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On Families of Subsets With a Forbidden Subposet

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Dedicated to Professor William T. Trotter on the occasion of his 65th birthday

Let $\mathcal{F} \subset 2^{[n]}$ be a family of subsets of $\{1, 2, \ldots, n\}$. For any poset H, we say $\mathcal{F}$ is H-free if $\mathcal{F}$ does not contain any subposet isomorphic to H. Katona and others have investigated the behaviour of $\mathrm{La}(n, H)$, which denotes the maximum size of H-free families $\mathcal{F} \subset 2^{[n]}$. Here we use a new approach, which is to apply methods from extremal graph theory and probability theory to identify new classes of posets H, for which $\mathrm{La}(n, H)$ can be determined asymptotically as $n \to \infty$ for various posets H, including two-end-forks, up-down trees, and cycles C_{4k} on two levels.

1. Introduction and results

A poset $(S, \leqslant)$ is a set S equipped with a partial ordering $\leqslant$. We say a poset $(S, \leqslant)$ contains another poset $(S', \leqslant')$ if there exists an injection $f : S' \to S$ which preserves the partial ordering, meaning that whenever $u, v \in S'$ satisfy $u \leqslant' v$ we have $f(u) \leqslant f(v)$. In this case, S' is called a subposet of S.

Let $\mathcal{F} \subset 2^{[n]}$ be a family of subsets of $[n] := \{1, 2, \ldots, n\}$. We can view $\mathcal{F}$ as a subposet of the Boolean lattice $\mathcal{B}_n = (2^{[n]}, \subseteq)$. For any poset H, we say $\mathcal{F}$ is H-free if the poset $(\mathcal{F}, \subseteq)$ does not contain H as a subposet. Let $\mathrm{La}(n, H)$ denote the largest size of H-free families of subsets of $[n]$. The fundamental result of this kind is for H being a chain P_2 of two elements. A P_2-free family is an antichain, and Sperner's theorem [12] from 1928 gives us that $\mathrm{La}(n, P_2) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. For small posets H in general, it is interesting to compare $\mathrm{La}(n, H)$ to $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Erdős [6] extended Sperner's theorem in 1945 to determine that $\mathrm{La}(n, P_k)$, where P_k is a chain (path) of k elements, is the sum of the $k - 1$ middle binomial coefficients in n. Consequently, $\mathrm{La}(n, P_k) \sim (k-1)\binom{n}{\lfloor \frac{n}{2} \rfloor}$, as $n \to \infty$. Let $h(P)$ denote the *height* of poset P, which is the largest

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cardinality of any chain in H. We are interested in the asymptotic behaviour of $\mathrm{La}(n,H)$ for other posets H of height k.

There have been several investigations of height-2 posets. Thanh [13] extended Sperner's theorem by showing that for all r, $\mathrm{La}(n,V_r)\sim\binom{n}{\lfloor\frac{n}{2}\rfloor}$, where V_r is the r-fork, the height-2 poset with one element at the bottom level below each of r elements at the top level. (Especially, V_1 is P_2, while V_2 looks like the letter V.) It is important to note that we are not only excluding 'induced' copies of a forbidden subposet H, *e.g.*, V_3 is a subposet of P_4, so excluding V_3 subposets also excludes P_4.

De Bonis, Katona and Swanepoel [5] determined that $\mathrm{La}(n,B)$, where B is the Butterfly poset on four elements A_1,A_2,B_1,B_2 with each $A_1,A_2\leqslant B_1,B_2$, is the sum of the two middle binomial coefficients in n. More generally, consider excluding the height-2 poset which is called (using graph-theoretic terminology) $K_{r,s}$, which has elements $A_i, 1\leqslant i\leqslant r$ at the bottom level, elements $B_j, 1\leqslant j\leqslant s$ at the top level, and for all i,j, $A_i\leqslant B_j$. De Bonis and Katona [4] extend the asymptotics for the butterfly B and show that $\mathrm{La}(n,K_{r,s})\sim 2\binom{n}{\lfloor\frac{n}{2}\rfloor}$ for all $r,s\geqslant 2$. Griggs and Katona [10] considered whether the asymptotics of excluding the N poset on four elements A_1,A_2,B_1,B_2 with $A_1\leqslant B_1, A_2\leqslant B_1, A_2\leqslant B_2$ is similar to excluding V_2 or B. It turns out to be the former: $\mathrm{La}(n,N)\sim\binom{n}{\lfloor\frac{n}{2}\rfloor}$.

One new class of posets considered here we call a *baton* $P_k(s,t)$, which is a path P_k on k elements, $k\geqslant 3$, such that the bottom element is replicated $s-1$ times and the top element is replicated $t-1$ times, $s,t\geqslant 1$. That is, we have a height-k poset with s (resp. t) independent elements on the bottom (resp. top) level. The particular case $P_k(1,r)$ (which resembles a palm tree), known as an r-fork with a k-shaft, ${}_{k-1}V_r$ has been examined by De Bonis and Katona [4]. They show that

$$\mathrm{La}(n,P_k(1,r))\geqslant\sum_{i=\lfloor\frac{n-(k-2)}{2}\rfloor}^{\lfloor\frac{n+(k-2)}{2}\rfloor}\binom{n}{i}+\binom{n}{\lfloor\frac{n+k+1}{2}\rfloor}\left(\frac{r-1}{n}+\Omega\left(\frac{1}{n^2}\right)\right), \tag{1.1}$$

$$\mathrm{La}(n,P_k(1,r))\leqslant\sum_{i=\lfloor\frac{n-(k-2)}{2}\rfloor}^{\lfloor\frac{n+(k-2)}{2}\rfloor}\binom{n}{i}+\binom{n}{\lfloor\frac{n+k+1}{2}\rfloor}\left(\frac{z(k)+2(r-1)}{n}+\Omega\left(\frac{1}{n^2}\right)\right), \tag{1.2}$$

where $z(k)=\lfloor\frac{k^2}{2}\rfloor$ if $n+k$ is even and $z(k)=\lfloor\frac{(k-1)^2}{2}\rfloor$ if $n+k$ is odd.

The previously known maximum sizes of families of subsets of $[n]$ without a given pattern are listed in Table 1.

In this paper we give new asymptotic upper bounds on $\mathrm{La}(n,H)/\binom{n}{\lfloor\frac{n}{2}\rfloor}$ for several classes of posets H, and identify some new ones for which this ratio goes to 1 as $n\to\infty$. We first 'roughly unify' the previous results on forks ${}_kV_r$ and on complete two-level posets $K_{s,t}$ by considering batons $P_k(s,t)$. Note that the summation term in the bound, which appears repeatedly, is just the sum of the $k-1$ middle binomial coefficients in n.

Theorem 1.1. *For any $s,t\geqslant 1$ and $k\geqslant 3$, we have*

$$\mathrm{La}(n,P_k(s,t))\leqslant\sum_{i=\lfloor\frac{n-(k-2)}{2}\rfloor}^{\lfloor\frac{n+(k-2)}{2}\rfloor}\binom{n}{i}+\binom{n}{\lfloor\frac{n+k}{2}\rfloor}\left(\frac{2k(s+t-2)}{n}+O(n^{-3/2}\sqrt{\ln n})\right). \tag{1.3}$$

Table 1. Previously known results in the literature.

Name	H	$\mathrm{La}(n,H)$	Reference
chain P_r	$A_1 \subset \cdots \subset A_r$	$(r-1+o_n(1))\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$	[6]
butterfly B	$A_i \subset B_j$, for $1 \leqslant i,j \leqslant 2$	$(2+o_n(1))\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$	[5]
$K_{r,s}$ $(r,s \geqslant 2)$	$A_i \subset B_j$, for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$	$(2+o_n(1))\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$	[4]
'N'	$A \subset B$, $C \subset B$, and $C \subset D$	$(1+o_n(1))\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$	[10]
'V_r'	$A \subset B_i$, for $i = 1,2,\ldots,r$	$(1+o_n(1))\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$	[13]
${}_kV_r$	$A_1 \subset \cdots \subset A_k \subset B_i$, for $i = 1,2,\ldots,r$	$(k+o_n(1))\binom{[n]}{\lfloor \frac{n}{2} \rfloor}$	[4]

Consequently, as $n \to \infty$,

$$\mathrm{La}(n, P_k(s,t)) / \binom{n}{\lfloor \frac{n}{2} \rfloor} \to k-1.$$

Remarks. **(1)** Theorem 1.1 (for $s=1$ and $t=r$) is better than inequality (1.2) for $k \geqslant 4r-3$. For small k and large r, inequality (1.2) gives a better constant in the second-order term.

(2) Note that $\mathrm{La}(n, P_k(s,t)) \geqslant \mathrm{La}(n, P_k(1, \max\{s,t\}))$. From inequality (1.1), we have

$$\mathrm{La}(n, P_k(s,t)) \geqslant \sum_{i=\lfloor \frac{n-(k-2)}{2} \rfloor}^{\lfloor \frac{n+(k-2)}{2} \rfloor} \binom{n}{i} + \binom{n}{\lfloor \frac{n+k}{2} \rfloor} \left(\frac{\max\{s,t\}-1}{n} + \Omega\left(\frac{1}{n^2}\right) \right). \tag{1.4}$$

This lower bound (1.4) can be compared to the upper bound (1.3).

(3) Note that $P_3(s,t)$ contains $P_2(s,t) = K_{s,t}$, the complete two-level poset. Theorem 1.1 implies

$$\mathrm{La}(n,H) \leqslant \left(2 + O\left(\frac{|H|}{n}\right)\right) \binom{n}{\lfloor \frac{n}{2} \rfloor} \tag{1.5}$$

for all posets H of height 2. The hidden constant in the second-order term is slightly worse than that given in [4]. If H is not a subposet of the two middle layers of $2^{[n]}$ (for example, H contains the butterfly B), then equality in (1.5) holds.

An *up-down tree T* is a poset of height 2 that is also a tree as an undirected graph; its order is the number of elements $|T|$.

Theorem 1.2. *For any up-down tree T with order t, we have*

$$\mathrm{La}(n,T) \leqslant \left(1 + \frac{16t}{n} + O\left(\frac{1}{n\sqrt{n \ln n}}\right)\right) \binom{n}{\lfloor \frac{n}{2} \rfloor}. \tag{1.6}$$

Consequently, as $n \to \infty$,

$$\mathrm{La}(n,T) \Big/ \binom{n}{\lfloor \frac{n}{2} \rfloor} \to 1.$$

The butterfly poset B has been solved, so it is interesting now to consider more generally the *crowns* $\mathcal{O}_{2k}$, $k \geqslant 2$, where $\mathcal{O}_{2k}$ is the poset of height 2 that is a cycle of length $2k$ as an undirected graph. Of course, $\mathcal{O}_4$ is the butterfly poset, while $\mathcal{O}_6$ is noteworthy for being the middle two levels of the Boolean lattice $\mathcal{B}_3$. We have the following theorem for crowns.

Theorem 1.3. *For $k \geqslant 2$, we have*

$$\mathrm{La}(n, \mathcal{O}_{4k}) = (1 + o_n(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}, \tag{1.7}$$

$$\mathrm{La}(n, \mathcal{O}_{4k-2}) \leqslant \left(1 + \frac{\sqrt{2}}{2} + o_n(1)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}. \tag{1.8}$$

So we see that for $k \geqslant 3$ the crowns $\mathcal{O}_{2k}$ have $\mathrm{La}(n, \mathcal{O}_{2k})/\binom{n}{\lfloor \frac{n}{2} \rfloor}$ staying strictly below 2 asymptotically, unlike the butterfly, the case $k = 2$, where the ratio goes to 2. For even $k \geqslant 4$, the ratio goes to 1, while for odd $k \geqslant 3$ we only have an asymptotic upper bound. This is evidently because two consecutive levels of the Boolean lattice $\mathcal{B}_n$ are $\mathcal{O}_4$-free but have many crowns $\mathcal{O}_{2k}$ for $k \geqslant 3$.

The theorem above for crowns is actually just a special case of the more general result which concerns a more general class of height-2 posets obtained from graphs in a natural way. The proof also relies on extremal graph theory. For a simple graph $G = (V, \mathrm{E})$, define a poset $P(G)$ on the set $V \cup \mathrm{E}$ with the partial ordering $v < e$ if the edge e is incident at vertex v in G. For example, the crown poset $\mathcal{O}_{2k}$ is $P(G)$ when graph G is a k-cycle.

Theorem 1.4. *For any non-empty simple graph G with chromatic number $\chi(G)$, we have*

$$\mathrm{La}(n, P(G)) \leqslant \left(1 + \sqrt{1 - \frac{1}{\chi(G) - 1}} + o_n(1)\right)\binom{n}{\lfloor \frac{n}{2} \rfloor}. \tag{1.9}$$

In particular, if G is a bipartite graph, then

$$\mathrm{La}(n, P(G)) = (1 + o_n(1))\binom{n}{\lfloor \frac{n}{2} \rfloor}. \tag{1.10}$$

Theorem 1.3 is a direct consequence of Theorem 1.4 by the observation $\mathcal{O}_{2k} = P(C_k)$.

In this theory we construct large families in the Boolean lattice that avoid a given subposet. This is analogous to the much-studied Turán theory of graphs, in which one seeks to maximize the number of edges on n vertices while avoiding a given subgraph. It is interesting that the theorem above applies the Turán theory of graphs to give a useful bound in our ordered set theory.

The rest of the paper is organized as follows. Three probabilistic lemmas are given in Section 2, and the proofs of the theorems are given in Section 3. We conclude with ideas for further research.

2. Lemmas

For any fixed poset H, $\mathrm{La}(n, H)$ is of magnitude $\Theta\left(\binom{n}{\lfloor \frac{n}{2} \rfloor}\right)$. The following lemma allows us to consider the families consisting only of subsets near the middle level. Similar ideas can also be found in [2] and [11].

Lemma 2.1. *For any positive integer n, we have*

$$\sum_{i>\frac{n}{2}+2\sqrt{n\ln n}} \binom{n}{i} < \frac{2^n}{n^2}, \tag{2.1}$$

$$\sum_{i<\frac{n}{2}-2\sqrt{n\ln n}} \binom{n}{i} < \frac{2^n}{n^2}. \tag{2.2}$$

Proof. Let $X_1, X_2, \ldots, X_n$ be n independent identically distributed $\{0,1\}$ random variables with

$$\Pr(X_i = 0) = \Pr(X_i = 1) = \frac{1}{2}$$

for any $1 \leqslant i \leqslant n$. Apply Chernoff's inequality [3] to $X = \sum_{i=1}^{n} X_i$. We have

$$\Pr(X - \mathrm{E}(X) > \lambda) < e^{-\frac{\lambda^2}{2n}}.$$

Choose $\lambda = 2\sqrt{n\ln n}$. We have

$$\begin{aligned}\sum_{i>\frac{n}{2}+2\sqrt{n\ln n}} \binom{n}{i} 2^{-n} &= \Pr\left(X > \frac{n}{2} + \lambda\right) \\ &< e^{-\frac{\lambda^2}{2n}} \\ &= \frac{1}{n^2}.\end{aligned}$$

Inequality (2.1) has been proved. Inequality (2.2) is equivalent to inequality (2.1) by the symmetry of binomial coefficients $\binom{n}{i} = \binom{n}{n-i}$. □

Apply Stirling's formula $n! = (1 + O(1/n))\sqrt{2\pi n}\frac{n^n}{e^n}$ to obtain the following approximation of $\binom{n}{\lfloor \frac{n}{2} \rfloor}$:

$$\begin{aligned}\binom{n}{\lfloor \frac{n}{2} \rfloor} &= \frac{n!}{\lfloor \frac{n}{2} \rfloor ! \lceil \frac{n}{2} \rceil !} \\ &= (1 + O(1/n)) \frac{\sqrt{2\pi n}\frac{n^n}{e^n}}{\sqrt{2\pi \left\lfloor \frac{n}{2} \right\rfloor}\frac{\lfloor \frac{n}{2} \rfloor^{\lfloor \frac{n}{2} \rfloor}}{e^{\lfloor \frac{n}{2} \rfloor}} \sqrt{2\pi \left\lceil \frac{n}{2} \right\rceil}\frac{\lceil \frac{n}{2} \rceil^{\lceil \frac{n}{2} \rceil}}{e^{\lceil \frac{n}{2} \rceil}}} \\ &= (1 + O(1/n)) \frac{\sqrt{2}}{\sqrt{\pi n}} 2^n.\end{aligned}$$

It implies that

$$\frac{2^n}{n^2} = (1 + O(1/n)) \frac{\sqrt{\pi/2}}{n^{3/2}} \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

For any family $\mathcal{F}$ of size $\Theta\left(\binom{n}{\lfloor \frac{n}{2} \rfloor}\right)$, we can delete all subsets of sizes not in $\left(\frac{n}{2} - 2\sqrt{n\ln n}, \frac{n}{2} + 2\sqrt{n\ln n}\right)$ from $\mathcal{F}$. We obtain a family of subsets that has about the same size as $\mathcal{F}$ and only contains subsets of sizes in $\left(\frac{n}{2} - 2\sqrt{n\ln n}, \frac{n}{2} + 2\sqrt{n\ln n}\right)$.

Lemma 2.2. *Suppose X is a random variable that takes on non-negative integer values. Let $f(x)$ and $g(x)$ be two non-decreasing functions defined for non-negative integers x. Then*

$$\mathrm{E}(f(X)g(X)) \geqslant \mathrm{E}(f(X))\mathrm{E}(g(X)).$$

Proof. Apply the FKG inequality (see Fortuin, Kasteleyn and Ginibre [9]) over the totally ordered set of non-negative integers. Alternatively, here we give a simple direct proof.

Note $\mathrm{E}(f(X)g(X)) - \mathrm{E}(f(X))\mathrm{E}(g(X))$ is invariant under any translations on $f(X)$ and $g(X)$. i.e., for any constants c_1 and c_2, we have

$$\mathrm{E}((f(X)+c_1)(g(X)+c_2) - \mathrm{E}(f(X)+c_1)\mathrm{E}(g(X)+c_2) = \mathrm{E}(f(X)g(X)) - \mathrm{E}(f(X))\mathrm{E}(g(X)).$$

Hence, without loss of generality, we can assume both $f(x)$ and $g(x)$ are non-negative for any $x \geqslant 0$.

For any integer $k \geqslant 0$, let h_k be the step function:

$$h_k(x) = \begin{cases} 0 & \text{if } 0 \leqslant x < k, \\ 1 & \text{if } x \geqslant k. \end{cases}$$

For integers $j \geqslant i \geqslant 0$, we observe that

$$\mathrm{E}(h_i(X)h_j(X)) \geqslant \mathrm{E}(h_i(X))\mathrm{E}(h_j(X)),$$

which holds since

$$\begin{aligned} \mathrm{E}(h_i(X)h_j(X)) &= \Pr(X \geqslant i \ \wedge \ X \geqslant j) \\ &= \Pr(X \geqslant j) \\ &\geqslant \Pr(X \geqslant i)\Pr(X \geqslant j) \\ &= \mathrm{E}(h_i(X))\mathrm{E}(h_j(X)). \end{aligned}$$

We write

$$f(x) = \sum_{k=0}^{\infty} a_k h_k(x)$$

with non-negative coefficients $a_0 = f(0)$ and $a_k = f(k) - f(k-1)$ for $k \geqslant 1$. Similarly

$$g(x) = \sum_{k=0}^{\infty} b_k h_k(x)$$

with non-negative coefficients $b_0 = g(0)$ and $b_k = g(k) - g(k-1)$ for $k \geqslant 1$. By linearity, we have

$$\begin{aligned} \mathrm{E}(f(X)g(X)) &= \sum_{i,j=0}^{\infty} a_i b_j \mathrm{E}(h_i(X)h_j(X)) \\ &\geqslant \sum_{i,j=0}^{\infty} a_i b_j \mathrm{E}(h_i(X))\mathrm{E}(h_j(X)) \\ &= \mathrm{E}(f(X))\mathrm{E}(g(X)). \end{aligned}$$

□

Lemma 2.3. *Suppose X is a random variable which takes on non-negative integer values. For integers $k > r \geqslant 1$, if $\mathrm{E}(X) > k-1$, then*

$$\mathrm{E}\binom{X}{k} \geqslant \mathrm{E}\binom{X}{r}\frac{r!}{k!}\prod_{i=0}^{k-r-1}(\mathrm{E}(X)-r-i). \tag{2.3}$$

Proof. Define

$$f(x) = \begin{cases} \frac{r!}{k!}\prod_{i=0}^{k-r}(x-r-i) & \text{if } x > k-1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{r!}\prod_{i=0}^{r-1}(x-i) & \text{if } x > r-1, \\ 0 & \text{otherwise.} \end{cases}$$

Both $f(x)$ and $g(x)$ are non-negative increasing functions. For each non-negative integer x, we have $g(x) = \binom{x}{r}$ and $f(x)g(x) = \binom{x}{k}$. By applying Lemma 2.2 we obtain

$$\begin{aligned} \mathrm{E}\binom{X}{k} &= \mathrm{E}(f(X)g(X)) \\ &\geqslant \mathrm{E}(f(X))\mathrm{E}(g(X)) \\ &= \mathrm{E}(f(X))\mathrm{E}\binom{X}{r} \\ &\geqslant f(\mathrm{E}(X))\mathrm{E}\binom{X}{r}, \end{aligned}$$

where the last inequality follows since $f(x)$ is concave upward. □

3. Proofs of the theorems

Proof of Theorem 1.1. We let

$$\epsilon = \frac{k(s+t-2)}{\frac{n}{2} - 2\sqrt{n\ln n}},$$

and

$$f = f(n,k,s,t) = \sum_{i=\lfloor\frac{n-(k-2)}{2}\rfloor}^{\lfloor\frac{n+(k-2)}{2}\rfloor}\binom{n}{i} + \binom{n}{\lfloor\frac{n+k}{2}\rfloor}\epsilon.$$

Suppose $\mathcal{F}$ is a family of subsets of $[n]$ with $|\mathcal{F}| > f + \frac{2^{n+1}}{n^2}$. By removing all subsets of size outside $(\frac{n}{2} - 2\sqrt{n\ln n}, \frac{n}{2} + 2\sqrt{n\ln n})$, we can assume $\mathcal{F}$ only contains subsets of sizes in $(\frac{n}{2} - 2\sqrt{n\ln n}, \frac{n}{2} + 2\sqrt{n\ln n})$ and $|\mathcal{F}| > f$.

We would like to show that $\mathcal{F}$ contains $P_k(s,t)$. We will prove this statement by contradiction. Suppose that $\mathcal{F}$ is $P_k(s,t)$-free. Take a random permutation $\sigma \in S_n$. Consider a random full (maximal) chain C_σ,

$$\emptyset \subset \{\sigma_1\} \subset \{\sigma_1,\sigma_2\} \subset \cdots \subset \{\sigma_1,\sigma_2,\ldots,\sigma_n\}.$$

Let X be the random number counting $|\mathcal{F} \cap C_\sigma|$. On the one hand, we have

$$\mathrm{E}(X) = \sum_{F\in\mathcal{F}} \frac{1}{\binom{n}{|F|}} \tag{3.1}$$
$$> k-1+\epsilon, \tag{3.2}$$

since the sum is minimized, for a family of subsets on $[n]$ of size f by taking the f sets closest to the middle size $n/2$, which means taking the $k-1$ middle levels and the remaining sets at the next closest level to the middle, $\lfloor \frac{n+k}{2} \rfloor$.

Apply Lemma 2.3 with $r = k-1$:

$$\mathrm{E}\binom{X}{k} \geqslant \frac{1}{k}\mathrm{E}\binom{X}{k-1}(\mathrm{E}(X)-k+1)$$
$$> \frac{\epsilon}{k}\mathrm{E}\binom{X}{k-1}. \tag{3.3}$$

On the other hand, we will compute $\mathrm{E}\binom{X}{k}$ directly. By counting chains, a subchain of length k in $\mathcal{F}$,

$$F_1 \subset F_2 \subset \cdots \subset F_k,$$

is in the random chain C_σ with probability

$$\frac{|F_1|!(|F_2|-|F_1|)!\cdots(n-|F_k|)!}{n!}.$$

By linearity, we have

$$\mathrm{E}\binom{X}{k} = \sum_{\substack{F_1,\dots,F_k\in\mathcal{F}\\ F_1\subset\cdots\subset F_k}} \frac{|F_1|!(|F_2|-|F_1|)!\cdots(n-|F_k|)!}{n!}. \tag{3.4}$$

We can rewrite equation (3.4) as

$$\mathrm{E}\binom{X}{k} = \sum_{\substack{F_2,\dots,F_{k-1}\in\mathcal{F}\\ F_2\subset\cdots\subset F_{k-1}}} \frac{|F_2|!\cdots(n-|F_{k-1}|)!}{n!} \sum_{\substack{F_1\in\mathcal{F}\\ F_1\subset F_2}} \frac{1}{\binom{|F_2|}{|F_1|}} \sum_{\substack{F_k\in\mathcal{F}\\ F_{k-1}\subset F_k}} \frac{1}{\binom{n-|F_{k-1}|}{n-|F_k|}}. \tag{3.5}$$

Since $\mathcal{F}$ is $P_k(s,t)$-free, for a fixed $F_2,\dots,F_{k-1}$, either 'the number of F_1 satisfying $F_1 \subset F_2$ is at most $s-1$' or 'the number of F_k satisfying $F_{k-1} \subset F_k$ is at most $t-1$'. Let $\mathcal{A}$ be the set of $(k-2)$-chains $F_2 \subset \cdots \subset F_{k-1}$ in $\mathcal{F}$ such that the number of $F_1 \in \mathcal{F}$, $F_1 \subset F_2$, is at most $s-1$. Let $\mathcal{B}$ be the set of $(k-2)$-chains $F_2 \subset \cdots \subset F_{k-1}$ in $\mathcal{F}$ such that the number of $F_k \in \mathcal{F}$, $F_{k-1} \subset F_k$, is at most $t-1$. The union of $\mathcal{A}$ and $\mathcal{B}$ covers all $(k-2)$-chains in $\mathcal{F}$. We have

$$\mathrm{E}\binom{X}{k} \leqslant \sum_{(F_2,\dots,F_{k-1})\in\mathcal{A}} \frac{|F_2|!\cdots(n-|F_{k-1}|)!}{n!} \sum_{\substack{F_1\in\mathcal{F}\\ F_1\subset F_2}} \frac{1}{\binom{|F_2|}{|F_1|}} \sum_{\substack{F_k\in\mathcal{F}\\ F_{k-1}\subset F_k}} \frac{1}{\binom{n-|F_{k-1}|}{n-|F_k|}}$$
$$+ \sum_{(F_2,\dots,F_{k-1})\in\mathcal{B}} \frac{|F_2|!\cdots(n-|F_{k-1}|)!}{n!} \sum_{\substack{F_1\in\mathcal{F}\\ F_1\subset F_2}} \frac{1}{\binom{|F_2|}{|F_1|}} \sum_{\substack{F_k\in\mathcal{F}\\ F_{k-1}\subset F_k}} \frac{1}{\binom{n-|F_{k-1}|}{n-|F_k|}}. \tag{3.6}$$

For the summation over $\mathcal{A}$, the number of F_1 satisfying $F_1 \subset F_2$ is at most $s-1$. We have

$$\sum_{\substack{F_1 \in \mathcal{F} \\ F_1 \subset F_2}} \frac{1}{\binom{|F_2|}{|F_1|}} \leqslant \frac{(s-1)}{\frac{n}{2} - 2\sqrt{n \ln n}}. \tag{3.7}$$

Apply inequality (3.7) to the first summation in (3.6):

$$\begin{aligned}
&\sum_{(F_2,\ldots,F_{k-1}) \in \mathcal{A}} \frac{|F_2|! \cdots (n-|F_{k-1}|)!}{n!} \sum_{\substack{F_1 \in \mathcal{F} \\ F_1 \subset F_2}} \frac{1}{\binom{|F_2|}{|F_1|}} \sum_{\substack{F_k \in \mathcal{F} \\ F_{k-1} \subset F_k}} \frac{1}{\binom{n-|F_{k-1}|}{n-|F_k|}} \\
&\leqslant \sum_{(F_2,\ldots,F_{k-1}) \in \mathcal{A}} \frac{|F_2|! \cdots (n-|F_{k-1}|)!}{n!} \sum_{\substack{F_k \in \mathcal{F} \\ F_{k-1} \subset F_k}} \frac{1}{\binom{n-|F_{k-1}|}{n-|F_k|}} \frac{(s-1)}{\frac{n}{2} - 2\sqrt{n \ln n}} \\
&\leqslant \sum_{\substack{F_2,\ldots,F_{k-1} \in \mathcal{F} \\ F_2 \subset \cdots \subset F_{k-1}}} \frac{|F_2|! \cdots (n-|F_{k-1}|)!}{n!} \sum_{\substack{F_k \in \mathcal{F} \\ F_{k-1} \subset F_k}} \frac{1}{\binom{n-|F_{k-1}|}{n-|F_k|}} \frac{(s-1)}{\frac{n}{2} - 2\sqrt{n \ln n}} \\
&= \mathrm{E}\binom{X}{k-1} \frac{(s-1)}{\frac{n}{2} - 2\sqrt{n \ln n}}.
\end{aligned} \tag{3.8}$$

For the summation over $\mathcal{B}$, the number of F_k satisfying $F_{k-1} \subset F_k$ is at most $t-1$. We have

$$\sum_{\substack{F_k \in \mathcal{F} \\ F_{k-1} \subset F_k}} \frac{1}{\binom{n-|F_{k-1}|}{n-|F_k|}} \leqslant \frac{(t-1)}{\frac{n}{2} - 2\sqrt{n \ln n}}. \tag{3.9}$$

An inequality similar to (3.8) can be obtained:

$$\begin{aligned}
&\sum_{(F_2,\ldots,F_{k-1}) \in \mathcal{B}} \frac{|F_2|! \cdots (n-|F_{k-1}|)!}{n!} \sum_{\substack{F_1 \in \mathcal{F} \\ F_1 \subset F_2}} \frac{1}{\binom{|F_2|}{|F_1|}} \sum_{\substack{F_k \in \mathcal{F} \\ F_{k-1} \subset F_k}} \frac{1}{\binom{n-|F_{k-1}|}{n-|F_k|}} \\
&\leqslant \mathrm{E}\binom{X}{k-1} \frac{(t-1)}{\frac{n}{2} - 2\sqrt{n \ln n}}.
\end{aligned} \tag{3.10}$$

Combining inequalities (3.6), (3.8) and (3.10), we have

$$\mathrm{E}\binom{X}{k} \leqslant \mathrm{E}\binom{X}{k-1} \frac{s+t-2}{\frac{n}{2} - 2\sqrt{n \ln n}}. \tag{3.11}$$

From inequalities (3.3) and (3.11), and the fact that $\mathrm{E}\binom{X}{k-1} > 0$, we have

$$\frac{\epsilon}{k} < \frac{s+t-2}{\frac{n}{2} - 2\sqrt{n \ln n}},$$

which contradicts our choice of ϵ. □

Proof of Theorem 1.2. Let $\mathcal{F}$ be a T-free family of subsets of $[n]$. By removing at most $\frac{2^{n+1}}{n^2}$ subsets, without loss of generality, we can assume $\mathcal{F}$ consists of subsets of sizes in $(\frac{n}{2} - 2\sqrt{n \ln n}, \frac{n}{2} + 2\sqrt{n \ln n})$ and $|\mathcal{F}| > (1+\epsilon)\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Here

$$\epsilon = \frac{8t}{\frac{n}{2} - n\sqrt{n \ln n}}.$$

Let X be the same variable as defined in the proof of Theorem 1.1. Recall

$$\mathrm{E}(X) = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}}. \tag{3.12}$$

We have

$$\begin{aligned} \mathrm{E}(X) &= \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \\ &\geqslant \frac{|\mathcal{F}|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} \\ &> 1 + \epsilon. \end{aligned} \tag{3.13}$$

Using that the variance of X is non-negative (or applying Lemma 2.3 with $r = 1$ and $k = 2$), we have

$$\mathrm{E}\binom{X}{2} \geqslant \frac{1}{2}\mathrm{E}(X)(\mathrm{E}(X) - 1). \tag{3.14}$$

From inequality (3.13) and (3.14), we get

$$\mathrm{E}\binom{X}{2} > \frac{\epsilon}{2}\mathrm{E}(X). \tag{3.15}$$

A simple case of (3.4) with $k = 2$ is

$$\mathrm{E}\binom{X}{2} = \sum_{\substack{F_1, F_2 \in \mathcal{F} \\ F_1 \subset F_2}} \frac{|F_1|!(|F_2| - |F_1|)!(n - |F_2|)!}{n!}. \tag{3.16}$$

Now partition $\mathcal{F}$ into $\mathcal{A} \cup \mathcal{B}$ randomly. With probability $\frac{1}{4}$, a pair (F_1, F_2) has $F_1 \in \mathcal{A}$ and $F_2 \in \mathcal{B}$. There is a partition $\mathcal{F} = \mathcal{A} \cup \mathcal{B}$ satisfying

$$\sum_{\substack{F_1 \in \mathcal{A}, F_2 \in \mathcal{B} \\ F_1 \subset F_2}} \frac{|F_1|!(|F_2| - |F_1|)!(n - |F_2|)!}{n!} > \frac{\epsilon}{8}\mathrm{E}(X). \tag{3.17}$$

Now we consider an edge-weighted bipartite graph G with $V(G) = \mathcal{A} \cup \mathcal{B}$, such that F_1F_2 is an edge of G if $F_1 \in \mathcal{A}$, $F_2 \in \mathcal{B}$, and $F_1 \subset F_2$. Each edge F_1F_2 has weight

$$\frac{|F_1|!(|F_2| - |F_1|)!(n - |F_2|)!}{n!}.$$

Inequality (3.17) states that the total sum of edge weights is greater than $\frac{\epsilon}{8}\mathrm{E}(X)$.

For any $F_1 \in \mathcal{A}$, the weighted degree of F_1 is

$$d_{F_1} = \frac{1}{\binom{n}{|F_1|}} \sum_{\substack{F_2 \in \mathcal{B} \\ F_1 \subset F_2}} \frac{1}{\binom{n-|F_1|}{n-|F_2|}}. \tag{3.18}$$

Similarly, the weighted degree of $F_2 \in \mathcal{B}$ is

$$d_{F_2} = \frac{1}{\binom{n}{|F_2|}} \sum_{\substack{F_1 \in \mathcal{A} \\ F_1 \subset F_2}} \frac{1}{\binom{|F_2|}{|F_1|}}. \tag{3.19}$$

We delete vertices F with weighted degree less than $\frac{\epsilon}{8}\frac{1}{\binom{n}{|F|}}$ recursively until all remaining vertices have weighted degree at least $\frac{\epsilon}{8}\frac{1}{\binom{n}{|F|}}$ in the remaining graph – call it G' – which has vertex partition $\mathcal{A}' \cup \mathcal{B}'$ with $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{B}' \subset \mathcal{B}$.

Every time a vertex F is removed, the edge-weighted sum drops by at most $\frac{\epsilon}{8}\frac{1}{\binom{n}{|F|}}$, so the total drop in edge sum when removing all these vertices is at most

$$\sum_{F\in\mathcal{F}} \frac{\epsilon}{8}\frac{1}{\binom{n}{|F|}} = \frac{\epsilon}{8}\mathrm{E}(X).$$

We have

$$\sum_{\substack{F_1\in\mathcal{A}',F_2\in\mathcal{B}'\\ F_1\subset F_2}} \frac{|F_1|!(|F_2|-|F_1|)!(n-|F_2|)!}{n!} \geqslant \sum_{\substack{F_1\in\mathcal{A},F_2\in\mathcal{B}\\ F_1\subset F_2}} \frac{|F_1|!(|F_2|-|F_1|)!(n-|F_2|)!}{n!} - \frac{\epsilon}{8}\mathrm{E}(X).$$

Since the last expression is positive by (3.17), both families $\mathcal{A}'$ and $\mathcal{B}'$ are non-empty.

By construction, every vertex in the remaining bipartite graph G' has weighted degree at least $\frac{\epsilon}{8}\frac{1}{\binom{n}{|F|}}$. For any $F_1 \in \mathcal{A}'$, by (3.18) we have

$$\sum_{\substack{F_2\in\mathcal{B}\\ F_1\subset F_2}} \frac{1}{\binom{n-|F_1|}{|F_2|-|F_1|}} \geqslant \frac{\epsilon}{8}. \tag{3.20}$$

Note that

$$\binom{n-|F_1|}{|F_2|-|F_1|} \geqslant n-|F_1| \geqslant \frac{n}{2} - 2\sqrt{n\ln n}. \tag{3.21}$$

Combining inequalities (3.20) and (3.21), we have

$$\sum_{\substack{F_2\in\mathcal{B}'\\ F_1\subset F_2}} 1 \geqslant \frac{\epsilon}{8}\left(\frac{n}{2} - 2\sqrt{n\ln n}\right). \tag{3.22}$$

Similarly, for any $F_2 \in \mathcal{B}'$,

$$\sum_{\substack{F_1\in\mathcal{A}'\\ F_1\subset F_2}} 1 \geqslant \frac{\epsilon}{8}\left(\frac{n}{2} - 2\sqrt{n\ln n}\right). \tag{3.23}$$

In other words, the minimum degree (in the usual sense) of G' is at least $\frac{\epsilon}{8}\left(\frac{n}{2} - 2\sqrt{n\ln n}\right)$, which equals t by the choice of ϵ.

A subgraph of G' which is isomorphic to T can be constructed as follows. For any $u \in V(T)$, map u to any vertex v of G'. Map the neighbours of u in T to the neighbours of v in G', and so on. Since the minimum degree is at least t, we can always find new vertices which have not been selected yet. This greedy algorithm finds a subposet isomorphic to T. □

Proof of Theorem 1.4. Let $\mathcal{F}$ be any $P(G)$-free family of subsets of $[n]$. By removing at most $\frac{2^{n+1}}{n^2}$ subsets, we can assume that $\mathcal{F}$ contains only subsets of sizes in the interval

$(\frac{n}{2} - 2\sqrt{n \ln n}, \frac{n}{2} + 2\sqrt{n \ln n})$. Let X be the random number defined in the proof of Theorem 1.1. We claim $\mathrm{E}(X) = 1 + o_n(1)$. Recall that

$$\mathrm{E}(X) = \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}}, \tag{3.24}$$

so that $|\mathcal{F}| \leqslant \mathrm{E}(X)\binom{n}{\lfloor \frac{n}{2} \rfloor}$. We obtain an upper bound on $\mathrm{E}(X)$. As before, we have

$$\mathrm{E}\binom{X}{2} \geqslant \frac{1}{2}\mathrm{E}(X)(\mathrm{E}(X) - 1). \tag{3.25}$$

We will bound $\mathrm{E}\binom{X}{2}$ in terms of $\mathrm{E}(X)$. Recall

$$\mathrm{E}\binom{X}{2} = \sum_{\substack{A,B \in \mathcal{F} \\ A \subset B}} \frac{|A|!(|B| - |A|)!(n - |B|)!}{n!}. \tag{3.26}$$

We split the summation into two parts, depending on whether $|B| - |A|$ is greater than 1 or equal to 1.

For the case that $|B| - |A| > 1$, let Y be the random variable that is the number of triples (A, S, B) in a random full (maximal) chain satisfying

$$A \subset S \subset B \quad A, B \in \mathcal{F}.$$

We can express $\mathrm{E}(Y)$ by summing over triples (A, S, B) the proportion of full chains containing A, S, B. We have

$$\begin{aligned} \mathrm{E}(Y) &= \sum_{\substack{A,B \in \mathcal{F}, S \\ A \subset S \subset B}} \frac{|A|!(|S| - |A|)!(|B| - |S|)!(n - |B|)!}{n!} \\ &= \sum_{\substack{A,B \in \mathcal{F} \\ A \subset B}} \frac{|A|!(|B| - |A|)!(n - |B|)!}{n!} \sum_{S\,:\, A \subset S \subset B} \frac{1}{\binom{|B|-|A|}{|S|-|A|}} \\ &= \sum_{\substack{A,B \in \mathcal{F} \\ A \subset B}} \frac{|A|!(|B| - |A|)!(n - |B|)!}{n!}(|B| - |A| - 1) \\ &\geqslant \sum_{\substack{A,B \in \mathcal{F} \\ A \subset B, |B|-|A|>1}} \frac{|A|!(|B| - |A|)!(n - |B|)!}{n!}. \end{aligned} \tag{3.27}$$

Denote the number of vertices in G by v and the number of edges in G by m. Since $\mathcal{F}$ is $P(G)$-free, there are no $v + m$ subsets $A_1, A_2, \ldots, A_v, B_1, \ldots, B_m \in \mathcal{F}$ satisfying $A_i \subset S \subset B_j$ for $1 \leqslant i \leqslant v$ and $1 \leqslant j \leqslant m$.

For any fixed subset S, either 'at most $m - 1$ subsets in $\mathcal{F}$ are supersets of S' or 'at most $v - 1$ subsets in $\mathcal{F}$ are subsets of S'. Define

$$\mathcal{G}_1 = \left\{ S : |S| \in \left(\frac{n}{2} - 2\sqrt{n \ln n}, \frac{n}{2} + 2\sqrt{n \ln n} \right),\ S \text{ has at most } v - 1 \text{ subsets in } \mathcal{F} \right\},$$

$$\mathcal{G}_2 = \left\{ S : |S| \in \left(\frac{n}{2} - 2\sqrt{n \ln n}, \frac{n}{2} + 2\sqrt{n \ln n} \right),\ S \text{ has at most } m - 1 \text{ supersets in } \mathcal{F} \right\}.$$

$\mathcal{G}_1 \cup \mathcal{G}_2$ covers all subsets with sizes in $\left(\frac{n}{2} - 2\sqrt{n \ln n}, \frac{n}{2} + 2\sqrt{n \ln n}\right)$. Rewrite $\mathrm{E}(Y)$ as

$$\mathrm{E}(Y) = \sum_{S\,:\,||S|-\frac{n}{2}|<2\sqrt{n\ln n}} \frac{1}{\binom{n}{|S|}} \sum_{\substack{A\in\mathcal{F}\\A\subset S}} \frac{1}{\binom{|S|}{|A|}} \sum_{\substack{B\in\mathcal{F}\\S\subset B}} \frac{1}{\binom{n-|S|}{n-|B|}}. \tag{3.28}$$

For $S \in \mathcal{G}_2$, we have

$$\sum_{B\in\mathcal{F}, S\subset B} \frac{1}{\binom{n-|S|}{n-|B|}} \leqslant \frac{m-1}{\frac{n}{2} - 2\sqrt{n\ln n}}. \tag{3.29}$$

It implies that

$$\begin{aligned}\sum_{S\in\mathcal{G}_1} \frac{1}{\binom{n}{|S|}} \sum_{\substack{A\in\mathcal{F}\\A\subset S}} \frac{1}{\binom{|S|}{|A|}} \sum_{\substack{B\in\mathcal{F}\\S\subset B}} \frac{1}{\binom{n-|S|}{n-|B|}} &\leqslant \sum_{S\in\mathcal{G}_1} \frac{1}{\binom{n}{|S|}} \sum_{\substack{A\in\mathcal{F}\\A\subset S}} \frac{1}{\binom{|S|}{|A|}} \frac{m-1}{\frac{n}{2} - 2\sqrt{n\ln n}} \\ &\leqslant \mathrm{E}(X) 4\sqrt{n\ln n} \frac{m-1}{\frac{n}{2} - 2\sqrt{n\ln n}}. \end{aligned} \tag{3.30}$$

Similarly, we have

$$\sum_{S\in\mathcal{G}_2} \frac{1}{\binom{n}{|S|}} \sum_{\substack{A\in\mathcal{F}\\A\subset S}} \frac{1}{\binom{|S|}{|A|}} \sum_{\substack{B\in\mathcal{F}\\S\subset B}} \frac{1}{\binom{n-|S|}{n-|B|}} \leqslant \mathrm{E}(X) 4\sqrt{n\ln n} \frac{v-1}{\frac{n}{2} - 2\sqrt{n\ln n}}. \tag{3.31}$$

Combining (3.28) with inequalities (3.30) and (3.31), we have

$$\mathrm{E}(Y) \leqslant \mathrm{E}(X) 4\sqrt{n\ln n} \frac{v+m-2}{\frac{n}{2} - 2\sqrt{n\ln n}}. \tag{3.32}$$

In particular, combining with inequality (3.27), we have

$$\sum_{\substack{A,B\in\mathcal{F},|B|-|A|>1\\A\subset B}} \frac{|A|!(|B|-|A|)!(n-|B|)!}{n!} \leqslant \mathrm{E}(X) 4\sqrt{n\ln n} \frac{v+m-2}{\frac{n}{2} - 2\sqrt{n\ln n}} = o_n(\mathrm{E}(X)). \tag{3.33}$$

Now we consider pairs (A, B) with additional property $|B| - |A| = 1$. For any subset S, we define

$$\begin{aligned} N^+(S) &= \{T \in \mathcal{F} \mid S \subset T, |T| = |S| + 1\}, \\ N^-(S) &= \{T \in \mathcal{F} \mid T \subset S, |T| = |S| - 1\}. \end{aligned}$$

Let $d^+(S) = |N^+(S)|$ and $d^-(S) = |N^-(S)|$. We have

$$\begin{aligned} \sum_{\substack{A,B\in\mathcal{F}\\A\subset B,|B|-|A|=1}} \frac{|A|!(|B|-|A|)!(n-|B|)!}{n!} &= \sum_{\substack{A,B\in\mathcal{F}\\A\subset B,|B|-|A|=1}} \frac{|A|!(n-|B|)!}{n!} \\ &= \sum_{A\in\mathcal{F}} \frac{d^+(A)}{\binom{n}{|A|}(n-|A|)} && (3.34) \\ &= \sum_{B\in\mathcal{F}} \frac{d^-(B)}{\binom{n}{|B|}|B|}. && (3.35) \end{aligned}$$

We will demonstrate that most contributions to the summation above are from pairs (A, B) with $d^+(A) \geqslant m$ and $d^-(B) \geqslant v$. We define two subfamilies of $\mathcal{F}$ as follows:

$$\mathcal{F}_1 = \{S \in \mathcal{F} \mid d^+(S) \geqslant m\},$$
$$\mathcal{F}_2 = \{S \in \mathcal{F} \mid d^-(S) \geqslant v\}.$$

We have

$$\begin{aligned}
&\sum_{\substack{A,B\in\mathcal{F}\\ A\subset B, |B|-|A|=1}} \frac{|A|!(n-|B|)!}{n!} \\
&\quad\leqslant \sum_{\substack{A\in\mathcal{F}_1, B\in\mathcal{F}_2\\ A\subset B, |B|-|A|=1}} \frac{|A|!(n-|B|)!}{n!} + \sum_{A\in\mathcal{F}\setminus\mathcal{F}_1} \frac{d^+(A)}{\binom{n}{|A|}(n-|A|)} + \sum_{B\in\mathcal{F}\setminus\mathcal{F}_2} \frac{d^-(B)}{\binom{n}{|B|}|B|} \\
&\quad\leqslant \sum_{\substack{A\in\mathcal{F}_1, B\in\mathcal{F}_2\\ A\subset B, |B|-|A|=1}} \frac{|A|!(n-|B|)!}{n!} + \sum_{A\in\mathcal{F}\setminus\mathcal{F}_1} \frac{m-1}{\binom{n}{|A|}(n-\sqrt{2n\ln n})} \\
&\qquad + \sum_{B\in\mathcal{F}\setminus\mathcal{F}_2} \frac{v-1}{\binom{n}{|B|}(n-\sqrt{2n\ln n})} \\
&\quad\leqslant \sum_{\substack{A\in\mathcal{F}_1, B\in\mathcal{F}_2\\ A\subset B, |B|-|A|=1}} \frac{|A|!(n-|B|)!}{n!} + \frac{v+m-2}{n-\sqrt{2n\ln n}} \mathrm{E}(X). \qquad (3.36)
\end{aligned}$$

Recall that C_σ is a random full chain of subsets of $[n]$. For $i = 1, 2$, let $X_i = |\mathcal{F}_i \cap C_\sigma|$, so

$$\mathrm{E}(X_i) = \sum_{F\in\mathcal{F}_i} \frac{1}{\binom{n}{|F|}}. \qquad (3.37)$$

Since $\mathcal{F}$ is $P(G)$-free, we have $\mathcal{F}_1 \cap \mathcal{F}_2 = \emptyset$. In particular,

$$\mathrm{E}(X_1) + \mathrm{E}(X_2) \leqslant \mathrm{E}(X). \qquad (3.38)$$

Let us consider a 'diamond' configuration $S \subset A_i \subset B$ for $(i = 1, 2)$ with $A_1, A_2 \in \mathcal{F}_1$, $B \in \mathcal{F}_2$, and $|B| = |S| + 2$. In other words, $S = A_1 \cap A_2$ and $B = A_1 \cup A_2 \in \mathcal{F}_2$, where $A_1, A_2 \in \mathcal{F}_1$ only differ by one element. For a fixed S, we define an auxiliary graph L_S with vertex set $N^+(S) \cap \mathcal{F}_1$ such that two subsets A_1, A_2 form an edge in L_S if $A_1 \cup A_2 \in \mathcal{F}_2$. We have:

(1) L_S is G-free since $\mathcal{F}$ is $P(G)$-free,
(2) each edge of L_S is in one-to-one correspondence with a diamond configuration as above.

Recall that the Turán number $t(n, G)$ is the maximum number of edges that a graph on n vertices can have without containing the subgraph G. The Erdős–Simonovits–Stone theorem [7, 8] states

$$t(n, G) = \left(1 - \frac{1}{\chi(G)-1} + o_n(1)\right)\frac{n^2}{2}, \qquad (3.39)$$

where $\chi(G)$ is the chromatic number of G.

Let $d_1^+(S) = |N^+(S) \cap \mathcal{F}_1|$ and $d_1^-(B) = |N^-(B) \cap \mathcal{F}_1|$. The number of edges in L_S is at most $t(d_1^+(S), G)$. We have

$$\sum_S f(|S|) t(d_1^+(S), G) \geqslant \sum_{B \in \mathcal{F}_2} f(|B|-2) \binom{d_1^-(B)}{2}. \tag{3.40}$$

Here $f(k)$ is any non-negative function over integers and the summation on the left is taken over all S with sizes in $\left(\frac{n}{2} - 2\sqrt{n \ln n} - 1, \frac{n}{2} + 2\sqrt{n \ln n} - 1\right)$. Choose

$$f(k) = \frac{1}{\binom{n}{k}(n-k)^2}$$

for $k \in \left(\frac{n}{2} - 2\sqrt{n \ln n} - 1, \frac{n}{2} + 2\sqrt{n \ln n} - 1\right)$. We have

$$\begin{aligned} \sum_S f(|S|) t(d_1^+(S), G) &\leqslant \frac{1}{2}\left(1 - \frac{1}{\chi(G)-1} + o_n(1)\right) \sum_S f(|S|) d_1^+(S)(n-|S|) \\ &= \frac{1}{2}\left(1 - \frac{1}{\chi(G)-1} + o_n(1)\right) \sum_S \frac{d_1^+(S)}{\binom{n}{|S|}(n-|S|)} \\ &= \frac{1}{2}\left(1 - \frac{1}{\chi(G)-1} + o_n(1)\right) \mathrm{E}(X_1) \end{aligned} \tag{3.41}$$

and

$$\begin{aligned} \sum_{B \in \mathcal{F}_2} f(|B|-2)\binom{d_1^-(B)}{2} &= \frac{1}{2} \sum_{B \in \mathcal{F}_2} \frac{1}{\binom{n}{|B|-2}(n-|B|+2)^2}\left((d_1^-(B))^2 - d_1^-(B)\right) \\ &= \frac{1}{2}\left(1 + O\left(\frac{\sqrt{n \ln n}}{n}\right)\right) \sum_{B \in \mathcal{F}_2} \frac{(d_1^-(B))^2 - d_1^-(B)}{\binom{n}{|B|}|B|^2} \\ &= \frac{1}{2}\left(1 + O\left(\frac{\sqrt{n \ln n}}{n}\right)\right) \sum_{B \in \mathcal{F}_2} \frac{(d_1^-(B))^2}{\binom{n}{|B|}|B|^2} - O\left(\frac{1}{n}\right)\mathrm{E}(X_2). \end{aligned} \tag{3.42}$$

Combining (3.40), (3.41) and (3.42), we have

$$\sum_{B \in \mathcal{F}_2} \frac{(d_1^-(B))^2}{\binom{n}{|B|}|B|^2} \leqslant \left(1 - \frac{1}{\chi(G)-1} + o_n(1)\right)\mathrm{E}(X_1) + O\left(\frac{1}{n}\right)\mathrm{E}(X_2).$$

Applying the Cauchy–Schwarz inequality, the inequality above, and the arithmetic–geometric mean inequality, we have

$$\begin{aligned} &\sum_{\substack{A \in \mathcal{F}_1, B \in \mathcal{F}_2 \\ A \subset B, |B|-|A|=1}} \frac{|A|!(n-|B|)!}{n!} \\ &\quad = \sum_{B \in \mathcal{F}_2} \frac{d_1^-(B)}{\binom{n}{|B|}|B|} \\ &\quad \leqslant \sqrt{\sum_{B \in \mathcal{F}_2} \frac{1}{\binom{n}{|B|}} \sum_{B \in \mathcal{F}_2} \frac{(d_1^-(B))^2}{\binom{n}{|B|}|B|^2}} \end{aligned}$$

$$\leqslant \sqrt{\mathrm{E}(X_2)\left(\left(1-\frac{1}{\chi(G)-1}+o_n(1)\right)\mathrm{E}(X_1)+O\left(\frac{1}{n}\right)\mathrm{E}(X_2)\right)}$$
$$= \left(\sqrt{1-\frac{1}{\chi(G)-1}}+o_n(1)\right)\sqrt{\mathrm{E}(X_1)\mathrm{E}(X_2)}+O\left(\frac{1}{\sqrt{n}}\right)\mathrm{E}(X_2)$$
$$\leqslant \left(\sqrt{1-\frac{1}{\chi(G)-1}}+o_n(1)\right)\frac{\mathrm{E}(X_1)+\mathrm{E}(X_2)}{2}+O\left(\frac{1}{\sqrt{n}}\right)\mathrm{E}(X_2)$$
$$\leqslant \left(\sqrt{1-\frac{1}{\chi(G)-1}}+o_n(1)\right)\frac{\mathrm{E}(X)}{2}. \tag{3.43}$$

Combining inequalities (3.33), (3.36) and (3.43) , we have

$$\mathrm{E}\binom{X}{2} = \sum_{\substack{A,B\in\mathcal{F},|B|-|A|>1\\ A\subset B}} \frac{|A|!(|B|-|A|)!(n-|B|)!}{n!}$$
$$+ \sum_{\substack{A,B\in\mathcal{F},|B|-|A|=1\\ A\subset B}} \frac{|A|!(|B|-|A|)!(n-|B|)!}{n!}$$
$$\leqslant o_n(\mathrm{E}(X)) + \left(\sqrt{1-\frac{1}{\chi(G)-1}}+o_n(1)\right)\frac{\mathrm{E}(X)}{2}$$
$$\leqslant \left(\sqrt{1-\frac{1}{\chi(G)-1}}+o_n(1)\right)\frac{1}{2}\mathrm{E}(X). \tag{3.44}$$

Combining inequalities (3.25) and (3.44), we have

$$\mathrm{E}(X) \leqslant 1+\sqrt{1-\frac{1}{\chi(G)-1}}+o_n(1). \tag{3.45}$$

The proof is finished by observing $|\mathcal{F}| \leqslant \mathrm{E}(X)\binom{n}{\lfloor \frac{n}{2}\rfloor}$. □

4. Further research

Let

$$\pi(H) := \lim_{n\to\infty} \frac{\mathrm{La}(n,H)}{\binom{n}{\lfloor \frac{n}{2}\rfloor}},$$

when this limit exists. Does this limit exist for all posets H, as we suspect, even though we have proved it for only a small family of posets?

When the limit $\pi(H)$ exists, how does it depend on H? For those few posets H where we know $\pi(H)$, it is an integer. Is $\pi(H)$ an integer for all H, as we suspect? In fact, Mike Saks and Pete Winkler have observed (unpublished) that for all examples where $\pi(H)$ is known, it equals the maximum number m such that the middle m levels of the Boolean lattice $\mathcal{B}_n$ do not contain H, no matter how large n is.

In particular, this observation is consistent with the values of $\pi(H)$ given here for batons and for up-down trees. After discovering our results, we learned of a preprint by Bukh [1] that describes the asymptotic behaviour of $\mathrm{La}(n, T)$ for *every* tree poset. Specifically, if T is any poset for which the Hasse diagram is a tree (connected and acyclic), then

$$\mathrm{La}(n, T) = (h(T) - 1)\binom{n}{\lfloor \frac{n}{2} \rfloor}(1 + O(1/n)). \tag{4.1}$$

This result, if correct, would be consistent with the observation of Saks and Winkler. It would imply the leading asymptotic behaviour for batons and up-down trees in Theorems 1.1 and 1.2, though the proofs and error terms are different.

For posets H of height 2, we know that the limit $\pi(H)$, when it exists, belongs to the interval $[1, 2]$. Are there any posets H of height 2 such that $\pi(H)$ is strictly between 1 and 2? (Of course, we expect not, but we do not have much evidence.)

At an early stage of this work, we asked whether there exists a number c_h such that, for all posets H of height h, $\pi(H) \leqslant c_h$. As we noted above, $c_2 = 2$. However, Lu and, independently, Tao Jiang, pointed out that no such c_h exists for $h \geqslant 3$ by reasoning similar to that above. The idea is that if one takes $\mathcal{F}$ to consist of the middle m levels in the Boolean lattice $\mathcal{B}_n$, then two sets $A, B \in \mathcal{F}$ with $A \subset B$ have at most $2^{m-1} - 2$ sets C with $A \subset C \subset B$. Hence, the family $\mathcal{F}$, which has size $\sim m\binom{n}{\lfloor \frac{n}{2} \rfloor}$, avoids the height-3 poset consisting of a minimum element, a maximum element, and an antichain of $2^{m-1} - 1$ elements in between. This forces c_3 to be larger than any m, so that no such c_3 exists. It seems that not just the height, but the width, of H affects $\pi(H)$.

It would be very interesting then to determine $\pi(B_n)$ for the Boolean lattice $\mathcal{B}_n$. Even for a poset as fundamental as the diamond poset B_2, we do not yet know the value of $\pi(B_2)$, or if it even exists. We believe one can show that if it exists, it must be in the interval $[2, 2.25]$. As for B_3, it is worth noting that the height-2 poset formed by its middle two levels is the poset $\mathcal{O}_6$, which is the smallest crown for which π is not yet determined. There is plenty of work still to be done in this subject!

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