Continuous Nowhere Differentiable Functions

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Summary

Continuous nowhere differentiable functions are functions that are continuous at every point in their domains but do not have a derivative at any point in their domains. Although continuous functions are usually presented in a way that leads students to assume that differentiability is the norm and that nowhere differentiable functions are the exception, one can make the case that most continuous functions are nowhere differentiable. Two examples of continuous nowhere differentiable functions on $[0, 1]$ are the Generalized van der Waerden–Takagi function and Kiesswetter's function. The Generalized van der Waerden–Takagi function is constructed by taking the infinite sum over $n = 0, 1, 2, \ldots$ of specially constructed functions $f_{b,n}$ on $[0, 1]$. The number b is a fixed integer greater than or equal to 2, and the functions $f_{b,n}$ have b^n peaks. Since the sum of the functions converges and the peaks do not cancel each other out in the summation, the resulting function has an "infinite number of peaks" and is thus nowhere differentiable. Similarly, Kiesswetter's function is constructed as a convergent sequence of functions g_n on $[0, 1]$ for $n = 0, 1, 2, \ldots$. For all n greater than 0, g_n has more than 4^{n-1} peaks. Since Kiesswetter's function is the limit of the sequence of g_n's as $g \to \infty$, Kiesswetter's function also has an "infinite number of peaks" and is nowhere differentiable on $[0, 1]$. We begin by defining these functions in detail and showing that they are in fact continuous and nowhere differentiable on $[0, 1]$.

Although we cannot fully visualize their graphs, we can determine certain properties of continuous nowhere differentiable functions. One quantity that we want to consider is Hölder continuity, which relates the difference in the values of a function

at two points to the distance between the two points. If there exists a positive constant, M, such that $|f(x) - f(y)| \leq M|x - y|^{\alpha}$ for all x and y in the domain of the function, then the function f is said to be Hölder continuous of exponent α. We determine the exponents for which the Generalized van der Waerden–Takagi function and Kiesswetter's function are Hölder continuous.

Since they are nowhere differentiable, we cannot use the existence of integer–order derivatives to determine the smoothness of continuous nowhere differentiable functions. However, these functions do have some level of smoothness. To be able to measure this smoothness and compare it to the smoothness of other functions, we use fractional derivatives. The existence of fractional derivatives of a particular order can then be used as our measure of smoothness for a nowhere differentiable function. Since fractional derivatives can be very difficult to calculate directly, we use the connections between fractional derivatives and Hölder continuity to be able to determine the existence of fractional derivatives of certain orders without direct calculation. Using these connections, we are able to determine the existence of fractional derivatives for the Generalized van der Waerden–Takagi function and Kiesswetter's function.

We also want to be able to measure and compare the sizes of the graphs of continuous nowhere differentiable functions. However, the graphs of these functions all have infinite length. Thus, we must refine our notion of size. We use the ideas of dimension from fractal geometry as a different way to measure the size of the graphs as subsets of the plane. The two ideas of dimension that we use are the Hausdorff dimension and the box–counting dimension. We apply both of these ideas to the graphs of the Generalized van der Waerden–Takagi function and Kiesswetter's function. Using the connections between Hölder continuity and dimension, we are able to obtain bounds, and in some cases exact values, for the Hausdorff and box–counting dimensions of the graphs of the two functions.

Continuous nowhere differentiable functions are a fascinating topic because they question preconceived notions about continuous functions. Although these functions are very difficult to visualize, it is possible to learn about their behavior. The properties of Hölder continuity, fractional derivatives, and dimension allow us to obtain a better understanding of continuous nowhere differentiable functions.

Abstract

In this thesis we investigate two classes of continuous nowhere differentiable functions on $[0,1]$: the Generalized van der Waerden–Takagi function, $f(x) = \sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$ where $a_0(x) = \operatorname{dist}(x, \mathbb{Z})$, $c > 1$, $b \in \mathbb{N}$, and $b \geq c$, and Kiesswetter's function. We investigate the Hölder continuity and fractional differentiability of the functions in these classes. We also bound, and in some cases fully determine, the Hausdorff and box–counting dimensions of the graphs of these functions.

Introduction

In this thesis, we consider continuous functions on $[0, 1]$ that are nowhere differentiable on $[0, 1]$. Two classic examples of continuous nowhere differentiable functions are the van der Waerden–Takagi function and Kiesswetter's function. The van der Waerden–Takagi function was introduced in 1903 by Teiji Takagi [10] and reintroduced with different parameter values in 1930 by B. L. van der Waerden [11]. Konrad Knopp proposed the general case of the function, but he only proved the nowhere differentiability of the function for restricted values of the parameter [6]. We relax the restrictions on the parameters and show that the resulting function is nowhere differentiable. Karl Kiesswetter introduced his example of a continuous nowhere differentiable function in 1966 [5].

Continuous nowhere differentiable functions are not Lipschitz, i.e. there does not exist a constant $M > 0$ so that $|f(x) - f(y)| \leq M|x - y|$ for all x and y in the domain of the function. However, there do exist relations between the absolute difference in the values of the function at two points, $|f(x) - f(y)|$, and the distance between these two points, $|x - y|$, raised to a power less than 1. These relations can be classified through the concept of Hölder continuity. We will study the Hölder continuity of both the Generalized van der Waerden–Takagi function and Kiesswetter's function.

Although continuous nowhere differentiable functions do not have first-order derivatives, they do have some level of smoothness. The smoothness of a continuous nowhere differentiable function can be measured by the existence of fractional derivatives. Since we are only interested in the existence of fractional derivatives of given exponents and not of the value of the fractional derivative, we will extend a result of

Hardy and Littlewood [3] to relate Hölder continuity and fractional differentiability. We can then apply our results about Hölder continuity to determine for which exponents there exist fractional derivatives for the Generalized van der Waerden–Takagi function and Kiesswetter's function.

It can be shown that the graphs of all continuous nowhere differentiable functions have infinite length. Thus to compare the sizes of graphs of continuous nowhere differentiable functions with the sizes of graphs of other continuous function, we will use the ideas of dimension from fractal geometry. We will consider the Hausdorff and box–counting dimensions and will bound, and in some cases fully determine, these quantities for the graphs of the Generalized van der Waerden–Takagi function and Kiesswetter's function.

Chapter 1

Nowhere Differentiable Functions

1.1. The Generalized van der Waerden–Takagi Function

An example of a continuous nowhere differentiable function is the following function defined on $[0,1]$. Let $a_0(x) = \operatorname{dist}(x, \mathbb{Z})$ and define $f : [0,1] \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}.$$

where $c > 1$ and $b \in \mathbb{N}$ such that $b \geq c$. In consideration of the historical context of this function, we note that if $b = c = 2$, then f is the function introduced by Takagi [10]. If $b = c = 10$, then f is the example given by van der Waerden [11]. Knopp considered the generalized form of the the function but only proved that the function was nowhere differentiable for $b \geq 4c$ [6]. We relax the restrictions on the values of the parameters to those given above. We require that $b \geq c$ in order to obtain a nowhere differentiable function on $[0,1]$. The following fact and proposition show that if $b < c$, then f is differentiable at points in $[0,1]$.

Fact 1.1.1. Weierstrass M-test. *Let (M_n) be a sequence of positive real numbers such that $|f_n(x)| \leq M_n$ for all $x \in D$ and $n \in \mathbb{N} \cup \{0\}$. If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on D.*

Proposition 1.1.2. *If $b \in \mathbb{N}$ and $c > 1$ such that $b < c$, then $f(x) = \sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$ is Lipschitz on $[0,1]$ and is differentiable at all points $x \in [0,1)$ such that $x \neq \frac{k}{2b^m}$ for all $k, m \in \mathbb{N} \cup \{0\}$.*

PROOF. Let $b \in \mathbb{N}$ and $c > 1$ such that $b < c$. We let $f_n(x) = \frac{a_0(b^n x)}{c^n}$ on $[0,1]$. Then $f(x) = \sum_{n=0}^{\infty} f_n(x)$. As we will show in Proposition 2.1.1, $|f_n(x) - f_n(y)| \leq \left(\frac{b}{c}\right)^n |x-y|$ for all $x, y \in [0,1]$ and $n \in \mathbb{N} \cup \{0\}$. Thus, for all $x, y \in [0,1]$,

$$|f(x) - f(y)| \leq \sum_{n=0}^{\infty} |f_n(x) - f_n(y)| \leq \sum_{n=0}^{\infty} \left(\frac{b}{c}\right)^n |x-y| = \frac{c}{b-c}|x-y|$$

since $\sum_{n=0}^{\infty} \left(\frac{b}{c}\right)^n$ is a geometric series with $b < c$. Therefore, f is Lipschitz on $[0,1]$.

It is well known that Lipschitz functions are differentiable almost everywhere [8, pages 108–112], but in this case, we can better determine specific points in $[0,1]$ at which the function has derivatives. For all $n \in \mathbb{N} \cup \{0\}$, we denote the right-hand derivative of f_n by $D^+ f_n(x) = \lim_{t \to x^+} \frac{f_n(t) - f_n(x)}{t-x}$. Let $g(x) = \sum_{n=0}^{\infty} D^+ f_n(x)$. For all $n \in \mathbb{N} \cup \{0\}$ and $x \in [0,1)$, $D^+ f_n(x)$ is either $\left(\frac{b}{c}\right)^n$ or $-\left(\frac{b}{c}\right)^n$. Thus $\sum_{n=0}^{\infty} D^+ f_n(x) = \sum_{n=0}^{\infty} \pm \left(\frac{b}{c}\right)^n$ is uniformly convergent on $[0,1)$ by the Weierstrass M-Test since $b < c$. Since $D^+ f_n$ is continuous at all points in $[0,1)$ except on the set $\{\{\frac{k}{2b^n}\} : k \in \mathbb{Z} \cap [0, 2b^n - 1]\}$, $D^+ f_n$ is Riemann integrable on $[0,1)$. Then

$$\int_0^x g(t)dt = \int_0^x \sum_{n=0}^{\infty} D^+ f_n(t)dt = \sum_{n=0}^{\infty} \int_0^x D^+ f_n(t)dt = \sum_{n=0}^{\infty} f_n(x) = f(x)$$

since $\sum_{n=0}^{\infty} D^+ f_n(x)$ is uniformly convergent on $[0,1)$. Then, by the Fundamental Theorem of Calculus, $f'(x) = \sum_{n=0}^{\infty} D^+ f_n(x)$ for all $x \in [0,1)$ such that $\sum_{n=0}^{\infty} D^+ f_n(x)$ is continuous at x. Therefore, $f(x)$ is differentiable at all points $x \in [0,1)$ such that $x \neq \frac{k}{2b^m}$ for all $k, m \in \mathbb{N} \cup \{0\}$. $\square$

The preceding proof does not tell us whether or not the function is differentiable at points in $[0,1]$ of the form $x = \frac{k}{2b^m}$ for some $k, m \in \mathbb{N} \cup \{0\}$. The differentiability of the function at these points is dependent on the specific values of b and c. In fact, for certain values of b and c such that $b < c$, the function $f(x) = \sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$ is everywhere differentiable on its domain. To see this, we consider the following proposition suggested by [12, pages 33–38].

PROPOSITION 1.1.3. *Let* $f : [0,1] \to \mathbb{R}$ *be defined by* $f(x) = \sum_{n=0}^{\infty} \frac{a_0(2^n x)}{4^n}$ *and* $g : [0,1] \to \mathbb{R}$ *be defined by* $g(x) = 2x - 2x^2$. *Then* $f = g$.

PROOF. For all $i, k \in \mathbb{N} \cup \{0\}$ such that $i \leq 2^k - 1$, the function $\frac{a_0(2^n x)}{4^n}$ is linear on $\left[\frac{i}{2^k}, \frac{i+1}{2^k}\right]$ for all $n \in \mathbb{N} \cup \{0\}$ such that $n \leq k-1$. Then since $\frac{2i+1}{2^{k+1}}$ is the midpoint of $\left[\frac{i}{2^k}, \frac{i+1}{2^k}\right]$,

$$\frac{a_0\left(2^n\left(\frac{2i+1}{2^{k+1}}\right)\right)}{4^n} = \frac{1}{2}\left(\frac{a_0\left(2^n\left(\frac{i}{2^k}\right)\right)}{4^n} + \frac{a_0\left(2^n\left(\frac{i+1}{2^k}\right)\right)}{4^n}\right)$$

for all $n \in \mathbb{N} \cup \{0\}$ such that $n \leq k-1$. Using these facts, we observe that f and g satisfy the same difference equation for $i, k \in \mathbb{N} \cup \{0\}$ such that $i \leq 2^k - 1$

$$\begin{aligned}
f\left(\frac{2i+1}{2^{k+1}}\right) - \frac{1}{2}\left(f\left(\frac{i}{2^k}\right) + f\left(\frac{i+1}{2^k}\right)\right) &= \sum_{n=0}^{k} \frac{a_0\left(\frac{2i+1}{2^{k+1-n}}\right)}{4^n} \\
&\quad - \frac{1}{2}\left(\sum_{n=0}^{k-1} \frac{a_0\left(\frac{i}{2^{k-n}}\right) + a_0\left(\frac{i+1}{2^{k-n}}\right)}{4^n}\right) \\
&= \frac{a_0\left(\frac{2i+1}{2}\right)}{4^k} \\
&= \frac{1}{2}\frac{1}{4^k} = \frac{2}{4^{k+1}}
\end{aligned}$$

and

$$\begin{aligned}
g\left(\frac{2i+1}{2^{k+1}}\right) - \frac{1}{2}\left(g\left(\frac{i}{2^k}\right) + g\left(\frac{i+1}{2^k}\right)\right) &= \frac{2i+1}{2^k} - \frac{4i^2+4i+1}{2^{2k+1}} \\
&\quad - \left(\frac{2i+1}{2^k} - \frac{2i^2+2i+1}{2^{2k}}\right) \\
&= \frac{4i^2+4i+2}{2^{2k+1}} - \frac{4i^2+4i+1}{2^{2k+1}} \\
&= \frac{1}{2^{2k+1}} = \frac{2}{4^{k+1}}.
\end{aligned}$$

Thus, f and g agree on the set $\left\{\frac{i}{2^k} : i, k \in \mathbb{N} \cup \{0\}, i \leq 2^k - 1\right\}$. Since f and g are continuous on $[0,1]$, $\left\{\frac{i}{2^k} : i, k \in \mathbb{N} \cup \{0\}, i \leq 2^k - 1\right\}$ is a dense subset of $[0,1]$, and $f(0) = f(1) = 0 = g(0) = g(1)$, then $f(x) = g(x)$ for all $x \in [0,1]$. $\square$

Now that we have established why we require that $b \geq c$, we will show that the Generalized van der Waerden–Takagi function is continuous on $[0,1]$.

PROPOSITION 1.1.4. *The series* $\sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$, *where* $c > 1$ *and* $b \in \mathbb{N}$ *such that* $b \geq c$, *is uniformly convergent on* $[0,1]$.

PROOF. Since $a_0(x) \leq \frac{1}{2}$ for all $x \in \mathbb{R}$, then $a_0(b^n x) \leq \frac{1}{2}$ for all $x \in [0,1], n \in \mathbb{N} \cup \{0\}$. Let $M_n = \frac{1}{2c^n}$ for all $n \in \mathbb{N} \cup \{0\}$. Then $\left|\frac{a_0(b^n x)}{c^n}\right| \leq M_n$ for all $n \in \mathbb{N} \cup \{0\}$. The series $\sum_{n=0}^{\infty} M_n = \frac{1}{2}\sum_{n=0}^{\infty} \left(\frac{1}{c}\right)^n$ is a geometric series and is convergent since $c > 1$. Thus, by the Weierstrass M-test, $\sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$ is uniformly convergent on $[0,1]$. $\square$

PROPOSITION 1.1.5. *The function* $a_0(x)$ *is continuous on* $\mathbb{R}$.

PROOF. Since $a_0(x)$ is linear on $\left\{\left(z, z+\frac{1}{2}\right) \cup \left(z+\frac{1}{2}, z+1\right) : z \in \mathbb{Z}\right\}$, then $a_0(x)$ is continuous on $\mathbb{R} \setminus \left\{\mathbb{Z} \cup \left\{\frac{z}{2} : z \in \mathbb{Z}\right\}\right\}$. It is obvious that $a_0(x)$ is continuous at t for all $t \in \left\{\mathbb{Z} \cup \left\{\frac{z}{2} : z \in \mathbb{Z}\right\}\right\}$. Therefore, $a_0(x)$ is continuous on $\mathbb{R}$. $\square$

PROPOSITION 1.1.6. *For all* $n \in \mathbb{N} \cup \{0\}$, *the function* $f_n(x) = \frac{a_0(b^n x)}{c^n}$, *where* $c > 1$ *and* $b \in \mathbb{N}$ *such that* $b \geq c$, *is continuous on* $[0,1]$.

PROOF. Since a continuous function multiplied by a constant is a continuous function, $b^n x$ is continuous on $[0,1]$ for all $n \in \mathbb{N} \cup \{0\}$. Since the composition of continuous functions is a continuous function and since $a_0(x)$ is continuous on $\mathbb{R}$ by Proposition 1.1.5, then $a_0(b^n x)$ is a continuous function on $[0,1]$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, for all $n \in \mathbb{N} \cup \{0\}$, $\frac{a_0(b^n x)}{c^n}$ is continuous on $[0,1]$ since c^n is a constant with respect to x. $\square$

FACT 1.1.7. *Let* $\{f_n\}$ *be a sequence of continuous functions on a set* $A \subseteq \mathbb{R}$ *and suppose that* $\{f_n\}$ *converges uniformly on* A *to a function* $f : A \to \mathbb{R}$. *Then* f *is continuous on* A.

PROPOSITION 1.1.8. $f(x) = \sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$ *is continuous on* $[0,1]$.

PROOF. By Proposition 1.1.6, we know that $\frac{a_0(b^n x)}{c^n}$ is continuous on $[0,1]$ for all $n \in \mathbb{N} \cup \{0\}$. By Proposition 1.1.4, $\sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$ is uniformly convergent on $[0,1]$. Thus, the sequence $\left(\sum_{n=0}^{m} \frac{a_0(b^n x)}{c^n}\right)_{m \in \mathbb{N}}$ is uniformly convergent on $[0,1]$, and $\sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$ is continuous on $[0,1]$ by Fact 1.1.7. □

To show that the Generalized van der Waerden–Takagi function is nowhere differentiable on $[0,1]$, we need the following lemma.

LEMMA 1.1.9. *If f is differentiable at $x = c$, then if $(u_n)_{n \in \mathbb{N}}$ is an increasing sequence and $(v_n)_{n \in \mathbb{N}}$ is a decreasing sequence such that $u_n \neq v_n$ and $u_n \leq c \leq v_n$ for all $n \in \mathbb{N}$ and such that $\lim_{n\to\infty} v_n - u_n = 0$, then $\lim_{n\to\infty} \frac{f(v_n)-f(u_n)}{v_n-u_n} = f'(c)$.*

PROOF. Fix $\varepsilon > 0$. Since f is differentiable at $x = c$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| < \frac{\varepsilon}{2}$.

Case 1. There exists $M \in \mathbb{N}$ such that $u_n = c < v_n$ for all $n > M$.

Let $(u_n)_{n \in \mathbb{N}}$ be an increasing sequence and $(v_n)_{n \in \mathbb{N}}$ be a decreasing sequence such that there exists $M \in \mathbb{N}$ such that $u_n = c < v_n$ for all $n \in \mathbb{N}$ such that $n > M$. Then since $\lim_{n\to\infty} v_n - u_n = 0$, there exists $N \in \mathbb{N}$ such that if $n \in \mathbb{N}$ and $n > N$, then $|v_n - u_n| < \delta$. Let $n \in \mathbb{N}$ such that $n > \max\{N, M\}$. Since $u_n = c$, then $|v_n - c| < \delta$. Thus

$$\left|\frac{f(v_n) - f(u_n)}{v_n - u_n} - f'(c)\right| = \left|\frac{f(v_n) - f(c)}{v_n - c} - f'(c)\right| < \frac{\varepsilon}{2}.$$

Therefore, $\lim_{n\to\infty} \frac{f(v_n)-f(u_n)}{v_n-u_n} = f'(c)$.

Case 2. There exists $M \in \mathbb{N}$ such that $u_n < c = v_n$ for all $n > M$.

By an argument similar to the one used in Case 1, it can be shown that the value of the limit $\lim_{n\to\infty} \frac{f(v_n)-f(u_n)}{v_n-u_n}$ is $f'(c)$.

Case 3. $u_n < c < v_n$ for all $n \in \mathbb{N}$.

Let $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ be sequences such that $u_n < c < v_n$ for all $n \in \mathbb{N}$. Then since $\lim_{n\to\infty} v_n - u_n = 0$, there exists $N \in \mathbb{N}$ such that if $n \in \mathbb{N}$ and $n > N$, then $|v_n - u_n| < \delta$. Let $n \in \mathbb{N}$ such that $n > N$. Since $|v_n - u_n| < \delta$ and $u_n < c < v_n$,

$|v_n - c| < \delta$ and $|u_n - c| < \delta$. Then

$$\begin{aligned}
\left|\frac{f(v_n) - f(u_n)}{v_n - u_n} - f'(c)\right| &= \left|\frac{f(v_n) - f(u_n) - (v_n - u_n)f'(c)}{v_n - u_n}\right| \\
&= \left|\frac{f(v_n) - f(u_n) - f(c) + f(c) - (v_n + c - c - u_n)f'(c)}{v_n - u_n}\right| \\
&\leq \left|\frac{f(v_n) - f(c) - (v_n - c)f'(c)}{v_n - u_n}\right| \\
&\qquad\qquad + \left|\frac{f(u_n) - f(c) - (u_n - c)f'(c)}{u_n - v_n}\right| \\
&\leq \left|\frac{f(v_n) - f(c) - (v_n - c)f'(c)}{v_n - c}\right| \\
&\qquad\qquad + \left|\frac{f(u_n) - f(c) - (u_n - c)f'(c)}{u_n - c}\right| \\
&= \left|\frac{f(v_n) - f(c)}{v_n - c} - f'(c)\right| + \left|\frac{f(u_n) - f(c)}{u_n - c} - f'(c)\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} \frac{f(v_n)-f(u_n)}{v_n-u_n} = f'(c)$. □

We now return to the Generalized van der Waerden–Takagi function to show that the function is nowhere differentiable on $[0, 1]$.

Proposition 1.1.10. *The function $f(x) = \sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$, where $c > 1$ and $b \in \mathbb{N}$ such that $b \geq c$, is nowhere differentiable on $[0, 1]$.*

Proof. Let $t \in [0, 1]$. We want to show that $f(x)$ is not differentiable at $x = t$.

Case 1. $t = 1$.

Let $(u_m)_{m\in\mathbb{N}}$ and $(v_m)_{m\in\mathbb{N}}$ be sequences such that $u_m = 1 - \frac{1}{b^m}$ for all $m \in \mathbb{N}$ and $v_m = 1$ for all $m \in \mathbb{N}$. Clearly, $\lim_{m\to\infty} v_m - u_m = 0$. Then

$$\frac{f(v_m) - f(u_m)}{v_m - u_m} = \frac{\sum_{n=0}^{\infty} \frac{a_0(b^n)}{c^n} - \sum_{n=0}^{\infty} \frac{a_0\left(\left(1-\frac{1}{b^m}\right)b^n\right)}{c^n}}{\frac{1}{b^m}}$$

$$= b^m \left(-\sum_{n=0}^{m-1} \frac{a_0\left(b^n - \frac{b^n}{b^m}\right)}{c^n} \right)$$

$$= -\sum_{n=0}^{m-1} \frac{b^{m+n}}{b^m c^n}$$

$$= -\sum_{n=0}^{m-1} \left(\frac{b}{c}\right)^n .$$

Since $b \geq c > 0$, $\lim_{n\to\infty} \left(\frac{b}{c}\right)^n \neq 0$. Thus the sequence $\left(\frac{f(v_m)-f(u_m)}{v_m-u_m}\right)_{m\,\epsilon\,\mathbb{N}}$, which is equivalent to the sequence $\left(\sum_{n=0}^{m-1} - \left(\frac{b}{c}\right)^n\right)_{m\,\epsilon\,\mathbb{N}}$, does not converge as $m \to \infty$. By Lemma 1.1.9, $f(x)$ is not differentiable at $x = 1$.

Case 2. $t\,\epsilon\,[0, 1)$.

Since $t\,\epsilon\,[0, 1)$, then for all $m\,\epsilon\,\mathbb{N}$ there exists $k_m\,\epsilon\,\mathbb{Z} \cap [0, 2b^m - 1]$ such that $\frac{k_m}{2b^m} \leq t < \frac{k_m+1}{2b^m}$. Let $u_m = \frac{k_m}{2b^m}$ for all $m\,\epsilon\,\mathbb{N}$ and let $v_m = \frac{k_m+1}{2b^m}$ for all $m\,\epsilon\,\mathbb{N}$. Then $(v_m)_{m\,\epsilon\,\mathbb{N}}$ is a decreasing sequence and $(u_m)_{m\,\epsilon\,\mathbb{N}}$ is an increasing sequence such that $\lim_{m\to\infty} v_m - u_m = 0$.

Let $m\,\epsilon\,\mathbb{N}$. Since $u_m = \frac{k_m}{2b^m}$ and $v_m = \frac{k_m+1}{2b^m}$, then for all n such that $0 \leq n \leq m-1$ there exists $z\,\epsilon\,\mathbb{Z}$ such that $b^n v_m$ and $b^n u_m$ are either both contained in $\left[z, z+\frac{1}{2}\right]$ or are both contained in $\left[z+\frac{1}{2}, z+1\right]$. Thus for all n such that $0 \leq n \leq m-1$, $\frac{a_0(b^n x)}{c^n}$ is linear on $[u_m, v_m]$ and has a slope of either $\left(\frac{b}{c}\right)^n$ or $-\left(\frac{b}{c}\right)^n$. Then for all n such that $0 \leq n \leq m-1$,

$$\frac{\frac{a_0(b^n v_m)}{c^n} - \frac{a_0(b^n u_m)}{c^n}}{v_m - u_m} = \pm \left(\frac{b}{c}\right)^n .$$

For $n \geq m$, we must consider 2 subcases.

Subcase 1. b is even.

Let $n = m$. If k_m is even, then $a_0\left(\frac{b^n k_m}{2b^m}\right) = 0$ and $a_0\left(\frac{b^n(k_m+1)}{2b^m}\right) = \frac{1}{2}$. Thus,

$$\frac{\frac{a_0(b^n v_m)}{c^n} - \frac{a_0(b^n u_m)}{c^n}}{v_m - u_m} = \frac{\frac{1}{2c^m} - 0}{\frac{1}{2b^m}} = \left(\frac{b}{c}\right)^m .$$

By a similar argument, if k_m is odd, then

$$\frac{\frac{a_0(b^n v_m)}{c^n} - \frac{a_0(b^n u_m)}{c^n}}{v_m - u_m} = -\left(\frac{b}{c}\right)^m .$$

Now let $n > m$. Then $b^{n-m}k_m$ and $b^{n-m}(k_m+1)$ are both even since b is even. Thus, $a_0\left(\frac{b^n k_m}{2b^m}\right) = 0$ and $a_0\left(\frac{b^n(k_m+1)}{2b^m}\right) = 0$. Then, for all $n \,\epsilon\, \mathbb{N}$ such that $n > m$,

$$\frac{\frac{a_0(b^n v_m)}{c^n} - \frac{a_0(b^n u_m)}{c^n}}{v_m - u_m} = 0.$$

Therefore if b is even, then

$$\frac{f(v_m) - f(u_m)}{v_m - u_m} = \sum_{n=0}^{m} \pm \left(\frac{b}{c}\right)^n. \tag{1}$$

Since $b \geq c > 0$, $\lim_{n\to\infty} \pm \left(\frac{b}{c}\right)^n \neq 0$. Thus $\lim_{m\to\infty} \sum_{n=0}^{m} \pm \left(\frac{b}{c}\right)^n$ does not converge. Then $\lim_{m\to\infty} \frac{f(v_m)-f(u_m)}{v_m-u_m}$ does not exist. By Lemma 1.1.9, $f(x)$ is not differentiable at $x = t$.

Subcase 2. b is odd.

Let $n \geq m$. If k_m is even, then $a_0\left(\frac{b^n k_m}{2b^m}\right) = 0$ and $a_0\left(\frac{b^n(k_m+1)}{2b^m}\right) = \frac{1}{2}$. Thus if k_m is even, then

$$\frac{\frac{a_0(b^n v_m)}{c^n} - \frac{a_0(b^n u_m)}{c^n}}{v_m - u_m} = \frac{\frac{1}{2c^n} - 0}{\frac{1}{2b^m}} = \frac{b^m}{c^n}.$$

for all $n \,\epsilon\, \mathbb{N}$ such that $n \geq m$. If k_m is odd, then by a similar argument

$$\frac{\frac{a_0(b^n v_m)}{c^n} - \frac{a_0(b^n u_m)}{c^n}}{v_m - u_m} = -\frac{b^m}{c^n}$$

for all $n \,\epsilon\, \mathbb{N}$ such that $n \geq m$. Thus,

$$\frac{f(v_m) - f(u_m)}{v_m - u_m} = \sum_{n=0}^{m-1} \pm \left(\frac{b}{c}\right)^n \pm b^m \sum_{n=m}^{\infty} \left(\frac{1}{c}\right)^n.$$

Since $c > 1$, $\sum_{n=m}^{\infty} \left(\frac{1}{c}\right)^n = \frac{c}{c-1}\left(\frac{1}{c}\right)^m$. Therefore if b is odd, then

$$\frac{f(v_m) - f(u_m)}{v_m - u_m} = \sum_{n=0}^{m-1} \pm \left(\frac{b}{c}\right)^n \pm \frac{c}{c-1}\left(\frac{b}{c}\right)^m. \tag{2}$$

Since $b \geq c > 1$, then $\lim_{n\to\infty} \pm \left(\frac{b}{c}\right)^n \neq 0$ and $\lim_{m\to\infty} \pm \frac{c}{c-1}\left(\frac{b}{c}\right)^m \neq 0$. Thus $\lim_{m\to\infty} \sum_{n=0}^{m-1} \pm \left(\frac{b}{c}\right)^n \pm \frac{c}{c-1}\left(\frac{b}{c}\right)^m$ does not converge. Then $\lim_{m\to\infty} \frac{f(v_m)-f(u_m)}{v_m-u_m}$ does not exist. By Lemma 1.1.9, $f(x)$ is not differentiable at $x = t$. $\square$

1.2. Kiesswetter's Function

Another example of a continuous nowhere differentiable function is Kiesswetter's function. The original construction of the function proposed by Kiesswetter defined the function using base 4 expansion [5]. The following alternate construction from [1, pages 201–202] uses Kiesswetter's curve to define the function. Kiesswetter's curve is constructed using the functions

$$
\begin{aligned}
f_1\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{-1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
f_2\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{-1}{2} \end{bmatrix} \\
f_3\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \\
f_4\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix}.
\end{aligned}
$$

To construct Kiesswetter's curve, let L_0 be the line segment from $(0,0)$ to $(1,1)$ in the Cartesian plane. At each stage of the construction, replace L_n with $L_{n+1} = f_1[L_n] \cup f_2[L_n] \cup f_3[L_n] \cup f_4[L_n]$. The sequence $(L_n)_{n \epsilon \mathbb{N}}$ is a convergent sequence of graphs, and the limit is the graph of Kiesswetter's function.

Let $g_n : [0,1] \to \mathbb{R}$ be the function whose graph is L_n. Then for all $t \,\epsilon\, [0,1]$, the point $(t, g_n(t)) \,\epsilon\, L_n$. We note that, from the definition of Kiesswetter's curve, the function $g_0 : [0,1] \to \mathbb{R}$ is defined by $g_0(t) = t$. To define $g_n(t)$ for $n \,\epsilon\, \mathbb{N}$, we develop the following recurrence relation.

Proposition 1.2.1. *For all $t \in [0,1]$ and all $n \in \mathbb{N}$,*

$$g_n(t) = \begin{cases} \frac{-1}{2} g_{n-1}(4t), & 0 \leq t \leq \frac{1}{4} \\ \frac{1}{2} g_{n-1}(4t-1) - \frac{1}{2}, & \frac{1}{4} < t \leq \frac{1}{2} \\ \frac{1}{2} g_{n-1}(4t-2), & \frac{1}{2} < t \leq \frac{3}{4} \\ \frac{1}{2} g_{n-1}(4t-3) + \frac{1}{2}, & \frac{3}{4} < t \leq 1. \end{cases} \tag{3}$$

Proof. Let $t \in [0,1]$ and $n \in \mathbb{N}$. Then $(t, g_{n-1}(t)) \in L_{n-1}$.

By the definition of the Kiesswetter's curve,

$$f_1 \begin{bmatrix} t \\ g_{n-1}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} t \\ g_{n-1}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t \\ \frac{-1}{2} g_{n-1}(t) \end{bmatrix} \in L_n.$$

Thus if $0 \leq t \leq \frac{1}{4}$, then $\left(t, \frac{-1}{2} g_{n-1}(4t)\right) \in L_n$.

Similarly, since

$$f_2 \begin{bmatrix} t \\ g_{n-1}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} t \\ g_{n-1}(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{4} \\ \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t + \frac{1}{4} \\ \frac{1}{2} g_{n-1}(t) - \frac{1}{2} \end{bmatrix} \in L_n,$$

then, for $\frac{1}{4} < t \leq \frac{1}{2}$, $\left(t, \frac{1}{2} g_{n-1}(4t-1) - \frac{1}{2}\right) \in L_n$.

Applying f_3, we show that

$$f_3 \begin{bmatrix} t \\ g_{n-1}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} t \\ g_{n-1}(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t + \frac{1}{2} \\ \frac{1}{2} g_{n-1}(t) \end{bmatrix} \in L_n.$$

Thus if $\frac{1}{2} < t \leq \frac{3}{4}$, then $\left(t, \frac{1}{2} g_{n-1}(4t-2)\right) \in L_n$.

Similarly, since the definition of Kiesswetter's curve states that

$$f_4 \begin{bmatrix} t \\ g_{n-1}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} t \\ g_{n-1}(t) \end{bmatrix} + \begin{bmatrix} \frac{3}{4} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4}t + \frac{3}{4} \\ \frac{1}{2} g_{n-1}(t) + \frac{1}{2} \end{bmatrix} \in L_n,$$

then, for $\frac{3}{4} < t \leq 1$, $\left(t, \frac{1}{2} g_{n-1}(4t-3) + \frac{1}{2}\right) \in L_n$.

Thus,

$$g_n(t) = \begin{cases} \frac{-1}{2} g_{n-1}(4t), & 0 \le t \le \frac{1}{4} \\ \frac{1}{2} g_{n-1}(4t-1) - \frac{1}{2}, & \frac{1}{4} < t \le \frac{1}{2} \\ \frac{1}{2} g_{n-1}(4t-2), & \frac{1}{2} < t \le \frac{3}{4} \\ \frac{1}{2} g_{n-1}(4t-3) + \frac{1}{2}, & \frac{3}{4} < t \le 1. \end{cases}$$

□

We now prove some properties of the functions, g_n for $n \in \mathbb{N} \cup \{0\}$ that will be used to show that the function whose graph is Kiesswetter's curve is continuous and nowhere differentiable.

PROPOSITION 1.2.2. *Let $g_0(t) = t$ on $[0,1]$ and let $g_n(t)$ be defined on $[0,1]$ by (3) for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N} \cup \{0\}$, $g_n(0) = 0$.*

PROOF. Since $g_0(t) = t$ for all $t \in [0,1]$, $g_0(0) = 0$. Let $k \in \mathbb{N} \cup \{0\}$ and assume $g_k(0) = 0$. Then

$$g_{k+1}(0) = \frac{-1}{2} g_k(4(0)) = \left(\frac{-1}{2}\right)(0) = 0.$$

Therefore, $g_n(0) = 0$ for all $n \in \mathbb{N} \cup \{0\}$. □

PROPOSITION 1.2.3. *Let $g_0(t) = t$ on $[0,1]$ and let $g_n(t)$ be defined on $[0,1]$ by (3) for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N} \cup \{0\}$, $g_n(1) = 1$.*

PROOF. Since $g_0(t) = t$ for all $t \in [0,1]$, $g_0(1) = 1$. Let $k \in \mathbb{N} \cup \{0\}$ and assume $g_k(1) = 1$. Then

$$g_{k+1}(1) = \frac{1}{2} g_k(4(1) - 3) + \frac{1}{2} = \frac{1}{2} g_k(1) + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1.$$

Thus, $g_n(1) = 1$ for all $n \in \mathbb{N} \cup \{0\}$. □

PROPOSITION 1.2.4. *Let $g_0(t) = t$ on $[0,1]$ and let $g_n(t)$ be defined on $[0,1]$ by (3) for all $n \in \mathbb{N}$. If $j, k \in \mathbb{N} \cup \{0\}$ such that $j \ge k$, then $g_j\left(\frac{m}{4^k}\right) = g_k\left(\frac{m}{4^k}\right)$ for all $m \in \mathbb{Z} \cap [0, 4^k]$.*

Proof. Let $k = 0$. Then $m = 0$ or $m = 1$. By Proposition 1.2.2, $g_j\left(\frac{0}{4^0}\right) = g_0\left(\frac{0}{4^0}\right) = 0$ for all $j \in \mathbb{N} \cup \{0\}$ such that $j \geq 0$. Similarly by Proposition 1.2.3, $g_j\left(\frac{1}{4^0}\right) = g_0\left(\frac{1}{4^0}\right) = 1$ for all $j \in \mathbb{N} \cup \{0\}$ such that $j \geq 0$. Thus, for all $m \in \mathbb{Z}$ such that $0 \leq m \leq 4^0$, $g_j\left(\frac{m}{4^0}\right) = g_0\left(\frac{m}{4^0}\right)$ for all $j \in \mathbb{N} \cup \{0\}$ such that $j \geq 0$.

Now let $k \in \mathbb{N}$ such that $k \geq 0$ and assume that for all $m \in \mathbb{Z} \cap [0, 4^k]$, $g_j\left(\frac{m}{4^k}\right) = g_k\left(\frac{m}{4^k}\right)$ for all $j \in \mathbb{N}$ such that $j \geq k$.

We now consider $k+1$. Fix $m \in \mathbb{Z}$ such that $0 \leq m \leq 4^{k+1}$ and let $j \in \mathbb{N}$ such that $j \geq k+1$.

Case 1. $0 \leq \frac{m}{4^{k+1}} \leq \frac{1}{4}$.

Using (3), we show that

$$g_j\left(\frac{m}{4^{k+1}}\right) = \frac{-1}{2} g_{j-1}\left(\frac{4m}{4^{k+1}}\right) = \frac{-1}{2} g_{j-1}\left(\frac{m}{4^k}\right).$$

Since $0 \leq \frac{m}{4^{k+1}} \leq \frac{1}{4}$, then $0 \leq m \leq 4^k$. Also since $j \geq k+1$, then $j - 1 \geq k$. Thus, by the inductive hypothesis, $g_{j-1}\left(\frac{m}{4^k}\right) = g_k\left(\frac{m}{4^k}\right)$. So

$$g_j\left(\frac{m}{4^{k+1}}\right) = \frac{-1}{2} g_k\left(\frac{m}{4^k}\right) = g_{k+1}\left(\frac{m}{4^{k+1}}\right).$$

Case 2. $\frac{1}{4} < \frac{m}{4^{k+1}} \leq \frac{1}{2}$.

Since $\frac{1}{4} < \frac{m}{4^{k+1}} \leq \frac{1}{2}$, then $0 < m - 4^k \leq 4^k$. Using (3) and the inductive hypothesis as in case 1, we show that

$$g_j\left(\frac{m}{4^{k+1}}\right) = \frac{1}{2} g_{j-1}\left(\frac{m - 4^k}{4^k}\right) - \frac{1}{2} = \frac{1}{2} g_k\left(\frac{m - 4^k}{4^k}\right) - \frac{1}{2} = g_{k+1}\left(\frac{m}{4^{k+1}}\right).$$

Case 3. $\frac{1}{2} < \frac{m}{4^{k+1}} \leq \frac{3}{4}$.

Since $\frac{1}{2} < \frac{m}{4^{k+1}} \leq \frac{3}{4}$, then $0 < m - 2(4^k) \leq 4^k$. Then by (3) and the inductive hypothesis,

$$g_j\left(\frac{m}{4^{k+1}}\right) = \frac{1}{2} g_{j-1}\left(\frac{m - 2(4^k)}{4^k}\right) = \frac{1}{2} g_k\left(\frac{m - 2(4^k)}{4^k}\right) = g_{k+1}\left(\frac{m}{4^{k+1}}\right).$$

Case 4. $\frac{3}{4} < \frac{m}{4^{k+1}} \leq 1$.

Since $\frac{3}{4} < \frac{m}{4^{k+1}} \leq 1$, then $0 < m - 3(4^k) \leq 4^k$. Thus using (3) and the inductive hypothesis, we show that

$$g_j\left(\frac{m}{4^{k+1}}\right) = \frac{1}{2}g_{j-1}\left(\frac{m-3(4^k)}{4^k}\right) + \frac{1}{2} = \frac{1}{2}g_k\left(\frac{m-3(4^k)}{4^k}\right) + \frac{1}{2} = g_{k+1}\left(\frac{m}{4^{k+1}}\right).$$

Thus, for all $m \in \mathbb{Z} \cap [0, 4^{k+1}]$, $g_j\left(\frac{m}{4^{k+1}}\right) = g_{k+1}\left(\frac{m}{4^{k+1}}\right)$ for all $j \in \mathbb{N}$ such that $j \geq k+1$. Therefore for all $j, k \in \mathbb{N} \cup \{0\}$ such that $j \geq k$, $g_j\left(\frac{m}{4^k}\right) = g_k\left(\frac{m}{4^k}\right)$ for all $m \in \mathbb{Z} \cap [0, 4^k]$. ☐

PROPOSITION 1.2.5. *Let $g_0(t) = t$ on $[0,1]$ and let $g_n(t)$ be defined on $[0,1]$ by (3) for all $n \in \mathbb{N}$. If $k \in \mathbb{N} \cup \{0\}$, then, for all $n, m \in \mathbb{Z}$ such that $n \geq k$ and $0 \leq m \leq 4^k - 1$,* $\left|g_n\left(\frac{m}{4^k}\right) - g_n\left(\frac{m+1}{4^k}\right)\right| = \frac{1}{2^k}$

PROOF. By Proposition 1.2.4, if $k, n \in \mathbb{N} \cup \{0\}$ such that $n \geq k$, then $g_n\left(\frac{m}{4^k}\right) = g_k\left(\frac{m}{4^k}\right)$ for all $m \in \mathbb{Z} \cap [0, 4^k]$. Thus, we need only show that if $k \in \mathbb{N} \cup \{0\}$, then $\left|g_k\left(\frac{m}{4^k}\right) - g_k\left(\frac{m+1}{4^k}\right)\right| = \frac{1}{2^k}$ for all $m \in \mathbb{Z}$ such that $0 \leq m \leq 4^k - 1$.

First let $k = 0$. Let $m \in [0, 4^0 - 1]$. Thus, $m = 0$. Then

$$\left|g_0\left(\frac{0}{4^0}\right) - g_0\left(\frac{1}{4^0}\right)\right| = |0 - 1| = \frac{1}{2^0}.$$

Now let $k = j \in \mathbb{N}$ and assume $\left|g_j\left(\frac{m}{4^j}\right) - g_j\left(\frac{m+1}{4^j}\right)\right| = \frac{1}{2^j}$ for all $m \in \mathbb{Z} \cap [0, 4^j - 1]$. Let $s \in \mathbb{Z}$ such that $0 \leq s \leq 4^{j+1} - 1$. We want to show that $\left|g_{j+1}\left(\frac{s}{4^{j+1}}\right) - g_{j+1}\left(\frac{s+1}{4^{j+1}}\right)\right| = \frac{1}{2^{j+1}}$.

Case 1. $0 \leq \frac{s}{4^{j+1}} \leq \frac{1}{4} - \frac{1}{4^{j+1}}$.

Since $0 \leq \frac{s}{4^{j+1}} \leq \frac{1}{4} - \frac{1}{4^{j+1}}$, then $0 \leq s \leq 4^j - 1$. Thus, by the inductive hypothesis, $\left|g_j\left(\frac{s}{4^j}\right) - g_j\left(\frac{s+1}{4^j}\right)\right| = \frac{1}{2^j}$. Then using (3), we show that

$$\begin{aligned}\left|g_{j+1}\left(\frac{s}{4^{j+1}}\right) - g_{j+1}\left(\frac{s+1}{4^{j+1}}\right)\right| &= \left|\frac{-1}{2}g_j\left(\frac{s}{4^j}\right) - \frac{-1}{2}g_j\left(\frac{s+1}{4^j}\right)\right| \\ &= \frac{1}{2}\left|g_j\left(\frac{s}{4^j}\right) - g_j\left(\frac{s+1}{4^j}\right)\right| \\ &= \frac{1}{2}\left(\frac{1}{2^j}\right) = \frac{1}{2^{j+1}}.\end{aligned}$$

Case 2. $\frac{1}{4} - \frac{1}{4^{j+1}} < \frac{s}{4^{j+1}} \le \frac{1}{2} - \frac{1}{4^{j+1}}$.

Since $\frac{1}{4} - \frac{1}{4^{j+1}} < \frac{s}{4^{j+1}} \le \frac{1}{2} - \frac{1}{4^{j+1}}$ and $s \,\epsilon\, \mathbb{Z}$, then $\frac{1}{4} \le \frac{s}{4^{j+1}} \le \frac{1}{2} - \frac{1}{4^{j+1}}$. Thus $0 \le s - 4^j \le 4^j - 1$. Then by the inductive hypothesis and (3),

$$\begin{aligned}
\left| g_{j+1}\left(\frac{s}{4^{j+1}}\right) - g_{j+1}\left(\frac{s+1}{4^{j+1}}\right) \right| &= \left| \frac{1}{2} g_j\left(\frac{s}{4^j} - 1\right) - \frac{1}{2} - \frac{1}{2} g_j\left(\frac{s+1}{4^j} + 1\right) + \frac{1}{2} \right| \\
&= \frac{1}{2} \left| g_j\left(\frac{s-4^j}{4^j}\right) - g_j\left(\frac{s-4^j+1}{4^j}\right) \right| \\
&= \frac{1}{2}\left(\frac{1}{2^j}\right) = \frac{1}{2^{j+1}}.
\end{aligned}$$

The other two cases are similar and are left to the reader. □

Now that we have established some of the properties of the functions whose limit is Kiesswetter's function, we will now show that Kiesswetter's function is continuous on $[0,1]$. We first show that the sequence of functions $(g_n)_{n \,\epsilon\, \mathbb{N} \cup \{0\}}$ is a sequence of continuous functions on $[0,1]$ and that the sequence converges uniformly on $[0,1]$.

PROPOSITION 1.2.6. *Let $g_0(t) = t$ on $[0,1]$ and let $g_n(t)$ be defined on $[0,1]$ by (3) for all $n \,\epsilon\, \mathbb{N}$. Then the function $g_n(t)$ is continuous on $[0,1]$ for all $n \,\epsilon\, \mathbb{N} \cup \{0\}$.*

PROOF. By the definition of Kiesswetter's curve, $g_0(t)$ is continuous on $[0,1]$. Let $k \,\epsilon\, \mathbb{N} \cup \{0\}$ and assume that $g_k(t)$ is continuous on $[0,1]$. Then, by (3), $g_{k+1}(t)$ is continuous on $\left[0, \frac{1}{4}\right) \cup \left(\frac{1}{4}, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{3}{4}\right) \cup \left(\frac{3}{4}, 1\right]$. Thus to show that $g_{k+1}(t)$ is continuous on $[0,1]$, we need only show that $g_{k+1}(t)$ is continuous at $t = \frac{1}{4}$, $t = \frac{1}{2}$, and $t = \frac{3}{4}$. We will show that $g_{k+1}(t)$ is continuous at $t = \frac{1}{4}$. The other proofs are similar and are left to the reader.

Since $g_k(t)$ is continuous on $[0,1]$ and $g_{k+1}(t) = \frac{1}{2} g_k(4t)$ for $0 \le t \le \frac{1}{4}$, thus $\lim_{t \to \frac{1}{4}^-} g_{k+1}(t) = g_{k+1}(\frac{1}{4})$. So we only need to show that the limit $\lim_{t \to \frac{1}{4}^-} g_{k+1}(t) = \lim_{t \to \frac{1}{4}^+} g_{k+1}(t)$. Since

$$\lim_{t \to \frac{1}{4}^-} g_{k+1}(t) = \lim_{t \to \frac{1}{4}^-} \frac{-1}{2} g_k(4t) = \lim_{t \to 1^-} \frac{-1}{2} g_k(t) = -\frac{1}{2}$$

and

$$\lim_{t\to\frac{1}{4}^+} g_{k+1}(t) = \lim_{t\to\frac{1}{4}^+} \frac{1}{2}g_k(4t-1) - \frac{1}{2} = \lim_{t\to 0^+} \frac{1}{2}g_k(t) - \frac{1}{2} = -\frac{1}{2},$$

then $\lim_{t\to\frac{1}{4}^-} g_{k+1}(t) = \lim_{t\to\frac{1}{4}^+} g_{k+1}(t)$. Therefore, $g_{k+1}(t)$ is continuous at $t = \frac{1}{4}$. □

PROPOSITION 1.2.7. *Let $g_0(t) = t$ on $[0,1]$ and let $g_n(t)$ be defined on $[0,1]$ by (3) for all $n \in \mathbb{N}$. Then the sequence $\{g_n\}_{n\in\mathbb{N}\cup\{0\}}$ converges uniformly on $[0,1]$.*

PROOF. Fix $\varepsilon > 0$. Pick $N \in \mathbb{N}$ such that $2^{N+1} > \frac{1}{\varepsilon}$. Let $j, n \in \mathbb{N}$ such that $j > n \geq N$. Then if $x \in [0,1]$, there exists $m \in \mathbb{Z} \cap [0, 4^n - 1]$ such that $\frac{m}{4^n} \leq x \leq \frac{m+1}{4^n}$. Since $j > n$, then $g_j\left(\frac{m}{4^n}\right) = g_n\left(\frac{m}{4^n}\right)$ by Proposition 1.2.4. Then

$$\begin{aligned} |g_j(x) - g_n(x)| &= \left|g_j(x) - g_j\left(\frac{m}{4^n}\right) + g_n\left(\frac{m}{4^n}\right) - g_n(x)\right| \\ &\leq \left|g_j(x) - g_j\left(\frac{m}{4^n}\right)\right| + \left|g_n(x) - g_n\left(\frac{m}{4^n}\right)\right| \\ &= \left|g_j(x) - g_j\left(\frac{4^{j-n}m}{4^j}\right)\right| + \left|g_n(x) - g_n\left(\frac{m}{4^n}\right)\right|. \end{aligned}$$

Since $\frac{m}{4^n} \leq x \leq \frac{m+1}{4^n}$ and g_n is linear on $[\frac{m}{4^n}, \frac{m+1}{4^n}]$, then, by Proposition 1.2.5,

$$\left|g_n(x) - g_n\left(\frac{m}{4^n}\right)\right| \leq \left|g_n\left(\frac{m}{4^n}\right) - g_n\left(\frac{m+1}{4^n}\right)\right| \leq \frac{1}{2^n}.$$

Since $\frac{4^{j-n}m}{4^j} \leq x \leq \frac{4^{j-n}m+4^{j-n}}{4^j}$, then there exists $s \in \mathbb{Z} \cap [0, 4^{j-n} - 1]$ so that $\frac{4^{j-n}m+s}{4^j} \leq x \leq \frac{4^{j-n}m+s+1}{4^j}$. Since $g_j(x)$ is linear on the interval $[\frac{4^{j-n}m+s}{4^j}, \frac{4^{j-n}m+s+1}{4^j}]$ and $x \in [\frac{4^{j-n}m+s}{4^j}, \frac{4^{j-n}m+s}{4^j}]$, then, by Proposition 1.2.5,

$$\left|g_j(x) - g_j\left(\frac{4^{j-n}m}{4^j}\right)\right| \leq \left|g_j\left(\frac{4^{j-n}m+s}{4^j}\right) - g_j\left(\frac{4^{j-n}m+s+1}{4^j}\right)\right| \leq \frac{1}{2^j}.$$

Thus,

$$\begin{aligned} |g_j(x) - g_n(x)| &\leq \left|g_j(x) - g_j\left(\frac{4^{j-n}m}{4^j}\right)\right| + \left|g_n(x) - g_n\left(\frac{m}{4^n}\right)\right| \\ &\leq \frac{1}{2^j} + \frac{1}{2^n} < \frac{1}{2^{n-1}} < \varepsilon. \end{aligned}$$

□

We recall that for all $n \in \mathbb{N} \cup \{0\}$ the function $g_n(t)$ is the function whose graph is L_n. Since the graph of Kiesswetter's curve is the limit of the sequence $(L_n)_{n \in \mathbb{N} \cup \{0\}}$, we can define Kiesswetter's function, $g : [0,1] \to \mathbb{R}$, by $g(t) = \lim_{n\to\infty} g_n(t)$ since the sequence $\{g_n\}_{n \in \mathbb{N}\cup\{0\}}$ converges. Now we can show that Kiesswetter's function is continuous and nowhere differentiable on $[0,1]$.

PROPOSITION 1.2.8. *The function $g(t) = \lim_{n\to\infty} g_n(t)$, where $g_0(t) = t$ and $g_n(t)$ is defined by (3) for all $n \in \mathbb{N}$, is continuous on* $[0,1]$.

PROOF. By Proposition 1.2.6, $g_n(x)$ is continuous on $[0,1]$ for all $n \in \mathbb{N} \cap \{0\}$. Then by Proposition 1.2.7, the sequence $\{g_n\}_{n \in \mathbb{N}\cup\{0\}}$ converges uniformly on $[0,1]$ to g. Thus, $g(x)$ is continuous on $[0,1]$ by Fact 1.1.7. $\square$

PROPOSITION 1.2.9. *The function $g(t) = \lim_{n\to\infty} g_n(t)$, where $g_0(t) = t$ and $g_n(t)$ is defined by (3) for all $n \in \mathbb{N}$, is nowhere differentiable on* $[0,1]$.

PROOF. Let $c \in [0,1]$. We want to show that $g(x)$ is not differentiable at c.

Case 1. $c = 1$.

Let $(u_m)_{m \in \mathbb{N}}$ and $(v_m)_{m \in \mathbb{N}}$ be sequences such that $u_m = \frac{4^m - 1}{4^m}$ for all $m \in \mathbb{N}$ and $v_m = \frac{4^m}{4^m} = 1$ for all $m \in \mathbb{N}$. Clearly, $\lim_{m\to\infty} v_m - u_m = 0$. Then by Propositions 1.2.4 and 1.2.5,

$$\begin{aligned}
\left|\frac{g(v_m) - g(u_m)}{v_m - u_m}\right| &= \left|\frac{\lim_{n\to\infty} g_n\left(\frac{4^m}{4^m}\right) - \lim_{n\to\infty} g_n\left(\frac{4^m-1}{4^m}\right)}{\frac{1}{4^m}}\right| \\
&= 4^m \left| g_m\left(\frac{4^m}{4^m}\right) - g_m\left(\frac{4^m - 1}{4^m}\right)\right| \\
&= 4^m \frac{1}{2^m} = 2^m.
\end{aligned}$$

Then $\left|\frac{g(v_m)-g(u_m)}{v_m-u_m}\right| \to \infty$ as $m \to \infty$ so the sequence $\left(\frac{g(v_m)-g(u_m)}{v_m-u_m}\right)_{m \in \mathbb{N}}$ does not converge. Thus, by Lemma 1.1.9, $g(x)$ is not differentiable at $x = c$.

Case 2. $c \in [0,1)$.

Since $c \,\epsilon\, [0,1)$, then for all $m \,\epsilon\, \mathbb{N}$ there exists $k_m \,\epsilon\, \mathbb{Z} \cap [0, 4^m - 1]$ such that $\frac{k_m}{4^m} \leq c < \frac{k_m+1}{4^m}$. Let $u_m = \frac{k_m}{4^m}$ for all $m \,\epsilon\, \mathbb{N}$ and let $v_m = \frac{k_m+1}{4^m}$ for all $m \,\epsilon\, \mathbb{N}$. Then $(u_m)_{m \,\epsilon\, \mathbb{N}}$ is an increasing sequence and $(v_m)_{m \,\epsilon\, \mathbb{N}}$ is a decreasing sequence. Clearly, $\lim_{m\to\infty} v_m - u_m = \lim_{m\to\infty} \frac{1}{4^m} = 0$.

Then by Propositions 1.2.4 and 1.2.5,

$$\begin{aligned}
\left|\frac{g(v_m) - g(u_m)}{v_m - u_m}\right| &= \left|\frac{\lim_{n\to\infty} g_n\left(\frac{k_m+1}{4^m}\right) - \lim_{n\to\infty} g_n\left(\frac{k_m}{4^m}\right)}{\frac{1}{4^m}}\right| \\
&= 4^m \left| g_m\left(\frac{k_m+1}{4^m}\right) - g_m\left(\frac{k_m}{4^m}\right)\right| \\
&= 4^m \frac{1}{2^m} = 2^m.
\end{aligned}$$

Since $\lim_{m\to\infty} \left|\frac{g(v_m)-g(u_m)}{v_m-u_m}\right| = \lim_{m\to\infty} 2^m = \infty$, the sequence $\left(\frac{g(v_m)-g(u_m)}{v_m-u_m}\right)_{m \,\epsilon\, \mathbb{N}}$ does not converge. Therefore, $g(x)$ is not differentiable at $x = c$ by Lemma 1.1.9. $\square$

Chapter 2

Hölder Continuity

A function $f : [0,1] \to \mathbb{R}$ is said to be a Hölder continuous function of exponent α if, for some constant M,

$$|f(x) - f(y)| \leq M|x - y|^{\alpha}$$

for all $x, y \,\epsilon\, [0,1]$. We note that if f is Hölder continuous of exponent 1, then f is Lipschitz. The maximal value of α such that f is a Hölder continuous function of exponent α is related to the existence of the derivative of f on $[0,1]$. We will investigate the Hölder continuity of the examples given in Chapter 1. Since the Generalized van der Waerden-Takagi function and Kiesswetter's function are nowhere differentiable on $[0,1]$, we know that the two functions are not Hölder continuous of exponent α for $\alpha \geq 1$. Lipschitz functions are differentiable almost everywhere [8, pages 108–112], and functions that are Hölder continuous of exponent $\alpha > 1$ have zero as a derivative everywhere. Thus to fully determine the exponents, α, for which the Generalized van der Waerden–Takagi function and Kiesswetter's function are Hölder continuous, we need only consider $\alpha < 1$.

2.1. The Generalized van der Waerden–Takagi Function

We first consider the Generalized van der Waerden–Takagi function. In order to be able to consider $\left|\sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n} - \sum_{n=0}^{\infty} \frac{a_0(b^n y)}{c^n}\right|$ for $x, y \,\epsilon\, [0,1]$, we need to obtain a bound for $\left|\frac{a_0(b^n x)}{c^n} - \frac{a_0(b^n y)}{c^n}\right|$ for $x, y \,\epsilon\, [0,1]$ and $n \,\epsilon\, \mathbb{N} \cup \{0\}$.

Proposition 2.1.1. *For all* $x, y \in [0,1]$ *and* $n \in \mathbb{N} \cup \{0\}$,

$$\left|\frac{a_0\left(b^n x\right)}{c^n}-\frac{a_0\left(b^n y\right)}{c^n}\right| \leq\left(\frac{b}{c}\right)^n|x-y|.$$

Proof. Let $n \in \mathbb{N} \cup \{0\}$ and $x, y \in [0,1]$. Let $k \in \mathbb{Z}$ such that k is the closest integer to $b^n y$. Then

$$a_0\left(b^n x\right) \leq\left|b^n x-k\right| \leq\left|b^n x-b^n y\right|+\left|b^n y-k\right|=\left|b^n x-b^n y\right|+a_0\left(b^n y\right).$$

Thus,

$$a_0\left(b^n x\right)-a_0\left(b^n y\right) \leq\left|b^n x-b^n y\right|.$$

By a similar argument, it can be shown that

$$a_0\left(b^n y\right)-a_0\left(b^n x\right) \leq\left|b^n y-b^n x\right|.$$

Thus,

$$\left|a_0\left(b^n x\right)-a_0\left(b^n y\right)\right| \leq b^n|x-y|.$$

Therefore,

$$\left|\frac{a_0\left(b^n x\right)}{c^n}-\frac{a_0\left(b^n y\right)}{c^n}\right| \leq\left(\frac{b}{c}\right)^n|x-y|.$$

□

We first consider the Hölder continuity of the Generalized van der Waerden–Takagi function with $b = c$. The following proof is along the lines of the proof given by Shidfar and Sabetfakhri in [9].

Theorem 2.1.2. *If* $c \in \mathbb{N}$ *such that* $c \geq 2$, *then* $f(x) = \sum_{n=0}^{\infty} \frac{a_0(c^n x)}{c^n}$ *is Hölder continuous of exponent* α *on* $[0,1]$ *if and only if* $\alpha < 1$.

Proof. Let $\alpha < 1$. Let x and $y \in [0,1]$ such that $x \neq y$. Thus $0 < |x-y| \leq 1$. Then there exists $k \in \mathbb{Z}$ such that $k \geq 0$ and $\frac{1}{c^{k+1}} < |x-y| \leq \frac{1}{c^k}$.

Then by Proposition 2.1.1,

$$\sum_{n=0}^{k-1}\left|\frac{a_0(c^n x)}{c^n}-\frac{a_0(c^n y)}{c^n}\right| \leq \sum_{n=0}^{k-1}\left(\frac{c}{c}\right)^n|x-y|=k|x-y| \leq k \frac{1}{c^{k(1-\alpha)}}|x-y|^\alpha. \tag{4}$$

Since $0 \leq \frac{a_0(c^n t)}{c^n} \leq \frac{1}{c^n}$ for all $n \,\epsilon\, \mathbb{N} \cup \{0\}$ and $t \,\epsilon\, [0,1]$, then

$$\left| \frac{a_0(c^n x)}{c^n} - \frac{a_0(c^n y)}{c^n} \right| \leq \frac{1}{c^n}$$

for all $n \,\epsilon\, \mathbb{N} \cup \{0\}$. Thus

$$\sum_{n=k}^{\infty} \left| \frac{a_0(c^n x)}{c^n} - \frac{a_0(c^n y)}{c^n} \right| \leq \sum_{n=k}^{\infty} \frac{1}{c^n} \leq \frac{1}{c^{k-1}} \leq c^2 |x-y| \leq c^2 |x-y|^\alpha \tag{5}$$

since $\frac{1}{c^{k+1}} < |x-y| \leq 1$.

Then combining (4) and (5), we obtain

$$\sum_{n=0}^{\infty} \left| \frac{a_0(c^n x)}{c^n} - \frac{a_0(c^n y)}{c^n} \right| \leq k \frac{1}{c^{k(1-\alpha)}} |x-y|^\alpha + c^2 |x-y|^\alpha = \left(c^2 + \frac{k}{c^{k(1-\alpha)}} \right) |x-y|^\alpha.$$

Since $\frac{k}{c^{k(1-\alpha)}} = \left(\frac{k^{\left(\frac{1}{1-\alpha}\right)}}{c^k} \right)^{(1-\alpha)}$ for all $k \,\epsilon\, \mathbb{N} \cup \{0\}$ and $c > 1$, $\lim_{k \to \infty} \frac{k}{c^{k(1-\alpha)}} = 0$. Thus, $\lim_{k \to \infty} c^2 + \frac{k}{c^{k(1-\alpha)}} = c^2$. Then the sequence $\left(c^2 + \frac{k}{c^{k(1-\alpha)}} \right)_{k \,\epsilon\, \mathbb{N} \cup \{0\}}$ is bounded, and there exists $M_\alpha > 0$ such that $\left| c^2 + \frac{k}{c^{k(1-\alpha)}} \right| \leq M_\alpha$ for all $k \,\epsilon\, \mathbb{N} \cup \{0\}$.

Then

$$\sum_{n=0}^{\infty} \left| \frac{a_0(c^n x)}{c^n} - \frac{a_0(c^n y)}{c^n} \right| \leq M_\alpha |x-y|^\alpha.$$

Thus, f is Hölder continuous of exponent α for all $\alpha < 1$. To prove the other direction of the theorem, we recall that f is nowhere differentiable on $[0,1]$ by Proposition 1.1.10. Therefore, f is not Hölder continuous of order α for all $\alpha \geq 1$. □

We now consider the Hölder continuity of the Generalized van der Waerden-Takagi function with $b > c$.

Proposition 2.1.3. *If $c > 1$ and $b \,\epsilon\, \mathbb{N}$ such that $b > c$, then $f(x) = \sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$ is Hölder continuous of exponent α on $[0,1]$ for all $\alpha \leq \frac{\log c}{\log b}$.*

Proof. Let $\alpha \leq \frac{\log c}{\log b} < 1$. Let x and $y \,\epsilon\, [0,1]$ such that $x \neq y$. Thus $0 < |x-y| \leq 1$. Then there exists $k \,\epsilon\, \mathbb{Z}$ such that $k \geq 0$ and $\frac{1}{b^{k+1}} < |x-y| \leq \frac{1}{b^k}$. Thus by Proposition

2.1.1,

$$\begin{aligned}
\sum_{n=0}^{k-1}\left|\frac{a_0(b^n x)}{c^n}-\frac{a_0(b^n y)}{c^n}\right| &\le \sum_{n=0}^{k-1}\left(\frac{b}{c}\right)^n |x-y| \\
&\le \left[\frac{1}{b^{k(1-\alpha)}}\sum_{n=0}^{k-1}\left(\frac{b}{c}\right)^n\right]|x-y|^\alpha \\
&= \left[\left(\frac{\left(\frac{b}{c}\right)^k-1}{\left(\frac{b}{c}\right)-1}\right)\frac{1}{b^{k(1-\alpha)}}\right]|x-y|^\alpha \\
&= \left[\frac{c}{b-c}\left(\left(\frac{b^\alpha}{c}\right)^k-\frac{1}{b^{k(1-\alpha)}}\right)\right]|x-y|^\alpha .
\end{aligned}$$

Since $0 \le \frac{a_0(b^n t)}{c^n} \le \frac{1}{c^n}$ for all $n \in \mathbb{N}\cup\{0\}$ and $t \in [0,1]$, then

$$\left|\frac{a_0(b^n x)}{c^n}-\frac{a_0(b^n y)}{c^n}\right| \le \frac{1}{c^n}.$$

for all $n \in \mathbb{N}\cup\{0\}$. Thus, by the same argument as in (5) in Proposition 2.1.2,

$$\sum_{n=k}^{\infty}\left|\frac{a_0(b^n x)}{c^n}-\frac{a_0(b^n y)}{c^n}\right| \le c^2|x-y|^\alpha .$$

Then

$$\sum_{n=0}^{\infty}\left|\frac{a_0(b^n x)}{c^n}-\frac{a_0(b^n y)}{c^n}\right| \le \left[\frac{c}{b-c}\left(\left(\frac{b^\alpha}{c}\right)^k-\frac{1}{b^{k(1-\alpha)}}\right)+c^2\right]|x-y|^\alpha .$$

Since $b>1$, $\lim_{k\to\infty}\frac{1}{b^{k(1-\alpha)}}=0$. Thus, the limit $\lim_{k\to\infty}\frac{c}{b-c}\left(\left(\frac{b^\alpha}{c}\right)^k-\frac{1}{b^{k(1-\alpha)}}\right)+c^2$ $=\lim_{k\to\infty}\frac{c}{b-c}\left(\frac{b^\alpha}{c}\right)^k+c^2$ exists if and only if $\lim_{k\to\infty}\left(\frac{b^\alpha}{c}\right)^k$ exists. Since $\alpha \le \frac{\log c}{\log b}$, then $\frac{b^\alpha}{c}\le 1$. Thus $\lim_{k\to\infty}\left(\frac{b^\alpha}{c}\right)^k$ is finite. The sequence $\left(\frac{c}{b-c}\left(\left(\frac{b^\alpha}{c}\right)^k-\frac{1}{b^{k(1-\alpha)}}\right)+c^2\right)_{k\in\mathbb{N}\cup\{0\}}$ is then bounded, and there exists $M_\alpha>0$ such that $\left|\frac{c}{b-c}\left(\left(\frac{b^\alpha}{c}\right)^k-\frac{1}{b^{k(1-\alpha)}}\right)+c^2\right|\le M_\alpha$ for all $k\in\mathbb{N}\cup\{0\}$.

Then

$$\sum_{n=0}^{\infty}\left|\frac{a_0(b^n x)}{c^n}-\frac{a_0(b^n y)}{c^n}\right| \le M_\alpha|x-y|^\alpha .$$

$\square$

PROPOSITION 2.1.4. *If $c>1$ and $b\in\mathbb{N}$ such that $b>c$, then $f(x)=\sum_{n=0}^{\infty}\frac{a_0(b^n x)}{c^n}$ is not Hölder continuous of exponent α on $[0,1]$ for $\alpha>\frac{\log c}{\log b}$.*

PROOF. Since b is an integer, $c > 1$, and $b > c$, f is a continuous nowhere differentiable function by Propositions 1.1.8 and 1.1.10. Thus, f is not Hölder continuous of exponent greater than or equal to 1.

To determine the Hölder continuity of f of exponent α for $\alpha < 1$, let $\frac{\log c}{\log b} < \alpha < 1$. Let $k \in \mathbb{N}$. Since $b \geq 2$, $\frac{1}{b^{k-n}} \leq \frac{1}{2}$ for all $n \in \mathbb{N} \cup \{0\}$ such that $n < k$. Thus,

$$\begin{aligned}
\left| f\left(\frac{1}{b^k}\right) - f(0) \right| = f\left(\frac{1}{b^k}\right) &= \sum_{n=0}^{\infty} \frac{a_0\left(b^n \frac{1}{b^k}\right)}{c^n} \\
&= \sum_{n=0}^{k-1} \frac{\left(\frac{b}{c}\right)^n}{b^k} \\
&= \frac{\sum_{n=0}^{k-1} \left(\frac{b}{c}\right)^n}{b^{k(1-\alpha)}} \left| \frac{1}{b^k} - 0 \right|^{\alpha} \\
&= \frac{\left(\left(\frac{b}{c}\right)^k - 1\right) c}{b^{k(1-\alpha)}(b-c)} \left| \frac{1}{b^k} - 0 \right|^{\alpha} \\
&= \frac{c}{b-c} \left(\left(\frac{b^{\alpha}}{c}\right)^k - \frac{1}{b^{k(1-\alpha)}} \right) \left| \frac{1}{b^k} - 0 \right|^{\alpha}.
\end{aligned}$$

Thus, $\lim_{k\to\infty} \frac{c}{b-c}\left(\left(\frac{b^{\alpha}}{c}\right)^k - \frac{1}{b^{k(1-\alpha)}}\right)$ must exist in order for f to be Hölder continuous of exponent α. However, since $\alpha > \frac{\log c}{\log b}$, $b^{\alpha} > c$ and $\left(\frac{b^{\alpha}}{c}\right)^k \to \infty$ as $k \to \infty$. Thus, $\frac{c}{b-c}\left(\left(\frac{b^{\alpha}}{c}\right)^k - \frac{1}{b^{k(1-\alpha)}}\right)$ is unbounded as $k \to \infty$ since $\lim_{k\to\infty} \frac{1}{b^{k(1-\alpha)}} = 0$. Therefore, f is not Hölder continuous of exponent α on $[0, 1]$. □

THEOREM 2.1.5. *Let $c > 1$ and $b \in \mathbb{N}$ such that $b > c$. Then $f(x) = \sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$ is Hölder continuous of exponent α on $[0, 1]$ if and only if $\alpha \leq \frac{\log c}{\log b}$.*

PROOF. This theorem is the immediate result of Propositions 2.1.3 and 2.1.4. □

COROLLARY 2.1.6. *For all $m \in \mathbb{N}$ such that $m \geq 2$, the function,*

$$f(x) = \sum_{n=0}^{\infty} \frac{a_0\left((c^m)^n x\right)}{c^n},$$

with $c > 1$ such that $c^m \in \mathbb{N}$, is Hölder continuous on $[0, 1]$ if and only if $\alpha \leq \frac{1}{m}$.

PROOF. Fix $m \in \mathbb{N} \setminus \{1\}$. Let $c > 1$ such that $c^m \in \mathbb{N}$. Since $m \geq 2$, $c^m > c$. Thus, f satisfies the hypotheses for Theorem 2.1.5. Therefore, f is Hölder continuous of exponent α on $[0,1]$ for $\alpha \leq \frac{\log c}{\log c^m} = \frac{1}{m}$, and f is not Hölder continuous of exponent α on $[0,1]$ for $\alpha > \frac{\log c}{\log c^m} = \frac{1}{m}$. □

2.2. Kiesswetter's Function

We now consider Kiesswetter's function.

PROPOSITION 2.2.1. *The function $g(t) = \lim_{n\to\infty} g_n(t)$, where $g_0(t) = t$ and $g_n(t)$ is defined by (3) for all $n \in \mathbb{N}$, is Hölder continuous of exponent α on $[0,1]$ for all $\alpha \leq \frac{1}{2}$.*

PROOF. Let $\alpha \leq \frac{1}{2}$. Let $x, y \in [0,1]$ so that $x \neq y$. Then there exists $k \in \mathbb{N} \cup \{0\}$ such that $\frac{1}{4^{k+1}} < |x - y| \leq \frac{1}{4^k}$.

Then by Proposition 1.2.7,

$$|g_j(x) - g_k(x)| \leq \frac{1}{2^k} + \frac{1}{2^j}$$

for all $j \in \mathbb{N}$ such that $j > k$. Thus,

$$|g(x) - g_k(x)| = \lim_{j\to\infty} |g_j(x) - g_k(x)| \leq \lim_{j\to\infty} \frac{1}{2^k} + \frac{1}{2^j} = \frac{1}{2^k}.$$

Similarly

$$|g(y) - g_k(y)| \leq \frac{1}{2^k}.$$

Then

$$\begin{aligned} |g(x) - g(y)| &= |g(x) - g_k(x) + g_k(x) - g_k(y) + g_k(y) - g(y)| \\ &\leq |g(x) - g_k(x)| + |g_k(x) - g_k(y)| + |g_k(y) - g(y)| \\ &\leq \frac{1}{2^k} + |g_k(x) - g_k(y)| + \frac{1}{2^k}. \end{aligned}$$

In order to bound $|g_k(x) - g_k(y)|$, we assume, without loss of generality, that $y > x$. Then there exists $m \in \mathbb{Z} \cap \left[0, 4^k - 1\right]$ such that $\frac{m}{4^k} \leq x < \frac{m+1}{4^k}$.

Case 1. $x < y \leq \frac{m+1}{4^k}$.

Then by Proposition 1.2.5,

$$|g_k(x) - g_k(y)| \leq \left| g_k\left(\frac{m}{4^k}\right) - g_k\left(\frac{m+1}{4^k}\right) \right| = \frac{1}{2^k}.$$

Case 2. $\frac{m+1}{4^k} < y < \frac{m+2}{4^k}$.

Then by Proposition 1.2.5,

$$\begin{aligned} |g_k(x) - g_k(y)| &\leq \left| g_k(x) - g_k\left(\frac{m+1}{4^k}\right) \right| + \left| g_k\left(\frac{m+1}{4^k}\right) - g_k(y) \right| \\ &\leq \left| g_k\left(\frac{m}{4^k}\right) - g_k\left(\frac{m+1}{4^k}\right) \right| + \left| g_k\left(\frac{m+1}{4^k}\right) - g_k\left(\frac{m+2}{4^k}\right) \right| \\ &= 2\left(\frac{1}{2^k}\right). \end{aligned}$$

So in both cases $|g_k(x) - g_k(y)| \leq 2\left(\frac{1}{2^k}\right)$. Thus

$$|g(x) - g(y)| \leq 4\left(\frac{1}{2^k}\right) = 8\left(\frac{1}{2^{k+1}}\right) < 8\,|x-y|^{\frac{1}{2}} \leq 8|x-y|^{\alpha}.$$

$\square$

PROPOSITION 2.2.2. *The function* $g(t) = \lim_{n\to\infty} g_n(t)$, *where* $g_0(t) = t$ *and* $g_n(t)$ *is defined by (3) for all* $n \in \mathbb{N}$, *is not Hölder continuous of exponent* α *on* $[0,1]$ *for all* $\alpha > \frac{1}{2}$.

PROOF. Let $\alpha > \frac{1}{2}$. Let $x_k = \frac{m}{4^k}$ and $y_k = \frac{m+1}{4^k}$ where $k \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{Z} \cap [0, 4^k - 1]$. Then $x_k, y_k \in [0,1]$ and $|x_k - y_k| = \frac{1}{4^k}$. By Propositions 1.2.4 and 1.2.5,

$$|g(x_k) - g(y_k)| = |g_k(x_k) - g_k(y_k)| = \frac{1}{2^k} = 4^{k\left(\alpha - \frac{1}{2}\right)}|x_k - y_k|^{\alpha}.$$

However $4^{k\left(\alpha-\frac{1}{2}\right)}$ is not bounded, so there does not exist a constant M such that $|g(x_k) - g(y_k)| \leq M|x_k - y_k|^{\alpha}$ for all $k \in \mathbb{N} \cup \{0\}$.

$\square$

THEOREM 2.2.3. *The function* $g(t) = \lim_{n\to\infty} g_n(t)$*, where* $g_0(t) = t$ *and* $g_n(t)$ *is defined by (3) for all* $n \in \mathbb{N}$*, is a Hölder continuous function of exponent* α *on* $[0,1]$ *if and only if* $\alpha \leq \frac{1}{2}$.

PROOF. This theorem is an immediate consequence of Propositions 2.2.1 and 2.2.2. $\square$

Chapter 3

Fractional Derivatives

Although continuous nowhere differentiable functions do not have first-order derivatives at any point in their domains, they do have some level of smoothness. We need a way to be able to measure this smoothness and to compare the smoothness of different examples of continuous nowhere differentiable functions. Fractional derivatives will provide us with such a measure of smoothness for these functions. We begin by defining fractional integrals and fractional derivatives.

If we denote the n-fold integral of a function f as $D^{-n}f$, then

$$D^{-1}f(t) = \int_0^t f(\xi)d\xi. \tag{6}$$

Proposition 3.0.1. *For all* $n \in \mathbb{N}$,

$$D^{-n}f(t) = \frac{1}{(n-1)!}\int_0^t (t-\xi)^{n-1}f(\xi)d\xi. \tag{7}$$

Proof. Let $n = 1$. Then by (6),

$$D^{-1}f(t) = \int_0^t f(\xi)d\xi = \int_0^t (t-\xi)^{1-1}f(\xi)d\xi.$$

Thus, (7) holds for $n = 1$. Now let $n = k \in \mathbb{N}$ and assume (7) holds. Then let $n = k+1$ and consider

$$\begin{aligned} D^{-(k+1)}f(t) &= \int_0^t D^{-k}f(x)dx \\ &= \int_0^t \frac{1}{(k-1)!}\int_0^x (x-\xi)^{k-1}f(\xi)d\xi dx \\ &= \frac{1}{(k-1)!}\int_0^t f(\xi)\int_\xi^t (x-\xi)^{k-1}dxd\xi \end{aligned}$$

$$= \frac{1}{(k-1)!}\int_0^t f(\xi)\left[\frac{(x-\xi)^k}{k}\right]_{x=\xi}^{x=t} d\xi$$
$$= \frac{1}{k!}\int_0^t f(\xi)(t-\xi)^k d\xi.$$

So (7) holds for $n = k+1$. Thus, by induction, (7) holds for all $n \in \mathbb{N}$. □

The definition of the n-fold integral can then be generalized to define fractional integrals of order $v > 0$ by

$$D^{-v}f(t) = \frac{1}{\Gamma(v)}\int_0^t (t-\xi)^{v-1} f(\xi)d\xi,$$

where $\Gamma(v)$ is the Gamma function. The fractional derivative of $f(t)$ of order $\mu > 0$, for t such that it exists, can be defined as

$$D^{\mu}f(t) = D^m\left[D^{-(m-\mu)}f(t)\right]$$

where $m \in \mathbb{N}$ such that $m \geq \lceil \mu \rceil$, where $\lceil x \rceil$ is the smallest integer greater than or equal to x. To show that the value of $D^{\mu}f(t)$ does not depend on the choice of m, provided $m \geq \lceil \mu \rceil$, we need the following facts from [4, pages 220–221] and [7, page 16].

FACT 3.0.2. *Let $a(t)$ and $b(t)$ be defined and have continuous derivatives for $t_1 < t < t_2$. Let $f(x,t)$ be continuous and have a continuous derivative $\frac{\partial f}{\partial t}$ in a domain of the $x-t$ plane which includes $\{(x,t) : t \in [t_1, t_2], x \in [a(t), b(t)]\}$. Then for $t_1 < t < t_2$,*

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = f(b(t),t)b'(t) - f(a(t),t)a'(t) + \int_{a(t)}^{b(t)}\left[\frac{\partial}{\partial t}f(x,t)\right]dx.$$

FACT 3.0.3. *Let $\Gamma(x)$ be the Gamma function. Then $\Gamma(x+1) = x\Gamma(x)$ for all $x \in \mathbb{R}$ such that x is not a negative integer.*

PROPOSITION 3.0.4. *Let $\mu > 0$. Then for all $m \in \mathbb{N}$ such that $m \geq \lceil \mu \rceil$,*

$$D^m\left[D^{-(m-\mu)}f(t)\right] = D^{\lceil \mu \rceil}\left[D^{-(\lceil \mu \rceil - \mu)}f(t)\right]. \tag{8}$$

Proof. Let $m = \lceil \mu \rceil$. Clearly, (8) holds for $m = \lceil \mu \rceil$. Now let $k \epsilon \mathbb{N}$ such that $k \geq \lceil \mu \rceil$. Let $m = k$ and assume (8) holds. Then let $m = k + 1$ and consider

$$D^{k+1}\left[D^{-(k+1-\mu)}f(t)\right].$$

By the inductive hypothesis and Facts 3.0.2 and 3.0.3,

$$\begin{aligned}
D^{k+1}\left[D^{-(k+1-\mu)}f(t)\right] &= D^{k+1}\left[\frac{1}{\Gamma(k+1-\mu)}\int_0^t (t-\xi)^{k-\mu}f(\xi)d\xi\right] \\
&= D^{k}\left[\frac{1}{\Gamma(k+1-\mu)}\frac{d}{dt}\int_0^t (t-\xi)^{k-\mu}f(\xi)d\xi\right] \\
&= D^{k}\left[\frac{1}{\Gamma(k+1-\mu)}\int_0^t (k-\mu)(t-\xi)^{k-\mu-1}f(\xi)d\xi\right] \\
&= D^{k}\left[\frac{1}{\Gamma(k-\mu)}\int_0^t (t-\xi)^{k-\mu-1}f(\xi)d\xi\right] \\
&= D^{k}\left[D^{-(k-\mu)}f(t)\right] \\
&= D^{\lceil \mu \rceil}\left[D^{-(\lceil \mu \rceil-\mu)}f(t)\right]
\end{aligned}$$

since $k \geq \lceil \mu \rceil$. So (8) holds for $m = k + 1$. Thus, (8) holds for all $m \epsilon \mathbb{N}$ such that $m \geq \lceil \mu \rceil$. □

To illustrate that fractional derivatives are indeed different from classical derivatives, we now calculate the fractional derivative of order $\frac{1}{2}$ of $f(t) = c$, where c is a constant.

Since $\lceil \frac{1}{2} \rceil = 1$, we write the fractional derivative as

$$D^{\frac{1}{2}}c = D^1\left[D^{\frac{-1}{2}}c\right]$$

and show that

$$D^{\frac{-1}{2}}c = \frac{1}{\Gamma(\frac{1}{2})}\int_0^t (t-\xi)^{\frac{-1}{2}}cd\xi = \frac{2c\sqrt{t}}{\sqrt{\pi}}.$$

Then the fractional derivative of f of order $\frac{1}{2}$ for $t \neq 0$ is

$$D^{\frac{1}{2}}c = D^1\left(\frac{2c}{\sqrt{\pi}}\sqrt{t}\right) = \frac{c}{\sqrt{\pi t}}, \tag{9}$$

which is not equal to zero provided $c \neq 0$. This indicates the difference between fractional derivatives and classical derivatives since all integer order derivatives of the constant function are 0.

Chapter 4

Hölder Continuity and Fractional Derivatives

Although fractional derivatives are very useful in determining the smoothness of functions that are not first–order differentiable, they can be very difficult to calculate directly. However, since we are only using fractional derivatives as a measure of smoothness, we are only concerned with the existence of fractional derivatives of a specific order and not of the value of the quantity. Thus, we only need a way to determine if a fractional derivative of a particular order exists for a function. To do this, we use the connections between Hölder continuity and fractional derivatives. We want to show that a function, $f^*(x)$, that is Hölder continuous of exponent $k \leq 1$ on $[0,1]$ has fractional derivatives of order β, where $0 < \beta < k$, at all points in $[0,1]$. However, this fact is not true without a normalization of the function. The constant function, $g(t) = c$, is Hölder continuous of exponent k for all $k \geq 0$, but, as we showed in (9), g does not have a fractional derivative of order $\frac{1}{2}$ at $t = 0$ unless $c = 0$. Thus, we must normalize f^* so that $f^*(0) = 0$. In order to obtain the uniform convergence necessary to complete the proof, we must also extend the function f^* on $[0,1]$ to the function f on $(-\infty, 1]$, where $f(x) = f^*(x)$ on $(0,1]$ and $f(x) = f^*(0)$ on $(-\infty, 0]$.

Proposition 4.0.1. *Let $\alpha \geq 0$. If f^* is a Hölder function for the exponent α on $[0,1]$, then the extension of f^* to $(-\infty, 1]$ defined by*

$$f(x) = \begin{cases} f^*(x), & 0 < x \leq 1 \\ f^*(0), & x \leq 0 \end{cases}$$

is Hölder continuous for α on $(-\infty, 1]$.

PROOF. Since f^* is Hölder continuous for α, there exists $M > 0$ such that for all $x,\ y \,\epsilon\, [0,1]$

$$|f^*(x) - f^*(y)| \leq M|x-y|^\alpha.$$

Let $x,\ y \,\epsilon\, (-\infty, 1]$.

Case 1. $x, y \,\epsilon\, (-\infty, 0]$.

Then $|f(x) - f(y)| = |f^*(0) - f^*(0)| = 0 \leq M|x-y|^\alpha$.

Case 2. $x, y \,\epsilon\, (0,1]$.

Then $|f(x) - f(y)| = |f^*(x) - f^*(y)| \leq M|x-y|^\alpha$.

Case 3. $x \,\epsilon\, (0,1]$, $y \,\epsilon\, (-\infty, 0]$.

Then $|f(x) - f(y)| = |f^*(x) - f^*(0)| \leq M|x-0|^\alpha \leq M|x-y|^\alpha$.

Case 4. $x \,\epsilon\, (-\infty, 0]$, $y \,\epsilon\, (0,1]$.

If we substitute $y = x$ and $x = y$ in Case 3, we find that $|f(x) - f(y)| \leq M|x-y|^\alpha$.

Thus, $|f(x) - f(y)| \leq M|x-y|^\alpha$ for all $x, y \,\epsilon\, (-\infty, 1]$. □

Using Proposition 4.0.1 and the ideas of Hardy and Littlewood [3], we are now able to establish the connection between Hölder continuity and fractional derivatives that will allow us to consider the smoothness of the Generalized van der Waerden–Takagi function and Kiesswetter's function without directly calculating fractional derivatives.

PROPOSITION 4.0.2. *Let $0 < \beta < k \leq 1$. Let f^* be a Hölder continuous function of exponent k on $[0,1]$ such that $f^*(0) = 0$. If $f : (-\infty, 1] \to \mathbb{R}$ is the extension of f^* defined by*

$$f(x) = \begin{cases} f^*(x), & 0 < x \leq 1 \\ f^*(0) = 0, & x \leq 0 \end{cases},$$

then $D^\beta f(x)$ exists for all $x \,\epsilon\, [0,1]$.

PROOF. For $\varepsilon > 0$ and $x \,\epsilon\, [0,1]$, we define

$$f_{1-\beta,\varepsilon}(x) = \frac{1}{\Gamma(1-\beta)} \int_0^{x-\varepsilon} f(t)(x-t)^{-\beta} dt.$$

CLAIM 4.0.3. *For* $x \in [0,1]$, *the limit* $\lim_{\varepsilon \to 0^+} \int_0^{x-\varepsilon} f(t)(x-t)^{-\beta} dt$ *exists.*

PROOF. Fix $\lambda > 0$. Since f is continuous on $(-\infty, 1]$ and constant on $(-\infty, 0]$, f is bounded. Thus, there exists $R > 0$ such that $|f(t)| \leq R$ for all $t \in (-\infty, 1]$. Let $(\varepsilon_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers that converges to 0. Then there exists $N \in \mathbb{N}$ such that if $n \in \mathbb{N}$ so that $n > N$, then $\varepsilon_n < \left(\frac{\lambda(1-\beta)}{2R}\right)^{\frac{1}{1-\beta}}$.

Let $n, m \in \mathbb{N}$ such that $n > m > N$ and let $x \in [0,1]$. Then since $\beta - 1 < 0$

$$\begin{aligned}
\left|\int_{x-\varepsilon_n}^{x-\varepsilon_m} f(t)(x-t)^{-\beta} dt\right| &\leq \left|\int_{x-\varepsilon_n}^{x-\varepsilon_m} |f(t)|(x-t)^{-\beta} dt\right| \\
&\leq \left|\int_{x-\varepsilon_n}^{x-\varepsilon_m} R(x-t)^{-\beta} dt\right| \\
&= \left|\frac{R(x-t)^{-\beta+1}}{\beta - 1}\right|_{t=x-\varepsilon_n}^{t=x-\varepsilon_m} \\
&= \left|\frac{R(\varepsilon_m)^{-\beta+1}}{\beta-1} - \frac{R(\varepsilon_n)^{-\beta+1}}{\beta-1}\right| \\
&\leq \frac{R(\varepsilon_m)^{-\beta+1}}{1-\beta} + \frac{R(\varepsilon_n)^{-\beta+1}}{1-\beta} \\
&< \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda.
\end{aligned}$$

Thus, $\left(\int_0^{x-\varepsilon_n} f(t)(x-t)^{-\beta} dt\right)_{n \in \mathbb{N}}$ is a Cauchy sequence and is uniformly convergent. Therefore, the uniform limit $\lim_{\varepsilon \to 0^+} \int_0^{x-\varepsilon} f(t)(x-t)^{-\beta} dt$ exists for $x \in [0,1]$ and is denoted $\int_0^x f(t)(x-t)^{-\beta} dt$. $\square$

Thus, $f_{1-\beta}$ can be defined on $[0,1]$ as the limit of $\frac{1}{\Gamma(1-\beta)} \int_0^{x-\varepsilon} f(t)(x-t)^{-\beta} dt$ as $\varepsilon \to 0^+$ which can be denoted as $f_{1-\beta}(x) = \frac{1}{\Gamma(1-\beta)} \int_0^x f(t)(x-t)^{-\beta} dt$. We note that, by the definition of the fractional derivative of order β, $f'_{1-\beta} = D^\beta f$.

Then for $\varepsilon > 0$

$$\Gamma(1-\beta) f'_{1-\beta,\varepsilon}(x) = f(x-\varepsilon)\varepsilon^{-\beta} - \beta \int_0^{x-\varepsilon} f(t)(x-t)^{-\beta-1} dt$$

for $x \in [0,1]$ by Fact 3.0.2. If $x \in (0,1]$, then

$$\Gamma(1-\beta)f'_{1-\beta,\varepsilon}(x) = \beta \int_0^{x-\varepsilon} (f(x)-f(t))(x-t)^{-\beta-1}dt - \varepsilon^{-\beta}(f(x)-f(x-\varepsilon)) + f(x)x^{-\beta}.$$

Since f is a Hölder continuous function of exponent k on $(-\infty,1]$, there exists $M>0$ so that

$$|f(x)-f(t)| \leq M|x-t|^k$$

for all $x,t \in (-\infty,1]$.

Claim 4.0.4. *As $\varepsilon \to 0^+$, $\varepsilon^{-\beta}(f(x)-f(x-\varepsilon))$ converges uniformly to 0 on $[0,1]$.*

Proof. Fix $\lambda > 0$. Let $\delta < \left(\frac{\lambda}{M}\right)^{\frac{1}{k-\beta}}$. Let $x \in [0,1]$. If $0<\varepsilon<\delta$, then

$$\left|\frac{f(x)-f(x-\varepsilon)}{\varepsilon^\beta} - 0\right| \leq \frac{M|\varepsilon|^k}{\varepsilon^\beta} = M\varepsilon^{k-\beta} < M\delta^{k-\beta} = \lambda.$$

□

Claim 4.0.5. *The limit $\lim_{\varepsilon\to 0^+}\int_0^{x-\varepsilon}(f(x)-f(t))(x-t)^{-\beta-1}dt$ exists for $x \in [0,1]$, and the convergence is uniform.*

Proof. Fix $\lambda>0$. Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers that converges to 0. Then there exists $N \in \mathbb{N}$ such that if $n \in \mathbb{N}$ so that $n > N$, then $\varepsilon_n < \left(\frac{\lambda(k-\beta)}{2M}\right)^{\frac{1}{k-\beta}}$.

Let $n,m \in \mathbb{N}$ such that $n>m>N$. Let $x \in [0,1]$. Then

$$\begin{aligned}
\left|\int_{x-\varepsilon_n}^{x-\varepsilon_m}(f(x)-f(t))(x-t)^{-\beta-1}dt\right| &\leq \left|\int_{x-\varepsilon_n}^{x-\varepsilon_m}|f(x)-f(t)||x-t|^{-\beta-1}dt\right| \\
&\leq \left|\int_{x-\varepsilon_n}^{x-\varepsilon_m}M|x-t|^{k-\beta-1}dt\right| \\
&= \left|\frac{-M}{k-\beta}(x-t)^{k-\beta}\right|_{t=x-\varepsilon_n}^{t=x-\varepsilon_m} \\
&= \frac{M}{k-\beta}\left|(\varepsilon_m)^{k-\beta}-(\varepsilon_n)^{k-\beta}\right| \\
&\leq \frac{M}{k-\beta}\left(\left|(\varepsilon_m)^{k-\beta}\right| + \left|(\varepsilon_n)^{k-\beta}\right|\right) \\
&< \frac{M}{k-\beta}\left(\frac{\lambda(k-\beta)}{2M}+\frac{\lambda(k-\beta)}{2M}\right) = \lambda.
\end{aligned}$$

So $\left(\int_0^{x-\varepsilon_n}(f(x)-f(t))(x-t)^{-\beta-1}dt\right)_{n\,\epsilon\,\mathbb{N}}$ is a Cauchy sequence and is uniformly convergent. Thus, the uniform limit $\lim_{\varepsilon\to 0^+}\int_0^{x-\varepsilon}(f(x)-f(t))(x-t)^{-\beta-1}dt$ exists and is denoted $\int_0^x(f(x)-f(t))(x-t)^{-\beta-1}dt$. $\square$

Thus, for $x\,\epsilon\,(0,1]$, we define

$$g(x)=\beta\int_0^x(f(x)-f(t))(x-t)^{-\beta-1}dt+f(x)x^{-\beta}$$

and note that g is the uniform limit of $\Gamma(1-\beta)f'_{1-\beta,\varepsilon}$ on $(0,1]$. Since for $\varepsilon>0$,

$$\begin{aligned}\Gamma(1-\beta)f'_{1-\beta,\varepsilon}(0) &= \Gamma(1-\beta)\left(f(-\varepsilon)\varepsilon^{-\beta}-\beta\int_0^{-\varepsilon}f(t)(-t)^{-\beta-1}dt\right)\\ &= \Gamma(1-\beta)\left(0-\beta\int_0^{-\varepsilon}0dt\right)=0,\end{aligned}$$

we define $g(0)=0$. Since $\Gamma(1-\beta)f'_{1-\beta,\varepsilon}$ converges uniformly to $g(x)$ on $(0,1]$ as $\varepsilon\to 0^+$ and $\Gamma(1-\beta)f'_{1-\beta,\varepsilon}(0)=g(0)$ for all $\varepsilon>0$, then $f'_{1-\beta,\varepsilon}(x)$ converges uniformly to $\frac{1}{\Gamma(1-\beta)}g(x)$ on $[0,1]$.

Claim 4.0.6. *g is continuous on* $[0,1]$.

Proof. Since f is continuous on $[0,1]$, g is continuous on $(0,1]$ by definition. To show that g is continuous at $x=0$, we show that $\lim_{x\to 0^+}g(x)=g(0)=0$.

Fix $\lambda>0$. Then there exists $0<\delta<\left(\frac{\lambda(k-\beta)}{Mk}\right)^{\frac{1}{k-\beta}}$. If $x\,\epsilon\,[0,1]$ such that $|x-0|<\delta$, then

$$\begin{aligned}|g(x)-0| &= \left|\beta\int_0^x(f(x)-f(t))(x-t)^{-\beta-1}dt+f(x)x^{-\beta}\right|\\ &\le \beta\int_0^x|f(x)-f(t)||x-t|^{-\beta-1}dt+|f(x)-f(0)|x^{-\beta}\\ &\le \beta\int_0^x M(x-t)^{k-\beta-1}dt+Mx^{k-\beta}\\ &\le \left[\frac{-M\beta}{k-\beta}(x-t)^{k-\beta}\right]_{t=0}^{t=x}+Mx^{k-\beta}\\ &= \frac{M\beta}{k-\beta}x^{k-\beta}+Mx^{k-\beta}\end{aligned}$$

$$= \left(\frac{Mk}{k-\beta}\right) x^{k-\beta}$$
$$< \left(\frac{Mk}{k-\beta}\right) \delta^{k-\beta} = \lambda.$$

Thus, g is continuous at $x = 0$ and is therefore continuous on $[0,1]$. $\square$

Then for $x \,\epsilon\, [0,1]$,

$$\begin{aligned} f_{1-\beta}(x) - f_{1-\beta}(0) &= \lim_{\varepsilon\to 0^+} (f_{1-\beta,\varepsilon}(x) - f_{1-\beta,\varepsilon}(0)) \\ &= \lim_{\varepsilon\to 0^+} \int_0^x f'_{1-\beta,\varepsilon}(t)dt \\ &= \int_0^x \lim_{\varepsilon\to 0^+} f'_{1-\beta,\varepsilon}(t)dt \\ &= \frac{1}{\Gamma(1-\beta)} \int_0^x g(t)dt. \end{aligned}$$

So

$$f_{1-\beta}(x) = \frac{1}{\Gamma(1-\beta)} \int_0^x g(t)dt - f_{1-\beta}(0).$$

Then by the Fundamental Theorem of Calculus,

$$f'_{1-\beta}(x) = \frac{1}{\Gamma(1-\beta)} g(x)$$

for $0 < x < 1$. Thus, $f'_{1-\beta}(x) = D^\beta f(x)$ exists for all $x \,\epsilon\, (0,1)$. However, we want to show the existence of the β–order fractional derivative of f on $[0,1]$, so we must consider the points $x = 0$ and $x = 1$.

To show that $f_{1-\beta}$ has a right–hand derivative at $x = 0$, we consider the convergence of $\frac{f_{1-\beta}(x) - f_{1-\beta}(0)}{x}$ as $x \to 0^+$.

Claim 4.0.7. $\lim_{x\to 0^+} \frac{f_{1-\beta}(x) - f_{1-\beta}(0)}{x} = \frac{g(0)}{\Gamma(1-\beta)} = 0$.

Proof. Fix $\lambda > 0$. Since g is continuous at $x = 0$, there exists $\delta > 0$ such that if $x \,\epsilon\, [0,1]$ and $|x - 0| < \delta$, then $|g(x) - g(0)| < \Gamma(1-\beta)\lambda$. Let $x \,\epsilon\, [0,1]$ such that

$|x - 0| < \delta$, then

$$\begin{aligned}
\left|\frac{f_{1-\beta}(x) - f_{1-\beta}(0)}{x} - \frac{g(0)}{\Gamma(1-\beta)}\right| &= \left|\frac{1}{x\Gamma(1-\beta)}\int_0^x g(x)dx\right| \\
&= \frac{1}{x\Gamma(1-\beta)}\int_0^x |g(x) - g(0)|dx \\
&< \frac{1}{x\Gamma(1-\beta)}\int_0^x \Gamma(1-\beta)\lambda dx \\
&= \left(\frac{1}{x\Gamma(1-\beta)}\right)(\Gamma(1-\beta)\lambda x) \\
&= \lambda.
\end{aligned}$$

□

Similarly to show that $f_{1-\beta}$ has a left–hand derivative at $x = 1$, we consider the convergence of $\frac{f_{1-\beta}(1)-f_{1-\beta}(x)}{1-x}$ as $x \to 1^-$.

CLAIM 4.0.8. $\lim_{x\to 1^-} \frac{f_{1-\beta}(1)-f_{1-\beta}(x)}{1-x} = \frac{g(1)}{\Gamma(1-\beta)}$.

PROOF. Fix $\lambda > 0$. Since g is continuous at $x = 1$, there exists $\delta > 0$ such that if $x \in [0, 1]$ and $|x - 1| < \delta$, then $|g(x) - g(1)| < \Gamma(1-\beta)\lambda$. Let $x \in [0, 1]$ such that $|x - 1| < \delta$, then

$$\begin{aligned}
\left|\frac{f_{1-\beta}(1) - f_{1-\beta}(x)}{1-x} - \frac{g(1)}{\Gamma(1-\beta)}\right| &= \frac{1}{\Gamma(1-\beta)}\left|\frac{1}{1-x}\int_x^1 g(x)dx - g(1)\right| \\
&= \frac{1}{\Gamma(1-\beta)}\left|\frac{\int_x^1 g(x) - g(1)dx}{1-x} + g(1) - g(1)\right| \\
&\le \frac{1}{(1-x)\Gamma(1-\beta)}\int_x^1 |g(x) - g(1)|dx \\
&< \frac{1}{(1-x)\Gamma(1-\beta)}\int_x^1 \Gamma(1-\beta)\lambda dx \\
&= \frac{1}{(1-x)\Gamma(1-\beta)}\Gamma(1-\beta)\lambda(1-x) \\
&= \lambda.
\end{aligned}$$

□

Thus, $f'_{1-\beta}(x)$ exists for all $x \epsilon [0,1]$. Furthermore, $f'_{1-\beta}(x) = \frac{1}{\Gamma(1-\beta)} g(x)$ for all $x \epsilon [0,1]$. Therefore, $D^\beta f(x)$ exists on $[0,1]$. $\square$

We now apply Proposition 4.0.2 to the extensions of the examples we defined in Chapter 1.

4.1. The Generalized van der Waerden–Takagi Function

PROPOSITION 4.1.1. *Let* $f(x) = \sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$, *where* $c > 1$ *and* $b \epsilon \mathbb{N}$ *such that* $b \geq c$. *Let* $f^*(x) = \begin{cases} f(x), & 0 < x \leq 1 \\ f(0), & x \leq 0 \end{cases}$. *Then for all* $0 < \beta < \frac{\log c}{\log b}$, $D^\beta f^*(x)$ *exists for* $x \epsilon [0,1]$.

PROOF. If $b = c$, then f is Hölder continuous of exponent k for all $0 \leq k < 1 = \frac{\log c}{\log b}$ by Theorem 2.1.2. If $b > c$, then f is a Hölder continuous function of exponent $\frac{\log c}{\log b}$ by Theorem 2.1.5. In both cases, $f(0) = 0$ so f satisfies the hypotheses of Proposition 4.0.2 for all $0 < \beta < \frac{\log c}{\log b}$. Thus, for all $0 < \beta < \frac{\log c}{\log b}$, $D^\beta f^*(x)$ exists for $x \epsilon [0,1]$. $\square$

4.2. Kiesswetter's Function

PROPOSITION 4.2.1. *Let* $g(t) = \lim_{n\to\infty} g_n(t)$, *where* $g_0(t) = t$ *and* $g_n(t)$ *is defined by (3) for all* $n \epsilon \mathbb{N}$. *Let* $g^*(t) = \begin{cases} g(t), & 0 < t \leq 1 \\ g(0), & t \leq 0 \end{cases}$. *Then for all* $0 < \beta < \frac{1}{2}$, $D^\beta g^*(t)$ *exists for* $t \epsilon [0,1]$.

PROOF. Since g is a Hölder continuous function of exponent $\frac{1}{2}$ by Theorem 2.2.3 and $g(0) = 0$, g satisfies the hypotheses of Proposition 4.0.2 for all $0 < \beta < \frac{1}{2}$. Thus, for all $0 < \beta < \frac{1}{2}$, $D^\beta g^*(t)$ exists for all $t \epsilon [0,1]$. $\square$

Chapter 5

Dimensions of Graphs of Nowhere Differentiable Functions

We have examined continuous nowhere differentiable functions in terms of Hölder continuity and fractional differentiability. Another way to study nowhere differentiable functions is to consider their graphs as subsets of the plane. We want to be able to compare the size of the graph of a continuous nowhere differentiable function to the sizes of the graphs of other continuous nowhere differentiable functions and to the sizes of the graphs of everywhere differentiable functions.

An obvious measure of size to consider is the length of a graph. However, the graphs of continuous nowhere differentiable functions have infinite length. To see this, we let f be a real-valued function on $[a,b]$ and let $\mathcal{P} : a = x_0 < x_1 < \ldots < x_n = b$ be a partition of $[a,b]$. Then we denote

$$L(\mathcal{P}) = \sum_{i=1}^{n} \left((x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2\right)^{\frac{1}{2}}$$

and define the length of the graph of f to be $\sup_{\mathcal{P}} L(\mathcal{P})$. The function f is of bounded variation on $[a,b]$, denoted $f \,\epsilon\, BV[a,b]$, if and only if $\sup_{\mathcal{P}} L(\mathcal{P}) < \infty$. If $f \,\epsilon\, BV[a,b]$, then, by Jordan's Theorem [8, page 103, theorem 5], f can be written as $f = g - h$, where g and h are monotone functions. Since Lebesgue's Theorem states that a monotone function on (a,b) is differentiable almost everywhere on (a,b) [8, page 100, theorem 3], $f'(x)$ exists almost everywhere on (a,b) if $f \,\epsilon\, BV[a,b]$. Since nowhere differentiable functions are not differentiable at any point in their domains, they cannot be of bounded variation. Thus, the graph of a continuous nowhere differentiable function must have infinite length.

Since we cannot compare the relative sizes of graphs of infinite length, we need to refine our notion of size. When we considered the smoothness of functions, we used fractional derivatives to refine our notion of smoothness into a concept that can distinguish differences in smoothness between examples of continuous nowhere differentiable functions. Similarly, we will use the ideas of dimension from fractal geometry to be able to compare the sizes of graphs that have infinite length.

5.1. Definitions and Facts

One important dimension is the Hausdorff dimension. Let $F \subset \mathbb{R}^n$. The diameter of a set V is defined to be $|V| = \sup\{|x - y| : x, y \in V\}$. Then a countable collection, $\{V_i\}$, of sets of diameter less than or equal to δ that covers F is a δ-cover of F. Now for all $\delta > 0$, the quantity $\mathcal{H}^s_\delta(F)$ is defined as

$$\mathcal{H}^s_\delta(F) = \inf\left\{\sum_{i=1}^{\infty} |V_i|^s : \{V_i\} \text{ is a } \delta\text{-cover of F}\right\}.$$

The s-dimensional Hausdorff measure of F, $\mathcal{H}^s(F)$, is defined to be

$$\mathcal{H}^s(F) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(F).$$

The Hausdorff dimension of F is then defined as

$$\dim_H F = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}$$

so that

$$\mathcal{H}^s(F) = \begin{cases} \infty, & s < \dim_H F \\ 0, & s > \dim_H F \end{cases}.$$

Another commonly used notion of dimension is the box–counting dimension. We again let $F \subset \mathbb{R}^n$. For $\delta > 0$, the collection of cubes

$$\{[M_1\delta, (M_1+1)\delta] \times \ldots \times [M_n\delta, (M_n+1)\delta] : M_1, \ldots, M_n \in \mathbb{Z}\}$$

is called the δ–mesh of $\mathbb{R}^n$. Let $N_\delta(F)$ be the number of δ–mesh cubes that intersect F. The lower box–counting dimension of F is defined as

$$\underline{\dim}_B F = \liminf_{\delta\to 0^+} \frac{\log N_\delta(F)}{-\log\delta}. \tag{10}$$

The upper box–counting dimension of F is defined by

$$\overline{\dim}_B F = \limsup_{\delta\to 0^+} \frac{\log N_\delta(F)}{-\log\delta}. \tag{11}$$

If the limits in (10) and (11) are equal, we call the value of the limit the box–counting dimension of F and denote it $\dim_B F$. It is sufficient to consider the limits in (10) and (11) as $\delta \to 0^+$ through any decreasing sequence δ_k such that $\delta_k = c^k$ for some constant $0 < c < 1$.

Before we consider the Hausdorff and box–counting dimensions of the graphs of specific functions, we note the following two facts from [2, pages 42, 146] relating these two concepts of dimension.

FACT 5.1.1. *For all* $F \subset \mathbb{R}^n$, $\dim_H F \le \underline{\dim}_B F \le \overline{\dim}_B F$.

FACT 5.1.2. *If* $f : [0,1] \to \mathbb{R}$ *has a continuous derivative and* F *is the graph of* f, *then* $\dim_H F = \dim_B F = 1$.

To bound the Hausdorff and upper box–counting dimensions of the graph of the Generalized van der Waerden–Takagi function and the graph of Kiesswetter's function, we use the following facts from [2, pages 147, 30].

FACT 5.1.3. *Let* $f : [0,1] \to \mathbb{R}$ *be a continuous function and let* F *be the graph of* f. *If* f *is Hölder continuous for the exponent* $2-s$ *on* $[0,1]$, *where* $1 \le s \le 2$, *then* $\mathcal{H}^s(F) < \infty$ *and* $\dim_H F \le \overline{\dim}_B F \le s$.

Fact 5.1.4. *Let $F \subset \mathbb{R}^n$. Let $g : F \to \mathbb{R}^m$. If $|g(x) - g(y)| \leq M|x - y|$ for all $x, y \,\epsilon\, F$, then* $\dim_H g(F) \leq \dim_H F$.

There are two corollaries of Fact 5.1.4 that are important for obtaining a lower bound for the Hausdorff dimension for the graph of a function.

Corollary 5.1.5. *Let F be the graph of a function, $f : \mathbb{R} \to \mathbb{R}$. Let T be the projection of F onto the x–axis. Then* $\dim_H T \leq \dim_H F$.

Proof. Let the function $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x, y) = x$ for all $x, y \,\epsilon\, \mathbb{R}$. Let $(x_1, y_1), (x_2, y_2) \,\epsilon\, F$. Then

$$\begin{aligned} |g(x_1, y_1) - g(x_2, y_2)| &= |x_1 - x_2| \\ &\leq \left((x_1 - x_2)^2 + (y_1 - y_2)^2\right)^{\frac{1}{2}} \\ &= |(x_1, y_1) - (x_2, y_2)| \end{aligned}$$

Thus, by Fact 5.1.4, $\dim_H g(F) \leq \dim_H F$. Since $T = g(F)$, $\dim_H T \leq \dim_H F$. □

Corollary 5.1.6. *Let $f : [0, 1] \to \mathbb{R}$. Let F be the graph of f. Then* $\dim_H F \geq 1$.

Proof. By Corollary 5.1.5, $\dim_H([0, 1]) \leq \dim_H F$. To determine $\dim_H([0, 1])$, we consider $\mathcal{H}^1([0, 1])$.

We first determine a lower bound for $\mathcal{H}^1([0, 1])$. Let $\delta > 0$. Let $\{V_i\}_{i \,\epsilon\, \mathbb{N}}$ be a δ–cover of $[0, 1]$. Since $[0, 1] \subset \bigcup_{i=1}^{\infty} V_i$, $\sum_{i=1}^{\infty} |V_i| \geq 1$. Thus, $\mathcal{H}^1_\delta([0, 1]) \geq 1$ for all $\delta > 0$. Then

$$\mathcal{H}^1([0, 1]) = \lim_{\delta \to 0^+} \mathcal{H}^1_\delta([0, 1]) \geq 1.$$

Now we determine an upper bound for $\mathcal{H}^1([0, 1])$. Let $\delta > 0$. Let $m = \lceil \frac{1}{\delta} \rceil$. Then [0,1] can be covered by m sets of diameter δ. Thus,

$$\begin{aligned} \mathcal{H}^1_\delta([0, 1]) &= \inf \left\{ \sum_{i=1}^{\infty} |V_i| : \{V_i\} \text{ is a } \delta\text{-cover of [0,1]} \right\} \\ &\leq m\delta \leq \left(1 + \frac{1}{\delta}\right) \delta = \delta + 1. \end{aligned}$$

Thus,

$$\mathcal{H}^1([0,1]) = \lim_{\delta\to 0^+} \mathcal{H}^1_\delta([0,1]) \le \lim_{\delta\to 0^+} \delta + 1 = 1.$$

So $\mathcal{H}^1([0,1]) = 1$. Since $0 < \mathcal{H}^1([0,1]) < \infty$, $\dim_H[0,1] = 1$. Therefore, $\dim_H F \ge 1$. $\square$

5.2. The Generalized van der Waerden–Takagi Function

We now apply the above general statements regarding the Hausdorff and box–counting dimensions to the specific example of the graph of the Generalized van der Waerden–Takagi function with $b = c$.

PROPOSITION 5.2.1. *Let F be the graph of the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sum_{n=0}^{\infty} \frac{a_0(c^n x)}{c^n}$, where $c \in \mathbb{N}$ and $c \ge 2$. Then $\dim_H F \le \overline{\dim}_B F \le 1$.*

PROOF. By Theorem 2.1.2, f is a Hölder continuous function of exponent α for all $\alpha < 1$. So f satisfies the hypothesis of Fact 5.1.3 for $1 < s \le 2$. Then, for all $s \in (1,2]$,

$$\dim_H F \le \overline{\dim}_B F \le s.$$

Thus,

$$\dim_H F \le \overline{\dim}_B F \le 1.$$

$\square$

THEOREM 5.2.2. *Let F be the graph of the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sum_{n=0}^{\infty} \frac{a_0(c^n x)}{c^n}$, where $c \in \mathbb{N}$ and $c \ge 2$. Then $\dim_H F = 1$.*

PROOF. By Proposition 5.2.1, $\dim_H F \le 1$. Since $f : [0,1] \to \mathbb{R}$, $\dim_H F \ge 1$ by Corollary 5.1.6. Therefore, $\dim_H F = 1$. $\square$

THEOREM 5.2.3. *Let F be the graph of the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sum_{n=0}^{\infty} \frac{a_0(c^n x)}{c^n}$, where $c \in \mathbb{N}$ and $c \ge 2$. Then $\dim_B F$ exists and equals 1.*

PROOF. By Fact 5.1.1 and Propositions 5.2.1 and 5.2.2,

$$1 = \dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F \leq 1.$$

Thus, $\underline{\dim}_B F = \overline{\dim}_B F = 1$. Therefore, $\dim_B F = 1$. □

We now consider the more general version of the Generalized van der Waerden-Takagi function with $b > c$.

PROPOSITION 5.2.4. *Let F be the graph of the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$, where $c > 1$ and $b \in \mathbb{N}$ such that $b > c$. Then $\dim_H F \leq \overline{\dim}_B F \leq 2 - \frac{\log c}{\log b}$.*

PROOF. By Theorem 2.1.5, f is a Hölder continuous function of exponent α for all $\alpha \leq \frac{\log c}{\log b}$. So f satisfies the hypothesis of Fact 5.1.3 for $2 - \frac{\log c}{\log b} \leq s \leq 2$. Thus,

$$\dim_H F \leq \overline{\dim}_B F \leq 2 - \frac{\log c}{\log b}.$$

□

Using Fact 5.1.1 and Corollary 5.1.6, we can obtain 1 as a lower bound for the Hausdorff and lower box–counting dimensions of the graph of the Generalized van der Waerden–Takagi function with $c > 1$ and $b \in \mathbb{N}$ such that $b > c$. Although obtaining a tighter bound for the Hausdorff dimension is beyond the scope of this work, we do want to obtain a tighter bound for the lower box–counting dimension of the graph. The following fact from [2, pages 146–147] will be important for obtaining this bound.

FACT 5.2.5. *Let $f : [0,1] \to \mathbb{R}$ be continuous. Let $0 < \delta < 1$ and let $m = \lceil \frac{1}{\delta} \rceil$. If N_δ is the number of squares of the δ–mesh that intersect the graph of f and $R_f[i\delta, (i+1)\delta]$ is the maximum range of f over the interval $[i\delta, (i+1)\delta]$, then*

$$\delta^{-1} \sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta] \leq N_\delta \leq 2m + \delta^{-1} \sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta].$$

By using Fact 5.2.5, the only difficulty in determining a lower bound for the box–counting dimension is the calculation of a lower bound for the range of the function over an interval of a given length δ. However, if we use $\delta_k = \frac{1}{2b^k}$, then the horizontal endpoints of each square in the δ_k–mesh are points of the form $\frac{i}{2b^k}$ and $\frac{i+1}{2b^k}$ where $i \in \mathbb{Z} \cap [0, 2b^k - 1]$. The value of $f\left(\frac{i+1}{2b^k}\right) - f\left(\frac{i}{2b^k}\right)$ has already been calculated in the proof of Proposition 1.1.10. If b is even, then (1) implies that

$$\left| f\left(\frac{i+1}{2b^k}\right) - f\left(\frac{i}{2b^k}\right) \right| = \left| \sum_{n=0}^{k} \pm \left(\frac{b}{c}\right)^n \right| \left| \frac{1}{2b^k} \right|. \tag{12}$$

If b is odd, then (2) implies that

$$\left| f\left(\frac{i+1}{2b^k}\right) - f\left(\frac{i}{2b^k}\right) \right| = \left| \sum_{n=0}^{k-1} \pm \left(\frac{b}{c}\right)^n \pm \frac{c}{c-1}\left(\frac{b}{c}\right)^k \right| \left| \frac{1}{2b^k} \right|. \tag{13}$$

Using Fact 5.2.5, we can now obtain a bound for the lower box–counting dimension of the graph of the Generalized van der Waerden-Takagi function for all values of b and c such that we can obtain a positive lower bound for the quantities given in (12) and (13).

Proposition 5.2.6. *Let F be the graph of the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$, where $c > 1$ and $b \in \mathbb{N}$ such that b is even and $b \geq 2c$. Then $2 - \frac{\log c}{\log b} \leq \underline{\dim}_B F$.*

Proof. For all $k \in \mathbb{N}$, let $\delta_k = \frac{1}{2b^k}$ and let $m_k = \frac{1}{\delta_k} = 2b^k$. Then for all $k \in \mathbb{N}$ and $i \in \mathbb{Z} \cap [0, m_k - 1]$,

$$\begin{aligned} R_f[i\delta_k, (i+1)\delta_k] &\geq \left| f\left(\frac{i+1}{2b^k}\right) - f\left(\frac{i}{2b^k}\right) \right| \\ &= \left| \sum_{n=0}^{k} \pm \left(\frac{b}{c}\right)^n \right| \left| \frac{1}{2b^k} \right| \\ &\geq \frac{\left(\frac{b}{c}\right)^k - \sum_{n=0}^{k-1} \left(\frac{b}{c}\right)^n}{2b^k} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\left(\frac{b}{c}\right)^k \left[\left(\frac{b}{c}\right) - 2\right] + 1}{2b^k\left(\left(\frac{b}{c}\right) - 1\right)} \\
&\geq \frac{\frac{b^{k-1}}{c^k}(b-2c)}{2b^k} \geq 0
\end{aligned}
$$

by (12) since $b \geq 2c$.

Then by Fact 5.2.5,

$$
\begin{aligned}
N_{\delta_k} &\geq \delta_k^{-1} \sum_{i=0}^{m_k - 1} R_f[i\delta_k, (i+1)\delta_k] \\
&\geq 2b^k \sum_{i=0}^{2b^k - 1} \frac{\frac{b^{k-1}}{c^k}(b-2c)}{2b^k} \\
&= \frac{2b^{2k-1}}{c^k}(b-2c).
\end{aligned}
$$

Then

$$
\begin{aligned}
\underline{\dim}_B F &= \liminf_{\delta \to 0^+} \frac{\log N_\delta(F)}{-\log \delta} \\
&\geq \liminf_{k \to \infty} \frac{\log \frac{2b^{2k-1}}{c^k}(b-2c)}{-\log \frac{1}{2b^k}} \\
&= \liminf_{k \to \infty} \frac{\log(b-2c) + \log 2 + (2k-1)\log b - k\log c}{\log 2 + k \log b} \\
&= 2 - \frac{\log c}{\log b}.
\end{aligned}
$$

$\square$

PROPOSITION 5.2.7. *Let F be the graph of the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$, where $c > 1$ and $b \,\epsilon\, \mathbb{N}$ such that b is odd and $b \geq 2c - 1$. Then $2 - \frac{\log c}{\log b} \leq \underline{\dim}_B F$.*

PROOF. For all $k \,\epsilon\, \mathbb{N}$, let $\delta_k = \frac{1}{2b^k}$ and let $m_k = \frac{1}{\delta_k} = 2b^k$. Then for all $k \,\epsilon\, \mathbb{N}$ and $i \,\epsilon\, \mathbb{Z} \cap [0, m_k - 1]$,

$$
R_f[i\delta_k, (i+1)\delta_k] \geq \left| f\left(\frac{i+1}{2b^k}\right) - f\left(\frac{i}{2b^k}\right) \right|
$$

$$
\begin{aligned}
&= \left|\sum_{n=0}^{k-1} \pm \left(\frac{b}{c}\right)^n \pm \frac{c}{c-1}\left(\frac{b}{c}\right)^k\right|\left|\frac{1}{2b^k}\right| \\
&\geq \frac{1}{2b^k}\left(\frac{c}{c-1}\left(\frac{b}{c}\right)^k - \sum_{n=0}^{k-1}\left(\frac{b}{c}\right)^n\right) \\
&= \frac{1}{2b^k}\frac{\left(\frac{b}{c}\right)^k\left(\frac{b-2c+1}{c-1}\right)+1}{\left(\frac{b}{c}\right)-1} \\
&\geq \frac{\frac{b^{k-1}}{c^k}(b-2c+1)}{2b^k} \geq 0
\end{aligned}
$$

by (13) since $b \geq 2c-1$.

Then by Fact 5.2.5,

$$
\begin{aligned}
N_{\delta_k} &\geq \delta_k^{-1}\sum_{i=0}^{m_k-1} R_f[i\delta_k,(i+1)\delta_k] \\
&\geq 2b^k \sum_{i=0}^{2b^k-1}\frac{\frac{b^{k-1}}{c^k}(b-2c+1)}{2b^k} \\
&= \frac{2b^{2k-1}}{c^k}(b-2c+1).
\end{aligned}
$$

Then

$$
\begin{aligned}
\underline{\dim}_B F &= \liminf_{\delta\to 0^+}\frac{\log N_\delta(F)}{-\log\delta} \\
&\geq \liminf_{k\to\infty}\frac{\log\frac{2b^{2k-1}}{c^k}(b-2c+1)}{-\log\frac{1}{2b^k}} \\
&= \liminf_{k\to\infty}\frac{\log(b-2c+1)+\log 2+(2k-1)\log b - k\log c}{\log 2 + k\log b} \\
&= 2-\frac{\log c}{\log b}.
\end{aligned}
$$

□

We know summarize the above discussion and propositions in the following theorem.

THEOREM 5.2.8. *Let F be the graph of the function $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \sum_{n=0}^{\infty} \frac{a_0(b^n x)}{c^n}$, where $c > 1$ and $b \in \mathbb{N}$ such that $b > c$. Then $1 \leq \dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F \leq 2 - \frac{\log c}{\log b}$. Moreover, if b is even and $b \geq 2c$ or b is odd and $b \geq 2c - 1$, then $\dim_B F$ exists and equals $2 - \frac{\log c}{\log b}$.*

5.3. KIESSWETTER'S FUNCTION

We now consider the Hausdorff and box–counting dimensions of the graph of Kiesswetter's function. The calculations are similar to those made for the graph of the Generalized van der Waerden–Takagi function.

PROPOSITION 5.3.1. *Let $g : [0,1] \to \mathbb{R}$ be defined by $g(t) = \lim_{n\to\infty} g_n(t)$, where $g_0(t) = t$ and $g_n(t)$ is defined by (3) for all $n \in \mathbb{N}$. Let G be the graph of g. Then $\dim_H G \leq \overline{\dim}_B G \leq \frac{3}{2}$.*

PROOF. By Theorem 2.2.3, g is a Hölder continuous function of exponent α for all $\alpha \leq \frac{1}{2}$. So g satisfies the hypothesis of Fact 5.1.3 for $\frac{3}{2} \leq s \leq 2$. Thus,

$$\dim_H G \leq \overline{\dim}_B G \leq \frac{3}{2}.$$

$\square$

PROPOSITION 5.3.2. *Let $g : [0,1] \to \mathbb{R}$ be defined by $g(t) = \lim_{n\to\infty} g_n(t)$, where $g_0(t) = t$ and $g_n(t)$ is defined by (3) for all $n \in \mathbb{N}$. Let G be the graph of g. Then $\frac{3}{2} \leq \underline{\dim}_B G$.*

PROOF. For all $k \in \mathbb{N}$, let $\delta_k = \frac{1}{4^k}$ and let $m_k = \frac{1}{\delta_k} = 4^k$. By Proposition 1.2.5, $\left|g\left(\frac{i+1}{4^k}\right) - g\left(\frac{i}{4^k}\right)\right| = \frac{1}{2^k}$ for all $k \in \mathbb{N}$ and $i \in \mathbb{Z} \cap [0, m_k - 1]$. Thus, for all $k \in \mathbb{N}$ and $i \in \mathbb{Z} \cap [0, m_k - 1]$,

$$R_g[i\delta_k, (i+1)\delta_k] \geq \frac{1}{2^k} = \delta_k^{\frac{1}{2}}.$$

Then by Fact 5.2.5,

$$\begin{aligned}
N_{\delta_k}(G) &\geq \delta_k^{-1} \sum_{i=0}^{m_k-1} R_g[i\delta_k, (i+1)\delta_k] \\
&\geq \delta_k^{-1} \sum_{i=0}^{m_k-1} \delta_k^{\frac{1}{2}} \\
&= \delta_k^{-1}\delta_k^{-1}\delta_k^{\frac{1}{2}} = \delta_k^{\frac{-3}{2}}.
\end{aligned}$$

Then

$$\begin{aligned}
\underline{\dim}_B G &= \liminf_{\delta \to 0^+} \frac{\log N_\delta(G)}{-\log \delta} \\
&\geq \liminf_{k \to \infty} \frac{\log(\frac{1}{4^k})^{\frac{-3}{2}}}{-\log \frac{1}{4^k}} \\
&= \liminf_{k \to \infty} \frac{-\frac{3}{2}\log \frac{1}{4^k}}{-\log \frac{1}{4^k}} \\
&= \frac{3}{2}.
\end{aligned}$$

□

THEOREM 5.3.3. *Let $g : [0,1] \to \mathbb{R}$ be defined by $g(t) = \lim_{n\to\infty} g_n(t)$, where $g_0(t) = t$ and $g_n(t)$ is defined by (3) for all $n \in \mathbb{N}$. Let G be the graph of g. Then $\dim_B G$ exists and equals $\frac{3}{2}$.*

PROOF. By Fact 5.1.1 and Propositions 5.3.1 and 5.3.2,

$$\frac{3}{2} \leq \underline{\dim}_B G \leq \overline{\dim}_B G \leq \frac{3}{2}.$$

Thus, $\underline{\dim}_B G = \overline{\dim}_B G = \frac{3}{2}$. Therefore, $\dim_B G = \frac{3}{2}$. □

Although we are not able to obtain an exact value for the Hausdorff dimension of the graph of Kiesswetter's function, the preceding propositions and corollaries provide bounds for the value. These bounds are stated explicitly in the following proposition for completeness.

Proposition 5.3.4. *Let* $g : [0, 1] \to \mathbb{R}$ *be defined by* $g(t) = \lim_{n\to\infty} g_n(t)$*, where* $g_0(t) = t$ *and* $g_n(t)$ *is defined by (3) for all* $n \in \mathbb{N}$*. Let* G *be the graph of* g*. Then* $1 \leq \dim_H G \leq \frac{3}{2}$.

Proof. Since $g : [0, 1] \to \mathbb{R}$, $\dim_H G \geq 1$ by Corollary 5.1.6. By Fact 5.1.1 and Proposition 5.3.3, $\dim_H G \leq \dim_B G \leq \frac{3}{2}$. Therefore, $1 \leq \dim_H G \leq \frac{3}{2}$. $\square$

Conclusion

In this thesis we investigated two special classes of continuous nowhere differentiable functions. One class consisted of the Generalized van der Waerden–Takagi function. Our methods allowed us to remove the restriction that $b \geq 4c$ imposed by Knopp and to show that the Generalized van der Waerden–Takagi function is nowhere differentiable on $[0,1]$ for $c > 1$ and $b \in \mathbb{N}$ such that $b \geq c$. The other class we considered was Kiesswetter's function, which we showed was nowhere differentiable on $[0,1]$.

Although these functions do not have first–order derivatives, they do share some smoothness properties. In order to study these properties, we had to refine our methods to reveal the finer structure that differentiability cannot see. One way we measured smoothness was by using Hölder continuity. We found that all of the functions in both classes were Hölder continuous of exponent α on $[0,1]$ for some $\alpha > 0$. In fact, the Generalized van der Waerden–Takagi function with $b = c$ is Hölder continuous of exponent α on $[0,1]$ for all $\alpha < 1$.

Another way of measuring the smoothness of the continuous nowhere differentiable functions was the existence of fractional derivatives. We used Hölder continuity to determine that the functions in both classes have fractional derivatives of order β for some $\beta > 0$. We found that the Generalized van der Waerden–Takagi function with $b = c$ has fractional derivatives of order β for all $0 \leq \beta < 1$.

Our final approach for measuring the smoothness of continuous nowhere differentiable functions utilized the ideas of dimension from fractal geometry to measure the size of the graphs of the functions as subsets of the plane. The potential range for the

Hausdorff and box–counting dimensions of the graphs of the functions is $[1, 2]$ since they are continuous functions on $[0, 1]$, but for all the functions in the two classes, we were able to show that these values are less than 2. For the graph of the Generalized van der Waerden–Takagi function with $b = c$, we found that the Hausdorff and box–counting dimensions equal 1.

Our measures of smoothness allow us to compare these classes of functions to continuous everywhere differentiable functions. The fact that the Generalized van der Waerden–Takagi function with $b = c$ is Hölder continuous of exponent α for all $\alpha < 1$, has fractional derivatives everywhere of order β for all $0 \leq \beta < 1$, and has a graph of dimension 1 reveals the strong similarities between this special case of the Generalized van der Waerden–Takagi function and continuous everywhere differentiable functions. With respect to these three measures, the Generalized van der Waerden–Takagi function with $b = c$ is as close to being everywhere differentiable as a nowhere differentiable function can be. The Hölder continuity of the other functions we studied and the dimensions of their graphs indicate that these functions differ much more from everywhere differentiable functions than the Generalized van der Waerden–Takagi function with $b = c$ does.

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