \leq k$) and Q_j ($1 \leq j \leq \ell$) be L-bounded posets with large intervals I_i and I'_j , resp. Here if $k = 0$ or $\ell = 0$, it means the corresponding collection is empty. Further assume each P_i has $\hat{0}$ and each Q_j has $\hat{1}$. Then the poset

$$Q_1 \oplus_I \cdots \oplus_I Q_\ell \oplus_I P_1 \oplus_I \cdots \oplus_I P_k$$

is L-bounded.

Proof. Suppose each P_i is m_i -L-bounded and each Q_j is m'_j -L-bounded. Let $m = \max\{m_1, \dots, m_k, m'_1, \dots, m'_\ell\}$. Hence all P_i 's and Q_j 's are m -L-bounded. We show $Q_1 \oplus_I \cdots \oplus_I Q_\ell \oplus_I P_1 \oplus_I \cdots \oplus_I P_k$ is m -L-bounded too.

Let us first see the case $\ell = 0$. We use induction on the number k . For $k \leq 1$, this is trivial. Suppose the theorem holds for some $k \geq 1$. Now let P_1, \dots, P_{k+1} be m -L-bounded posets such that each P_i has $\hat{0}$ and the large interval I_i . By induction, $P = P_1 \oplus_I \cdots \oplus_I P_k$ is m -L-bounded. Furthermore, $e(P) = \sum_{i=1}^k e(P_i)$ and $e(P \oplus_I P_{k+1}) = \sum_{i=1}^{k+1} e(P_i)$ by Lemma 4.1. Also note that $I_i \oplus_I \cdots \oplus_I I_k$ is a large interval in P . Consider any $(P \oplus_I P_{k+1})$ -free family \mathcal{F} of subsets of $[n]$, where for each $F \in \mathcal{F}$, $m \leq |\mathcal{F}| \leq n - m$. We partition \mathcal{F} into \mathcal{F}_1 and \mathcal{F}_2 as we did in Lemma 4.1 so that \mathcal{F}_1 is P -free and \mathcal{F}_2 is P_{k+1} -free. This gives $\bar{h}(\mathcal{F}_1) \leq e(P)$ and $\bar{h}(\mathcal{F}_2) \leq e(P_{k+1})$, and so, $\bar{h}(\mathcal{F}) \leq e(P_{k+1}) + e(P) = e(P \oplus_I P_{k+1})$. Therefore, $P \oplus_I P_{k+1}$ is m -L-bounded.

If $k = 0$, then we can use the same induction argument to show $Q = Q_1 \oplus_I \cdots \oplus_I Q_\ell$ is m -L-bounded by considering the dual case.

Finally, if $k, \ell > 0$, then let P be $P_1 \oplus_I \cdots \oplus_I P_k$ and Q be $Q_1 \oplus_I \cdots \oplus_I Q_\ell$. For any $(Q \oplus_I P)$ -free family \mathcal{F} of subsets of sizes in $[m, n - m]$, we partition \mathcal{F} into \mathcal{F}_1 and \mathcal{F}_2 , such that \mathcal{F}_1 is Q -free and \mathcal{F}_2 is P -free, as before. Since P and Q are both m -L-bounded, we have $\bar{h}(\mathcal{F}) = \bar{h}(\mathcal{F}_1) + \bar{h}(\mathcal{F}_2) \leq e(P) + e(Q) = e(P \oplus_I Q)$. This proves $P \oplus_I Q$ is m -L-bounded. \square

As an example, the poset \mathcal{D}_3 is uniformly L-bounded and is itself a large interval. We identify the maximal element in a \mathcal{D}_3 to the minimal element in another copy of \mathcal{D}_3 consecutively for several \mathcal{D}_3 's to obtain a ‘‘diamond-chain’’ that is uniformly L-bounded.

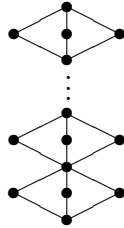


Figure 3: The poset $\mathcal{D}_3 \oplus_I \cdots \oplus_I \mathcal{D}_3$ is uniformly L-bounded.

In Section 5, we mentioned that there are L-bounded posets that are not lower-L-bounded. For any $m \geq 1$, consider $P_1 = \mathcal{V}(3, 2, \dots, 2)$, where there are $m + 1$ 2's, and P_2 , the dual of P_1 . By Theorem 6.2, the “crab poset” $P = P_2 \oplus_I P_1$ is m -L-bounded. The family $\mathcal{F} = \bigcup_{i=0}^4 \binom{[n]}{i}$ is P -free, and $\bar{h}(\mathcal{F}) = 5 > e(P)$ for any $n \geq 4$. Thus, P is not lower-L-bounded.

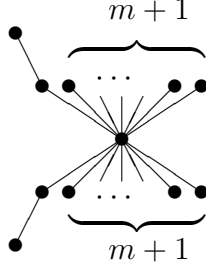


Figure 4: An L-bounded poset which is not lower-L-bounded.

We can construct new L-bounded posets not only “vertically” as above but also “horizontally”. Let P_1, \dots, P_k be posets with $\hat{0}$. Then we define the *wedge* $\mathcal{V}(P_1, \dots, P_k)$ to be the poset obtained by identifying the $\hat{0}$'s of the posets. The fan poset we introduced earlier is the special case where the posets P_i are paths.

Lemma 6.3 *Let P_1, \dots, P_k be posets with $\hat{0}$, ordered so that $e_1 \geq \dots \geq e_k$, where $e_i = e(P_i)$. Then for the wedge $P = \mathcal{V}(P_1, \dots, P_k)$, $e(P) = e_1$.*

Proof. We have $e(P) \geq e_1$ since P contains P_1 . Let n be large enough so that for each P_i , there exists integer s_i such that the family $\mathcal{F}_i = \bigcup_{j=0}^{e_i} \binom{[n]}{s_i+j}$ contains P_i as a subposet. We claim that the family $\mathcal{F} = \bigcup_{j=0}^{e_1} \binom{[kn]}{s+j}$ contains P , where $s = s_1 + \dots + s_k$.

For each \mathcal{F}_i we relabel the elements in the underlying set by $1 + (i - 1)n, 2 + (i - 1)n, \dots, in$. Let $A_i \in \mathcal{F}_i$ be the $\hat{0} \in P_i$. For any i , if $S \in \mathcal{F}_i$ is an element $p \in P_i$, then the set $(S \cup A_1 \cup \dots \cup A_k) \in \mathcal{F}$ will be the element $p \in P_i \subset P$. This shows that \mathcal{F} contains P . Hence, $e(P) \leq e_1$. \square

Theorem 6.4 *For $k \geq 2$ let P_1, \dots, P_k be uniformly L-bounded posets with $\hat{0}$, such that $e_1 \geq \dots \geq e_k$, where $e_i = e(P_i)$. Then the wedge $P = \mathcal{V}(P_1, \dots, P_k)$ is lower-L-bounded. In addition, if $e_1 > e_2$, then P is L-bounded.*

Proof. This is a generalization of Theorem 5.1 (iii) and (iv), and the proof is similar. Here we show the special case but omit the details of the proof of lower-L-boundedness.

First, assume $e_1 > e_2$. This result We have $e(P) = e_1$ by Lemma 6.3. Let $p_i = |P_i|$, and let m be an integer $\geq \sum_{i=1}^{k-1} (p_i - 1)$. Consider any P -free family \mathcal{F} of subsets of $[n]$ with sizes in $[m, n - m]$. If $\bar{h}(\mathcal{F}) \leq e_1$, then we done with this part.

Otherwise, $\bar{h}(\mathcal{F}) > e_1$. Apply the min partition on \mathcal{C}_n , and let \mathcal{C}_A be a block with $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}| > e_1$. Because P_1 is uniformly L-bounded, we conclude that $\mathcal{F} \cap [A, [n]]$

contains P as a subposet. Define $\mathcal{F}_1 = (\mathcal{F} \cap [A, [n]]) \setminus \{A\}$. Now let \mathcal{G}_1 be a subfamily of \mathcal{F}_1 with size $p_1 - 1$ such that $\mathcal{G}_1 \cup \{A\}$ contains P_1 as a subposet and such that A is the $\hat{0}$. Routinely, we can find disjoint subfamilies \mathcal{G}_i from each \mathcal{F}_i , where $\mathcal{F}_{i+1} = \mathcal{F}_i \setminus \mathcal{G}_i$, such that each \mathcal{G}_i together with A contains P_i as a subposet. This is because $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |(\mathcal{F}_i \cup A) \cap \mathcal{C}| > e_1 - \sum_{j=1}^{i-1} (p_j - 1)/(n - |A|) > e_2 \geq e_i$ for $2 \leq i \leq k$, and because P_i is uniformly L-bounded. To guarantee the P -freeness, no block \mathcal{C}_A can have $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}| > e_1$. Hence, $\bar{h}(\mathcal{F}) < e_1$, completing the proof. \square

Remark. In [3], the authors define a construction method similar to ours here to produce a class of posets, satisfying $e(P) = \frac{|P| + h(P)}{2}$, that contains some of our L-bounded posets.

7 Constructions with lower-L-bounded posets

The operations of the last section on L-bounded posets are also useful for lower L-bounded posets, to produce additional posets that satisfy the $\pi = e$ conjecture. Note that by duality we can obtain similar results for upper-L-bounded posets. The first two results are analogous to those in the last section.

Theorem 7.1 *For $1 \leq i \leq k$ let P_i be a lower-L-bounded poset with $\hat{0}$ and with a large interval I_i . Then*

$$P_1 \oplus_I \cdots \oplus_I P_k$$

is lower-L-bounded.

Proof. We only need to show the case $k = 2$. Let $P = P_1 \oplus_I P_2$. By Lemma 4.1, we have $e(P) = e(P_1) + e(P_2)$. Suppose P is not lower-L-bounded. Then for some β and 2ε with $\frac{1}{2} < \beta < 1$ and $2\varepsilon > 0$, we can, for infinitely many n , find a P -free family \mathcal{F} of subsets of $[n]$ with $\bar{h}(\mathcal{F}) > e(P) + 2\varepsilon$ such that every set in \mathcal{F} has size less than βn . Now use the idea in Lemma 4.1 to split \mathcal{F} into \mathcal{F}_1 and \mathcal{F}_2 , such that \mathcal{F}_1 and \mathcal{F}_2 are P_1 -free and P_2 -free, respectively. Either we have $\bar{h}(\mathcal{F}_1) > e(P_1) + \varepsilon$ or $\bar{h}(\mathcal{F}_2) > e(P_2) + \varepsilon$, for infinitely many n 's, and both families \mathcal{F}_1 and \mathcal{F}_2 satisfy the size condition. This then contradicts the assumption of lower-L-boundedness of P_1 and P_2 . So, P is lower-L-bounded. \square

Theorem 7.2 *For $1 \leq i \leq k$ let P_i be a lower-L-bounded poset with $\hat{0}$. Then the wedge $P = \mathcal{V}(P_1, \dots, P_k)$ is lower-L-bounded.*

Proof. Again we only need to show the case $k = 2$. Let $e_i = e(P_i)$. We may assume $e_1 \geq e_2$. By Lemma 6.3, $e(P) = e_1$. Suppose P is not lower-L-bounded. Then for some β and ε , with $\frac{1}{2} < \beta < 1$ and $2\varepsilon > 0$, we can find infinitely many n and a P -free family \mathcal{F} of subsets of $[n]$ for each n with $\bar{h}(\mathcal{F}) > e_1 + 2\varepsilon$, such that every set in \mathcal{F} has size less than βn . Apply the min partition on \mathcal{C}_n and let \mathcal{C}_A be a block with $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}| > e_1 + 2\varepsilon$. As before, we claim $[A, [n]]$ contains P , if n is sufficiently large. Recall that $\text{ave}_{\mathcal{C} \in \mathcal{C}_A} |\mathcal{F} \cap \mathcal{C}|$

is equal to $\bar{h}_{n'}(\mathcal{F}')$, where $n' = n - |A|$ and $\emptyset \in \mathcal{F}' \subseteq 2^{[n] \setminus A}$; also, \mathcal{F}' and $\mathcal{F} \cap [A, [n]]$ have the same structure. The size condition gives $n' > (1 - \beta)n$, and every set in \mathcal{F}' has size at most $\beta n - |A| < \beta(n - |A|) = \beta n'$. The definition of lower-L-boundedness then shows \mathcal{F}' contains P_1 when n , hence n' , is large enough. Now let \mathcal{G} be a subfamily of \mathcal{F}' with size $|P_1| - 1$ such that $\mathcal{G} \cup \{\emptyset\}$ contains P_1 as a subposet. Meanwhile, when n is large enough, removing \mathcal{G} from \mathcal{F}' reduces no more than ε from $\bar{h}_{n'}(\mathcal{F}')$. Thus $\mathcal{F}' \setminus \mathcal{G}$ would contain P_2 . Without loss of generality, we can assume the $\hat{0}$ of P_2 is the empty set. All these sets together with sets in \mathcal{G} contain P as a subposet, a contradiction. As a consequence, P must be lower-L-bounded. \square

Examples. We have seen that the fan posets $\mathcal{V}(2, 2)$ and $\mathcal{V}(2, 2, 2)$ are lower-L-bounded. The left poset in Figure 5 is a poset obtained by identifying the $\hat{0}$ of $\mathcal{V}(2, 2)$ to the maximal element of a large interval of another $\mathcal{V}(2, 2)$. The middle poset in Figure 5 is similar but the $\mathcal{V}(2, 2)$'s are replaced by $\mathcal{V}(2, 2, 2)$. The right poset in Figure 5 is obtained by wedging the other two posets. All these posets are lower-L-bounded by above theorems.

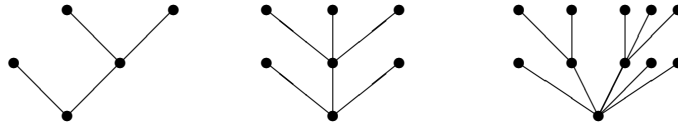


Figure 5: Three lower-L-bounded trees.

The trees above are just obtained by repeatedly applying Theorem 7.1 and Theorem 7.2 to poset \mathcal{P}_2 . In fact, we have many more interesting instances. We can expand each \mathcal{P}_2 in the tree posets by L-bounded posets. For example, the poset in Figure 6 is $\mathcal{V}(P_1, P_2, P_3)$, where $P_1 = \mathcal{P}_2$, $P_2 = \mathcal{D}_3 \oplus_I \mathcal{V}(\mathcal{H}(4, 3), \mathcal{P}_2)$, and $P_3 = \mathcal{P}_2 \oplus_I \mathcal{V}(2, 2, 2)$. This poset is lower-L-bounded.

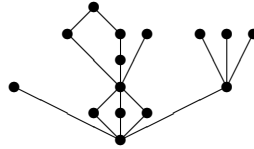


Figure 6: The poset $\mathcal{V}(\mathcal{P}_2, \mathcal{D}_3 \oplus_I \mathcal{V}(\mathcal{H}(4, 3), \mathcal{P}_2), \mathcal{P}_2 \oplus_I \mathcal{V}(2, 2, 2))$.

The posets in the figures have a tree-structure rooted at the bottom. We may combine such a poset with the dual of one, joined at the roots. The *baton* poset [10] $\mathcal{P}_k(s, t) = d(\mathcal{V}_s) \oplus_I \mathcal{P}_k \oplus_I \mathcal{V}_t$ is special case of this result.

Theorem 7.3 *Let P_1 and P_2 be lower-L-bounded posets with $\hat{0}$. Let $d(P_2)$ be the dual of P_2 . Then poset obtained by identifying the $\hat{0}$ of P_1 to the $\hat{1}$ of $d(P_2)$ satisfies $\pi = e$.*

Proof. Since a family \mathcal{F} of subsets of $[n]$ is P_2 -free if and only if the family $\mathcal{F}' = \{F \mid ([n] \setminus F) \in \mathcal{F}\}$ is $d(P_2)$ -free and P_2 is lower-L-bounded, we have

$$\pi(d(P_2)) = \lim_{n \rightarrow \infty} \frac{\text{La}(n, d(P_2))}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = \lim_{n \rightarrow \infty} \frac{\text{La}(n, P_2)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = e(P_2) = e(d(P_2)).$$

By Lemma 4.1, we have

$$e(P_1) + e(d(P_2)) = e(P) \leq \pi(P) \leq \pi(P_1) + \pi(d(P_2)) = e(P_1) + e(d(P_2)).$$

Hence $\pi(P) = e(P)$. □

8 Concluding Remarks

While we have verified Conjecture 1.1 for many new posets, it is far from proven. Beyond the conjecture, there are many problems on Lubell bounded posets that are interesting in their own right.

The uniformly L-bounded posets have the property $e(P) = \pi(P) = \lambda(P)$. It is not clear whether $\lambda(P)$ exists for every poset P (we believe it does exist). There are some centrally L-bounded posets such that if $\lambda(P)$ exists, then $\lambda(P) > \pi(P)$. A good example is the butterfly poset \mathcal{B} . We have already seen a \mathcal{B} -free family \mathcal{F} with $\bar{h}(\mathcal{F}) = 3$. On the other hand, the butterfly \mathcal{B} is a subposet of \mathcal{P}_4 . Thus, $|\mathcal{F} \cap \mathcal{C}| \leq 3$ for any \mathcal{B} -free family \mathcal{F} . We conclude that $\lambda_n(\mathcal{B}) = 3$ for all $n \geq 2$, and hence, $\lambda(\mathcal{B}) = 3$.

Another example is $\mathcal{J} = \mathcal{V}(3, 2)$. The family $\mathcal{F} = \{[n]\} \cup \{[n] \setminus \{i\} \mid i \text{ is odd.}\} \cup \{[n] \setminus \{i, j\} \mid \text{At least one of } i, j \text{ is not odd.}\}$ is $\mathcal{V}(3, 2)$ -free since every set in \mathcal{F} has at most two supersets. We have $\lambda_n(\mathcal{V}(3, 2)) \geq \bar{h}(\mathcal{F}) \geq \frac{9}{4}$ for all n , which is strictly larger than $\pi(\mathcal{V}(3, 2)) = 2$.

In addition to the centrally L-bounded posets above, no poset P that contains more than one large interval can have $\lambda(P) = e(P)$. Since two large intervals of P cannot share both the maximal and minimal elements, either $\bigcup_{i=0}^e \binom{[n]}{i}$ or $\bigcup_{i=0}^e \binom{[n]}{n-i}$ is P -free. We suspect that only uniformly L-bounded posets satisfy $e(P) = \pi(P) = \lambda(P)$.

Question 8.1 *Do there exist posets P that are not uniformly L-bounded, such that $\lambda(P) = e(P)$?*

For L-bounded posets, our suspicion is that for large enough n , the largest P -free families of subsets of $[n]$ cluster near the middle ranks where most subsets are located. Specifically, we ask

Question 8.2 *For any m -L-bounded poset P , does there exist $N = N(m, e, P)$ such that for all $n \geq N$, $\text{La}(n, P) = \Sigma(n, e(P))$?*

For $m = 0$ this holds for all n . For $m = 1$, we proved that if P is centrally L-bounded, $\text{La}(n, P) = \Sigma(n, e(P))$, whenever $n \geq e(P) + 3$, so it holds for $N = e(P) + 3$. For all m , when P is m -L-bounded, so far we have the general bound $\text{La}(n, P) \leq \Sigma(n, e(P)) + 2 \sum_{i=0}^{m-1} \binom{n}{i}$.

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