THE GENERAL LEBESGUE INTEGRAL

by

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1. Thesis Summary

This paper expands upon Robert Bartle's exploration of the General Lebesgue Integral in his text *Elements of Integration*. Following Bartle's example, the paper opens with a discussion of the groundwork on which the theory of the Lebesgue Integral stands. As a house sits on cement and brick footings, the Lebesgue Integral is propped on solid mathematical concepts such as algebras and measures.

After setting the basics, the framework or the skeleton of the Integral is presented. Mathematicians are fond of handling the simple cases first then extending the results to more complicated and detailed cases. Obeying this methodology, the paper initially establishes the Integral for a limited class of functions, namely those measurable functions with non-negative values. Then later the Integral is defined for all measurable functions.

Having completed the base and the frame of the Integral, the paper then focuses on some of the beautiful and elegant theories that adorn the General Lebesgue Integral, like the Monotone Convergence Theorem and the Lebesgue Dominated Convergence Theorem. Indeed, convergence is an extremely important idea in mathematics and many times mathematicians are interested in sequences of functions and the convergence (if it does in fact converge) of these sequences.. Sometimes these sequences of functions converge or get very close to another function. And other times these sequences diverge or don't get very close to another function. In a convergent sequence of functions, no matter how far you go out in the sequence it still stays very close to one unchanging function or what mathematicians call the limit of the sequence. In a divergent sequence, as you move through the sequence you never get close to one particular function. What mathematicians want to know is if you take the limit of the sequence and then integrate will you get the same value as if you integrate each function in the sequence and then take the limit of the integrals? In other words, with two operations, does

it matter in which order you perform them? It turns out that with some precautions, the order may be switched.

Furthermore, the paper explores spaces of functions. In layman's terms, through modifications of the Lebesgue Integral mathematicians are able to categorize functions that share similar properties. Once a definition of these spaces has been set, then an analysis of the interactions between these categories can be performed. For example, we can determine what properties are needed for a function to live in more than one space or what properties are needed so that a function lives in every space.

Of course, mathematicians combine these concepts of sequences of functions and spaces of functions. More specifically, they want to know if given a sequence of functions that converge and all of the functions in that sequence live in a particular space, does the limit live in that same space?

By the way, did I mention that there are a handful of ways that sequences of functions can converge? Well, the last section of paper dicusses these different modes of convergence. Of particular interest is the order of implication. In simpler terms, if a sequence converges in a given manner, does it then converge in another manner or manners.

2. Introduction

The French mathematician Henri Léon Lebesgue developed the Lebesgue integral as a consequence of the problems associated with the Riemann Integral. In particular, whole classes of important functions could not be integrated with the Riemann Integral. For example, the function on the interval $[0, 1]$ that maps all the rational numbers to zero and all the irrational numbers to themselves cannot be integrated using Riemannian Integration. In fact, Lebesgue provided necessary and sufficient conditions for a function to be Riemann Integrable.

Lebesgue's major insight was to leave the x-axis and find the area under the curve by partitioning the y-axis. To see the differing approaches of the two mathematicians, imagine that Riemann and Lebesgue were both merchants. Let's say that both men sold six items at the following prices: 5,10,15,10,5,3. Riemann would total his sales by adding the numbers as they appear (5+10+15+10+5+3). Lebesgue, on the other hand, would sum them like this: 2(5) + 2(10) + 15 + 3. Of course, in the end, the values are the same.

As expected, it will be seen that the Lebesgue integral of Riemann integrable functions equals the Riemann Integral. By approaching integration in this manner Lebesgue generalized the Riemann Integral which provided mathematicians with a gateway into modern mathematics. Indeed, the Lebesgue Integral has been instrumental in the theory of trigonometric series, curve rectification, calculus, and probability.

However, the most immediate consequence of the Lebesgue Integral is that it relaxes the requirements needed for the interchange of the limit and the integral in a sequence of functions. In the Riemann Integral, only uniform convergence of a sequence of a functions implies that the limit of the sequence will be Riemann Integrable. With Lebesgue integrable functions we find that almost everywhere convergence and boundedness are sufficient for integrability of the limit. This theorem is the Lebesgue Dominated Convergence Theorem (LCDT).

This paper expands on Robert G. Bartle's presentation of the Lebesgue Integral in his book *Elements of Integration*. Throughout the remainder of the paper I shall denote *Elements of Integration* as (EOI).The third section of this paper deals with measurable functions and measurable spaces. The fourth section presents the notion of a measure of a set. The fifth section applies to the General Lebesgue Integral for nonnegative functions and the Monotone Convergence Theorem. In section six, the General Lebesgue integral is extended to functions with positive and negative values. Section seven deals with the Lebesgue Spaces. Lastly section eight encapsulates all the various modes of convergence.

Finally, when referring to a theorem, lemma, proposition, fact, or definition from another text, I use the following convention: the number of the theorem, lemma, proposition, fact, or definition and the text's author enclosed in brackets. For example, if I use 3.6 Lemma from *Elements of Integration*, I would write " … 3.6 Lemma [Bartle] … ". If I refer to a theorem, lemma, proposition, fact, or definition from this paper, I simply write its number. Thus, if I refer to 2.3 Lemma, I would write " … 2.3 Lemma … ".

3. Measurable Functions

In defining the Lebesgue Integral we need to first discuss classes of real-valued functions on a set $\mathcal{X}$. This set $\mathcal{X}$ could be the unit interval, the natural numbers, or the entire real line.

From $\mathcal{X}$, we take a family $\boldsymbol{\mathcal{X}}$ of subsets of $\mathcal{X}$ which behave "nicely" in a technical sense. First, we'll develop the idea of behaving "nicely" for a finite number of subsets, then we'll introduce the case for a countable number of subsets.

Definition 3.1. A non-empty collection $\boldsymbol{\mathcal{X}}$ of subsets of $\mathcal{X}$ is called an **algebra** of sets or a **Boolean algebra** if the following hold:

1. $(\mathtt{A} \cup \mathtt{B}) \in \boldsymbol{\mathcal{X}}$ whenever $\mathtt{A} \in \boldsymbol{\mathcal{X}}$ and $\mathtt{B} \in \boldsymbol{\mathcal{X}}$.
2. $(\mathtt{A})^c \in \boldsymbol{\mathcal{X}}$ whenever $\mathtt{A} \in \boldsymbol{\mathcal{X}}$. Note: $(\mathtt{A})^c$ denotes the complement of $\mathtt{A}$
3. $(\mathtt{A} \cap \mathtt{B}) \in \boldsymbol{\mathcal{X}}$ whenever $\mathtt{A}$, $\mathtt{B} \in \boldsymbol{\mathcal{X}}$

Proposition 3.2. *If in Definition 3.1, a non-empty collection $\boldsymbol{\mathcal{X}}$ of subsets of $\mathcal{X}$ satisfies (2) and (3), then it satisfies (1) and thus, is a **Boolean algebra**. Similarly, if it satisfies (1) and (2), then it satisfies (3).*

Proof. Let $\mathtt{A}$, $\mathtt{B} \in \boldsymbol{\mathcal{X}}$. Then, $(\mathtt{A})^c, (\mathtt{B})^c \in \boldsymbol{\mathcal{X}}$. By (3), $(\mathtt{A})^c \cap (\mathtt{B})^c \in \boldsymbol{\mathcal{X}}$. Applying De Morgan's laws, $(\mathtt{A})^c \cap (\mathtt{B})^c = (\mathtt{A} \cup \mathtt{B})^c$. Therefore, $(\mathtt{A} \cup \mathtt{B})^c \in \boldsymbol{\mathcal{X}}$. Using (2) gives, $[(\mathtt{A} \cup \mathtt{B})^c]^c = (\mathtt{A} \cup \mathtt{B}) \in \boldsymbol{\mathcal{X}}$. In a similar fashion it is seen that (1) and (2), imply (3). □

It is clear from (2) of Definition 3.1 that the $\emptyset$ and the whole set $\mathcal{X}$ are in $\boldsymbol{\mathcal{X}}$. Also, by taking unions two at a time, it is evident that if $\mathtt{A}_1, \ldots, \mathtt{A}_n$ are sets in $\boldsymbol{\mathcal{X}}$, then $\mathtt{A}_1 \cup \mathtt{A}_2 \cup \cdots \cup \mathtt{A}_n$ is also in $\boldsymbol{\mathcal{X}}$. Several useful theorems concerning algebras of sets will follow.

Proposition 3.3. *Given any collection $\mathcal{C}$ of subsets of $\mathcal{X}$, there is a smallest algebra $\mathcal{A}$ which contains $\mathcal{C}$; that is, there is an algebra $\mathcal{A}$ containing $\mathcal{C}$ such that if $\mathcal{B}$ is any algebra containing $\mathcal{C}$, then $\mathcal{B}$ contains $\mathcal{A}$.*

Proof. Let $\mathfrak{F}$ be the family of all algebras (of subsets of $\mathcal{X}$) that contain $\mathcal{C}$. Let $\mathcal{A} = \bigcap\{\mathcal{B} : \mathcal{B} \in \mathfrak{F}\}$. Then $\mathcal{C}$ is a subcollection of $\mathcal{A}$ since each $\mathcal{B} \in \mathfrak{F}$ contains $\mathcal{C}$. Moreover, $\mathcal{A}$ is an algebra. For if $\mathtt{A}$ and $\mathtt{B}$ are in $\mathcal{A}$, then for each $\mathcal{B} \in \mathcal{F}$ we have $\mathtt{A} \in \mathcal{B}$ and $\mathtt{B} \in \mathcal{B}$. Since $\mathcal{B}$ is an algebra, $(\mathtt{A} \cap \mathtt{B}) \in \mathcal{B}$. Since this is true for all $\mathcal{B} : \mathcal{B} \in \mathcal{F}$, we have $(\mathtt{A} \cap \mathtt{B}) \in \bigcap\{\mathcal{B} : \mathcal{B} \in \mathcal{F}\}$. Likewise, if $\mathtt{A} \in \mathcal{A}$, then $(\mathtt{A})^c \in \mathcal{A}$. From the definition of $\mathcal{A}$, it follows that if $\mathcal{B}$ is an algebra containing $\mathcal{C}$, then $\mathcal{B} \supseteq \mathcal{A}$. □

The smallest algebra containing $\mathcal{C}$ is called the algebra generated by $\mathcal{C}$.

An algebra $\mathcal{A}$ of sets is called a $\sigma-$**algebra** or a $\sigma-$**field**, if every union of a countable collection of sets in $\mathcal{A}$ is again in $\mathcal{A}$. From De Morgan's laws it follows that the intersection of a countable collection of sets in $\mathcal{A}$ is again in $\mathcal{A}$. Modifying Proposition 3.3 gives the following:

Proposition 3.4. *Given any collection $\mathcal{C}$ of subsets of $\mathcal{X}$, there is a smallest σ-algebra $\mathcal{A}$ which contains $\mathcal{C}$; that is, there is a σ-algebra $\mathcal{A}$ containing $\mathcal{C}$ and such that if $\mathcal{B}$ is any σ-algebra containing $\mathcal{C}$, then $\mathcal{B}$ contains $\mathcal{A}$.*

Proof. Let $\mathfrak{F}$ be the family of all σ-algebras (of subsets of $\mathcal{X}$) that contain $\mathcal{C}$. Let $\mathcal{A} = \bigcap\{\mathcal{B} : \mathcal{B} \in \mathfrak{F}\}$. Then $\mathcal{C}$ is a subcollection of $\mathcal{A}$ since each $\mathcal{B} \in \mathcal{F}$ contains $\mathcal{C}$. Moreover, $\mathcal{A}$ is an σ-algebra. For if $\langle \mathtt{A_i} \rangle$ belongs to $\mathcal{A}$, then for each $\mathcal{B} \in \mathcal{F}$ we have $\langle \mathtt{A_i} \rangle \in \mathcal{B}$. Since $\mathcal{B}$ is an σ-algebra, $\bigcup_{i=1}^{\infty} \mathtt{A}_i \in \mathcal{B}$.

Since this is true for all $\mathcal{B} : \mathcal{B} \in \mathfrak{F}$, we have

$$\bigcup_{i=1}^{\infty} \mathtt{A}_i \in \bigcap\{\mathcal{B} : \mathcal{B} \in \mathcal{F}\}.$$

From the definition of $\mathcal{A}$, it follows that if $\mathcal{B}$ is a σ-algebra containing $\mathcal{C}$, then $\mathcal{B} \supseteq \mathcal{A}$. □

As with algebras, the smallest σ-algebra containing $\mathcal{C}$ is called the σ-algebra generated by $\mathcal{C}$ and is denoted by $\sigma(\mathcal{C})$.

Proposition 3.5. *If $\mathcal{A}$ is the algebra generated by $\mathcal{C}$, then $\mathcal{A}$ and $\mathcal{C}$ generate the same σ-algebra.*

Proof. Let $\sigma(\mathcal{C})$ be the smallest σ-algebra generated by $\mathcal{C}$. Let $\mathcal{A}(\mathcal{C})$ be the smallest algebra generated by $\mathcal{C}$. Certainly, $\mathcal{A}(\mathcal{C}) \subseteq \sigma(\mathcal{C})$. Therefore, $\sigma[\mathcal{A}(\mathcal{C})] \subseteq \sigma[\sigma(\mathcal{C})]$. But that means $\sigma[\mathcal{A}(\mathcal{C})] \subseteq \sigma(\mathcal{C})$. Also, $\mathcal{A}(\mathcal{C}) \supseteq \mathcal{C}$, since the smallest algebra generated by $\mathcal{C}$ contains $\mathcal{C}$. Thus, $\sigma(\mathcal{C}) \subseteq \sigma[(\mathcal{A}(\mathcal{C})]$. □

The next proposition develops the idea of disjunctification. Given a sequence of sets in a σ-algebra the sets in the sequence can be separated such that no two sets share a common element.

Proposition 3.6. *Let $\boldsymbol{\mathcal{X}}$ be a σ-algebra of subsets and $\{\mathcal{A}_n\}$ a sequence of sets in $\boldsymbol{\mathcal{X}}$. Then there is a sequence $\{\mathcal{B}_n\}$ of subsets in $\boldsymbol{\mathcal{X}}$ such that $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for $i \neq j$ and*

$$\bigcup_{n=1}^{\infty} \mathcal{B}_n = \bigcup_{n=1}^{\infty} \mathcal{A}_n.$$

Proof. Since the theorem works similarly for $\{\mathcal{A}_n\}$ finite and infinite, assume $\{\mathcal{A}_n\}$ to be an infinite sequence. Let $\mathcal{B}_1 = \mathcal{A}_1$ and for $n \in \mathbb{N}$ with $n \geq 2$, let $\mathcal{B}_n = \mathcal{A}_n \backslash (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_{n-1})$. Thus

$$\mathcal{B}_n = \mathcal{A}_n \cap \left(\bigcup_{k<n} \mathcal{A}_k \right)^c.$$

Clearly, from the definition of $\mathcal{B}_n$, $\mathcal{B}_n \subseteq \mathcal{A}_n$, for each $n \in \mathbb{N}$. Also since complements and intersections of sets in $\boldsymbol{\mathcal{X}}$ are in $\boldsymbol{\mathcal{X}}$, each $\mathcal{B}_n \in \boldsymbol{\mathcal{X}}$. If $i > j \geq 1$, then

$$\mathcal{B}_i \subseteq \left(\bigcup_{k<i} \mathcal{A}_k \right)^c = \bigcap_{k<i} (\mathcal{A}_k)^c \subseteq (\mathcal{A}_j)^c \subseteq (\mathcal{B}_j)^c.$$

and so $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$. Now, $\bigcup_{n=1}^{\infty} \mathcal{B}_n \subset \bigcup_{n=1}^{\infty} \mathcal{A}_n$ since $\mathcal{B}_n \subseteq \mathcal{A}_n$ for each $n \in \mathbb{N}$.

Now take an $x \in \bigcup_{n=1}^{\infty} \mathcal{A}_n$, then there exist a smallest $i \in \mathbb{N}$ such that x belongs to $\mathcal{A}_i$ and $x \notin \bigcup_{n<i} \mathcal{A}_n$. Therefore,

$$x \in \mathcal{A}_i \backslash \bigcup_{n<i} \mathcal{A}_n = \mathcal{B}_i$$

and thus, $x \in \bigcup_{n=1}^{\infty} \mathcal{B}_n$. This demonstrates, $\bigcup_{n=1}^{\infty} \mathcal{B}_n \supseteq \bigcup_{n=1}^{\infty} \mathcal{A}_n$. Therefore,

$$\bigcup_{n=1}^{\infty} \mathcal{B}_n = \bigcup_{i=1}^{\infty} \mathcal{A}_n$$

□

Proposition 3.7. *Let f be a function defined on a set $\mathcal{X}$ with values in a set $\mathcal{Y}$. If $\mathcal{E}$ is any subset of $\mathcal{Y}$, let*

$$f^{-1}(\mathcal{E}) = \{x \in \mathcal{X} : f(x) \in \mathcal{E}\}.$$

Show that $f^{-1}(\emptyset) = \emptyset, f^{-1}(\mathcal{Y}) = \mathcal{X}$. If $\mathcal{E}$ and $\mathcal{F}$ are subsets of $\mathcal{Y}$, then

$$f^{-1}(\mathcal{E} \backslash \mathcal{F}) = f^{-1}(\mathcal{E}) \backslash f^{-1}(\mathcal{F}).$$

If $\{\mathcal{E}_\alpha\}$ is any non-empty collection of subsets of $\mathcal{Y}$, then

$$f^{-1}\left(\bigcup_{\alpha} \mathcal{E}_\alpha\right) = \bigcup_{\alpha} f^{-1}\left(\mathcal{E}_\alpha\right), \qquad f^{-1}\left(\bigcap_{\alpha} \mathcal{E}_\alpha\right) = \bigcap_{\alpha} f^{-1}\left(\mathcal{E}_\alpha\right).$$

In particular, it follows that if $\boldsymbol{\mathcal{Y}}$ is a σ-algebra of subsets of $\mathcal{Y}$, then $\{f^{-1}(\mathcal{E}) : \mathcal{E} \in \boldsymbol{\mathcal{Y}}\}$ is a σ-algebra of subsets of $\mathcal{X}$.

Proof. First, $f^{-1}(\emptyset) = \{x \in \mathcal{X} : f(x) \in \emptyset\} = \emptyset$. Take $x \in f^{-1}(\mathcal{Y})$ thus $x \in \mathcal{X}$ and so $f^{-1}(\mathcal{Y}) \subseteq \mathcal{X}$. Taking $x \in \mathcal{X}$ and applying f gives the other containment, therefore $f^{-1}(\mathcal{Y}) = \mathcal{X}$.

Next, show that $f^{-1}(\mathcal{E} \backslash \mathcal{F}) = f^{-1}(\mathcal{E}) \backslash f^{-1}(\mathcal{F})$. By definition $x \in f^{-1}(\mathcal{E} \backslash \mathcal{F})$ if and only if $f(x) \in \mathcal{E} \backslash \mathcal{F}$ which is equivalent to $f(x) \in \mathcal{E}$ and $f(x) \notin \mathcal{F}$. Now $f(x) \in \mathcal{E}$ and $f(x) \notin \mathcal{F}$ if and only if $x \in f^{-1}(\mathcal{E})$ and $x \notin f^{-1}(\mathcal{F})$ or equivalently $x \in f^{-1}(\mathcal{E}) \backslash f^{-1}(\mathcal{F})$. Therefore, $f^{-1}(\mathcal{E} \backslash \mathcal{F}) = f^{-1}(\mathcal{E}) \backslash f^{-1}(\mathcal{F})$.

Let $\{\mathcal{E}_\alpha\}_{\alpha \in \gamma}$ be any collection of subsets of $\mathcal{Y}$. By definition $x \in f^{-1}\left(\bigcup_{\alpha} \mathcal{E}_\alpha\right)$ if and only if $f(x) \in \bigcup_{\alpha} \mathcal{E}_\alpha$ or equivalently $f(x) \in \mathcal{E}_\alpha$ for some α. Now $f(x) \in \mathcal{E}_\alpha$ for some α if and only if $x \in f^{-1}(\mathcal{E}_\alpha)$ for some α which is the same as $x \in \bigcup_{\alpha} f^{-1}(\mathcal{E}_\alpha)$. Thus, $f^{-1}\left(\bigcup_{\alpha} \mathcal{E}_\alpha\right) = \bigcup_{\alpha} f^{-1}(\mathcal{E}_\alpha)$. Likewise, $f^{-1}\left(\bigcap_{\alpha} \mathcal{E}_\alpha\right) = \bigcap_{\alpha} f^{-1}(\mathcal{E}_\alpha)$ □

Proposition 3.8. *Let f be function defined on a set, $\mathcal{X}$, with values in a set $\mathcal{Y}$. Let $\boldsymbol{\mathcal{X}}$ be a σ-algebra of subsets of $\mathcal{X}$ and let $\boldsymbol{\mathcal{Y}} = \{\mathcal{E} \subseteq \mathcal{Y} : f^{-1}(\mathcal{E}) \in \boldsymbol{\mathcal{X}}\}$. Show that $\boldsymbol{\mathcal{Y}}$ is a σ-algebra.*

Proof. We know that the empty set is a subset of $\mathcal{Y}$ and from Proposition 3.7, $f^{-1}(\emptyset) = \emptyset \in \boldsymbol{\mathcal{X}}$. Thus, the empty set is in $\boldsymbol{\mathcal{Y}}$.

Let $\mathcal{E} \in \boldsymbol{\mathcal{Y}}$. From definition of $\boldsymbol{\mathcal{Y}}$, $\mathcal{E}$ is a subset of $\mathcal{Y}$ and $f^{-1}(\mathcal{E}) \in \boldsymbol{\mathcal{X}}$. Since $\mathcal{E}$ is a subset of $\mathcal{Y}$, then $(\mathcal{E})^c$ (the complement of $\mathcal{E}$) is a subset of $\mathcal{Y}$. Furthermore, $f^{-1}(\mathcal{E}) \in \boldsymbol{\mathcal{X}}$ implies that $\left[f^{-1}(\mathcal{E})\right]^c \in \boldsymbol{\mathcal{X}}$ From Proposition 3.7, $\left[f^{-1}(\mathcal{E})\right]^c = f^{-1}\left[(\mathcal{E})^c\right] \in \boldsymbol{\mathcal{X}}$. Thus, $(\mathcal{E})^c \in \boldsymbol{\mathcal{Y}}$.

To conclude, let $\{\mathcal{E}_\alpha\}_{\alpha\in\gamma}$ be a countable collection of subsets of $\boldsymbol{\mathcal{Y}}$. For each $\alpha \in \gamma$, $\mathcal{E}_\alpha \in \mathcal{Y}$ and $f^{-1}(\mathcal{E}_\alpha) \in \boldsymbol{\mathcal{X}}$. Thus $\bigcup_{\alpha\in\gamma} \mathcal{E}_\alpha \subseteq \mathcal{Y}$. Since $\boldsymbol{\mathcal{X}}$ is an σ-algebra , using Proposition 3.7 gives

$$f^{-1}\left(\bigcup_{\alpha\in\gamma} \mathcal{E}_\alpha\right) = \bigcup_{\alpha\in\gamma} f^{-1}(\mathcal{E}_\alpha) \in \boldsymbol{\mathcal{X}}.$$

Thus, $\bigcup_{\alpha\in\gamma} \mathcal{E}_\alpha \in \boldsymbol{\mathcal{Y}}$.

Therefore, we have demonstrated that $\boldsymbol{\mathcal{Y}}$ satisfies the definition of a σ-algebra. □

Proposition 3.9. *Let $a, b \in \mathbb{R}$ with $a < b$. Then:*

1. $[a, b] = \bigcap_{n=1}^{\infty} (a - 1/n, b + 1/n)$
2. $(a, b) = \bigcup_{n=1}^{\infty} [a + 1/n, b - 1/n]$
3. $[a, b) = \bigcap_{n=1}^{\infty} (a - 1/n, b)$
4. $(a, b] = \bigcap_{n=1}^{\infty} (a, b + 1/n)$

5. $(a\,,\infty) = \bigcup_{n=1}^{\infty} (a\,, b+n)$.

Proof.

1. By definition, an element x is in $[a\,, b]$ if and only if $a \leq x \leq b$. Then we certainly can say $a - 1/n < a \leq x \leq b < b + 1/n$ for all $n \in \mathbb{N}$. Therefore, $x \in (a - 1/n\,, b + 1/n)$ for all $n \in \mathbb{N}$. And now we can conclude $x \in \bigcap_{n=1}^{\infty} (a - 1/n\,, b + 1/n)$. Containment the other way follows easily.
2. An element x is in $(a\,, b)$ if and only if $a < x < b$ However, there exists $n \in \mathbb{N}$ such that $a < a + 1/n \leq x \leq b - 1/n < b$. Therefore, $x \in [a + 1/n\,, b - 1/n]$ for some $n \in \mathbb{N}$ which implies that $x \in \bigcup_{n=1}^{\infty} [a + 1/n\,, b - 1/n]$. Containment the other way follows.
3. (3), (4) and (5) follow similarly.

□

Definition 3.10. The Borel Algebra is the σ-algebra generated by all open intervals (a, b) in $\mathbb{R}$.

The next corollary shows that in Definition 3.10 the open intervals can be replaced by any one of the other types of intervals.

Corollary 3.11. *The Borel Algebra, $\mathcal{B}$, is also generated by all the closed intervals $[a\,, b]$, or by all the half open \ half closed intervals $[a\,, b)$, $(a\,, b]$, or by all the half-rays $(a\,, \infty)$.*

Proof. Proposition 3.9 shows that the intervals can be rewritten as the countable unions or intersections of open or closed intervals. The Corollary follows from the fact that σ-algebra s are closed under countable unions and intersections. □

In the following propositions the notion of the limit (if it exists) of sets will be formalized.

Proposition 3.12. *Let* $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ *be a sequence of subsets of a set* $\mathcal{X}$. *If* $\mathcal{A}$ *consists of all the* $x \in \mathcal{X}$ *which belong to infinitely many of the sets* $\mathcal{A}_n$, *then,*

$$\mathcal{A} = \bigcap_{m=1}^{\infty} \left[\bigcup_{n=m}^{\infty} \mathcal{A}_n \right].$$

$\mathcal{A}$ *is called the* $\limsup(\mathcal{A}_n)$.

Proof. By definition of $\mathcal{A}$: $x \in \mathcal{A}$ if and only if $x \in \bigcup_{n=m}^{\infty} \mathcal{A}_n$ for each $m \in \mathbb{N}$ if and only if $x \in \bigcap_{m=1}^{\infty} \left[\bigcup_{n=m}^{\infty} \mathcal{A}_n \right]$ □

Proposition 3.13. *Let* $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ *be a sequence of subsets of a set* $\mathcal{X}$. *If* $\mathcal{B}$ *consists of all the* $x \in \mathcal{X}$ *which belong to all but a finite number of sets* $\mathcal{A}_n$, *then*

$$\mathcal{B} = \bigcup_{m=1}^{\infty} \left[\bigcap_{n=m}^{\infty} \mathcal{A}_n \right].$$

$\mathcal{B}$ *is called the* $\liminf(\mathcal{A})$.

Proof. By definition of $\mathcal{B}$: $x \in \mathcal{B}$ if and only if $x \in \bigcap_{n=m_\circ} \mathcal{A}_n$ for some $m_\circ \in \mathbb{N}$ if and only if $x \in \bigcup_{m=1}^{\infty} \left[\bigcap_{n=m}^{\infty} \mathcal{A}_n \right]$. □

Proposition 3.14. *If* $\{\mathcal{E}_n\}$ *is sequence of subsets of a set* $\mathcal{X}$ *which is monotone increasing (that is,* $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \mathcal{E}_3 \subseteq \cdots$*), show that*

$$\limsup \mathcal{E}_n = \bigcup_{n=1}^{\infty} \mathcal{E}_n = \liminf \mathcal{E}_n$$

Proof. Clearly, $\bigcap_{m=1}^{\infty} \left[\bigcup_{n=m}^{\infty} \mathcal{E}_n \right] \subseteq \bigcup_{n=1}^{\infty} \mathcal{E}_n$. Now, $\bigcap_{m=1}^{\infty} \left[\bigcup_{n=m}^{\infty} \mathcal{E}_n \right] \supseteq \bigcup_{n=1}^{\infty} \mathcal{E}_n$ follows from the fact if x is in the countable union of $\mathcal{E}_n$'s, then there exists some $j \in \mathbb{N}$ such that x is in $\mathcal{E}_j$. Therefore,

$$x \in \bigcup_{n=1}^{\infty} \mathcal{E}_n \cap \bigcup_{n=2}^{\infty} \mathcal{E}_n \cap \bigcup_{n=3}^{\infty} \mathcal{E}_n \cap \cdots \cap \bigcup_{n=j}^{\infty} \mathcal{E}_n.$$

But, x an element of $\mathcal{E}_j$ implies that x is in $\mathcal{E}_j \cap \mathcal{E}_{j+1} \cap \mathcal{E}_{j+2} \cdots$ since $\{\mathcal{E}_n\}$ monotone increasing. Therefore, $x \in \bigcup_{n=j+1}^{\infty} \mathcal{E}_n \cap \bigcup_{n=j+2}^{\infty} \mathcal{E}_n \cap \cdots$. Whence,

$$x \in \bigcap_{m=1}^{\infty} \left[\bigcup_{n=m}^{\infty} \mathcal{E}_n \right].$$

Now, we can conclude $\bigcup_{n=1}^{\infty} \mathcal{E}_n = \limsup E$.

To demonstrate $\bigcup_{n=1}^{\infty} \mathcal{E}_n = \liminf \mathcal{E}_n$, we notice that in a monotone increasing function,

$$\bigcap_{n=m}^{\infty} \mathcal{E}_n = \mathcal{E}_m$$

It follows then that

$$\bigcup_{m=1}^{\infty} \left[\bigcap_{n=m}^{\infty} \mathcal{E}_n \right] = \bigcup_{m=1}^{\infty} \mathcal{E}_m = \bigcup_{n=1}^{\infty} \mathcal{E}_n.$$

Therefore, $\liminf \mathcal{E}_n = \bigcup_{n=1}^{\infty} \mathcal{E}_n = \limsup \mathcal{E}_n$. □

Proposition 3.15. *If $\{\mathcal{F}_n\}$ is sequence of subsets of a set $\mathcal{X}$ which is monotone decreasing (that is, $\mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \mathcal{F}_3 \supseteq \cdots$), show that*

$$\limsup \mathcal{F}_n = \bigcap_{n=1}^{\infty} \mathcal{F}_n = \liminf \mathcal{F}_n$$

Proof. In a monotone decreasing sequence, $\bigcup_{n=m}^{\infty} \mathcal{F}_n = \mathcal{F}_m$. Consequently,

$$\bigcap_{m=1}^{\infty} \left[\bigcup_{n=m}^{\infty} \mathcal{F}_n \right] = \bigcap_{m=1}^{\infty} \mathcal{F}_m = \bigcap_{n=1}^{\infty} \mathcal{F}_n.$$

Now to show

$$\bigcup_{m=1}^{\infty} \left[\bigcap_{n=m}^{\infty} \mathcal{F}_n \right] = \bigcap_{n=1}^{\infty} \mathcal{F}_n.$$

Let $x \in \bigcap_{n=1}^{\infty} \mathcal{F}_n$, then $x \in \bigcup_{m=1}^{\infty} \left[\bigcap_{n=m}^{\infty} \mathcal{F}_n \right]$. Now, take $x \in \bigcup_{m=1}^{\infty} \left[\bigcap_{n=m}^{\infty} \mathcal{F}_n \right]$. Thus for some $j \in \mathbb{N}$, $x \in \bigcap_{n=j}^{\infty} \mathcal{F}_n$. Since $\{\mathcal{F}_n\}$ is monotone decreasing

$x \in \mathcal{F}_{j-1} \cap \cdots \cap \mathcal{F}_1$. Consequently, $x \in \bigcap_{n=1}^{\infty} \mathcal{F}_n$. Therefore, we conclude that

$$\limsup \mathcal{F}_n = \bigcap_{n=1}^{\infty} \mathcal{F}_n = \liminf \mathcal{F}_n.$$

□

Proposition 3.16. *If $\{\mathcal{A}_n\}$ is a sequence of subsets of $\mathcal{X}$, then*

$$\emptyset \subseteq \liminf \mathcal{A}_n \subseteq \limsup \mathcal{A}_n \subseteq \mathcal{X}.$$

Proof. Clearly, $\emptyset \subseteq \liminf \mathcal{A}_n$ and $\limsup \mathcal{A}_n \subseteq \mathcal{X}$ are clear. Now, let x be in $\liminf \mathcal{A}_n$. Thus, there exists a $j \in \mathbb{N}$ such that $x \in \bigcap_{n=j}^{\infty} \mathcal{A}_n$. Therefore, we can say that

$$x \in \bigcup_{n=1}^{\infty} \mathcal{A}_n \cap \cdots \cap \bigcup_{n=j}^{\infty} \mathcal{A}_n \cap \cdots \cap \bigcup_{n=j+1}^{\infty} \mathcal{A}_n \cap \cdots .$$

Therefore, $x \in \limsup \mathcal{A}_n$. And thus, $\emptyset \subseteq \liminf \mathcal{A}_n \subseteq \limsup \mathcal{A}_n \subseteq \mathcal{X}$. □

Example 3.17. Let $\mathcal{X} = [0, 1)$. Define $\{\mathcal{A}_n\}$ a sequence in $\mathcal{X}$ as $\left[0, \frac{1}{n}\right)$ when n is odd and $\left[0, 1 - \frac{1}{n}\right]$ when n is even. Notice that for all even n, $\{\mathcal{A}_n\}$ is monotone increasing and $\{\mathcal{A}_n\}$ is monotone decreasing when n is odd. Therefore,

$$\bigcap_{m=1}^{\infty} \left[\bigcup_{n=m}^{\infty} \mathcal{A}_{2n} \right] = \bigcup_{n=1}^{\infty} \mathcal{A}_{2n} = [0, 1) = \mathcal{X}$$

by Proposition 3.14. Similarly,

$$\bigcup_{m=1}^{\infty} \left[\bigcap_{n=m}^{\infty} \mathcal{A}_{2n-1} \right] = \bigcap_{n=1}^{\infty} \mathcal{A}_{2n-1} = \emptyset$$

by Proposition 3.15. Since, $\{\mathcal{A}_{2n}\}$ and $\{\mathcal{A}_{2n-1}\}$ are subsequences of $\{\mathcal{A}_n\}$ with limits $\mathcal{X}$ and $\emptyset$, respectively then $\limsup A = \mathcal{X}$ and $\liminf A = \emptyset$.

Example 3.18. Let $\mathcal{X} = (-1, 1)$ and let $\{\mathcal{A}_n\} = \left(-\frac{1}{n}, 0\right]$ when n is even and $\{\mathcal{A}_n\} = \left[0, \frac{1}{n}\right)$ when n is odd. Arguing in a similar fashion as above we quickly see that $\limsup \mathcal{A}_n = \{0\} = \liminf \mathcal{A}_n$. Thus, the $\lim \mathcal{A}_n = \{0\}$.

Proposition 3.19. *If a,b,c are real numbers and mid(a,b,c) denotes the "value in the middle", then*

$$mid(a, b, c) = \inf\{\sup\{a, b\}, \sup\{a, c\}, \sup\{b, c\}\}.$$

In addition, if f_1, f_2, f_3 *are* $\boldsymbol{\mathcal{X}} -$ *measurable functions on* $\mathcal{X}$ *to* $\mathbb{R}$ *and if* g *is defined for* $x \in \mathcal{X}$ *by*

$$g(x) = mid\left(f_1(x), f_2(x), f_3(x)\right),$$

then g *is* $\boldsymbol{\mathcal{X}} -$ *measurable.*

Proof. By pairing each number with the other numbers and taking the sup of the three pairs, the largest number will appear twice and the next largest number will appear once. Now, taking the inf of these three numbers gives the second largest number of the trio which is exactly the "value in the middle".

To prove the second part, we must show that $\mathcal{A} = \{x \in \mathcal{X} : g(x) > \alpha\} \in \boldsymbol{\mathcal{X}}$, where $\alpha \in \mathbb{R}$. To do this we will show that $\mathcal{A}$ is obtained from the intersections and unions of $\boldsymbol{\mathcal{X}}$-measurable sets.

$$\{x \in \mathcal{X} : g(x) > \alpha\} = \{x \in \mathcal{X} : mid\left(f_1(x), f_2(x), f_3(x)\right) > \alpha\}$$

$$= \{x \in \mathcal{X} : f_1(x) > \alpha \text{ or } f_2(x) > \alpha\}$$

$$\cap \{x \in \mathcal{X} : f_1(x) > \alpha \text{ or } f_3(x) > \alpha\}$$

$$\cap \{x \in \mathcal{X} : f_2(x) > \alpha \text{ or } f_3(x) > \alpha\}$$

Now we see that

$$A = [\{x \in \mathcal{X} : f_1 > \alpha\} \cup \{x \in \mathcal{X} : f_2 > \alpha\}]$$

$$\cap [\{x \in \mathcal{X} : f_1 > \alpha\} \cup \{x \in \mathcal{X} : f_3 > \alpha\}]$$

$$\cap [\{x \in \mathcal{X} : f_2 > \alpha\} \cup \{x \in \mathcal{X} : f_3 > \alpha\}]$$

□

Proposition 3.20. *If f is a measurable function and $A > 0$, then the truncation f_A defined by*

$$f_A(x) = \begin{cases} f(x), & \text{if } |f(x)| \leq A \\ A, & \text{if } f(x) > A \\ -A, & \text{if } f(x) < -A \end{cases} \tag{3.1}$$

is measurable. (Show directly without using Proposition 3.19).

Proof. For $\alpha \geq A$,

$$\{x \in \mathcal{X} : f_A(x) > \alpha\} = \emptyset \in \boldsymbol{\mathcal{X}}.$$

For $\alpha < -A$,

$$\{x \in \mathcal{X} : f_A(x) > \alpha\} = \mathcal{X} \in \boldsymbol{\mathcal{X}}.$$

For $-A \leq \alpha < A$,

$$\{x \in \mathcal{X} : f_A(x) > \alpha\} = \{x \in \mathcal{X} : f(x) > \alpha\} \in \boldsymbol{\mathcal{X}}.$$

□

Remark 1. Notice that the above proposition could have been shown by defining f_1, f_2, f_3 as

$$f_1(x) = A, \;\; f_2(x) = f(x), \;\; f_3 = -A$$

Now, $f_A = \text{mid}\,(f_1, f_2, f_3)$ and thus by applying Proposition 3.19, we see that f_A is measurable.

Proposition 3.21. *Let f be a nonnegative $\boldsymbol{\mathcal{X}}$-measurable function on $\mathcal{X}$ which is bounded (that is, there exists a constant K such that $0 \leq f(x) \leq K$ for all $x \in \mathcal{X}$). Then, the sequence $\{\varphi_n\}$ defined in Lemma 2.11* [Bartle] *converges uniformly on $\mathcal{X}$ to f.*

Proof. From Lemma 2.11, $\varphi_n(x) = \frac{k}{2^n}$ for $x \in \mathcal{E}_{kn}$ where

$$\mathcal{E}_{kn} = \left\{x \in \mathcal{X} : \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}\right\}$$

if $k = 0, 1, \cdots, n2^n - 1$ and

$$\mathcal{E}_{kn} = \{x \in \mathcal{X} : f(x) \geq n\} \text{ if } k = n2^n.$$

Since $f(x) \leq K$ then there exists a smallest $n_\circ \in \mathbb{N}$ where $K < n_\circ$ such that for all $x \in \mathcal{X}$,

$$x \in \bigcup_{k=0}^{n_\circ 2^{n_\circ}} \mathcal{E}_{kn_\circ}$$

Thus, for all $x \in \mathcal{X}$ and for all $n \in \mathbb{N}$ where $n \geq n_\circ$, there exists $k \in \mathbb{N}$ such that

$$\frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}$$

which implies that $\left|f(x) - \frac{k}{2^n}\right| < \frac{1}{2^n}$. Therefore, given $\varepsilon > 0$ there exists an $n_1 \in \mathbb{N}$ such that

$$|f(x) - \varphi_n(x)| < \frac{1}{2^n} < \varepsilon \text{ for all } n \geq n_1 \geq n_0 \text{ and for all } x \in \mathcal{X}.$$

□

Proposition 3.22. *Let $(\mathcal{X}, \boldsymbol{\mathcal{X}})$ be a measurable space and f a function from $\mathcal{X}$ to $\mathcal{Y}$. Let $\boldsymbol{\mathcal{A}}$ be a collection of subsets of $\mathcal{Y}$ such that $f^{-1}(\mathcal{E}) \in \boldsymbol{\mathcal{X}}$ for every set $\mathcal{E} \in \boldsymbol{\mathcal{A}}$, then $f^{-1}(\mathcal{F}) \in \boldsymbol{\mathcal{X}}$ for any set $\mathcal{F}$ which belongs to the $\sigma-algebra$ generated by $\boldsymbol{\mathcal{A}}$.*

Proof. By assumption $\boldsymbol{\mathcal{A}} = \{\mathcal{E} \subseteq \mathcal{Y} : f^{-1}(\mathcal{E}) \in \boldsymbol{\mathcal{X}}\}$. From Proposition 3.8, $\boldsymbol{\mathcal{A}}$ is a σ-algebra . Thus, for any $\mathcal{F}$ in the σ-algebra generated by $\boldsymbol{\mathcal{A}}$, $\mathcal{F}$ is also in $\boldsymbol{\mathcal{A}}$ since the σ-algebra generated by $\boldsymbol{\mathcal{A}}$ is $\boldsymbol{\mathcal{A}}$. Therefore, $f^{-1}(\mathcal{F}) \in \boldsymbol{\mathcal{X}}$ for any set $\mathcal{F}$ that belongs to the σ-algebra generated by $\boldsymbol{\mathcal{A}}$. □

Proposition 3.23. *Let $(\mathcal{X}, \boldsymbol{\mathcal{X}})$ be a measurable space and f be a real valued function defined on $\mathcal{X}$. Then, f is measurable if and only if $f^{-1}(\mathcal{E}) \in \boldsymbol{\mathcal{X}}$ for every Borel set $\mathcal{E}$.*

Proof. First, assume $f^{-1}(\mathcal{E}) \in \boldsymbol{\mathcal{X}}$ for every Borel set $\mathcal{E}$. Let $\mathcal{A}_\alpha = \{x \in \mathcal{X} : f(x) > \alpha\} = f^{-1}(\alpha, \infty)$ where $\alpha \in \mathbb{R}$. Now $(\alpha, \infty) = \bigcup_{n=1}^{\infty} (\alpha, \alpha + n)$. But $(\alpha, \alpha + n)$ is a Borel set and thus by assumption $f^{-1}(\alpha, \alpha + n) \in \boldsymbol{\mathcal{X}}$ and therefore,

$$f^{-1}(\alpha,\infty) = f^{-1}\left[\bigcup_{n=1}^{\infty}(\alpha,\alpha+n)\right] = \bigcup_{n=1}^{\infty} f^{-1}(\alpha,\alpha+n) \in \boldsymbol{\mathcal{X}}.$$

Consequently, since α was arbitrary, $\mathcal{A}_\alpha \in \boldsymbol{\mathcal{X}}$ for all $\alpha \in \mathbb{R}$.

Now, assume f is measurable. Let $\boldsymbol{\mathcal{A}} = \{\mathcal{E} \subseteq \mathbb{R} : f^{-1}(\mathcal{E}) \in \boldsymbol{\mathcal{X}}\}$. From Proposition 3.8, $\boldsymbol{\mathcal{A}}$ is a σ-algebra . Since f measurable, $f^{-1}(\alpha,\infty) \in \boldsymbol{\mathcal{X}}$ for all $\alpha \in \mathbb{R}$. It is seen that for all open intervals in $\mathbb{R}$, $\mathcal{O}$, $f^{-1}(\mathcal{O}) \in \boldsymbol{\mathcal{X}}$, thus by Proposition 3.22, $\mathcal{O} \in \boldsymbol{\mathcal{A}}$. Let $\boldsymbol{\mathcal{O}}$ denote the collection of all open intervals in $\mathbb{R}$, therefore $\boldsymbol{\mathcal{O}} \subseteq \boldsymbol{\mathcal{A}}$. Now, the σ-algebra generated by $\boldsymbol{\mathcal{O}}$, $\boldsymbol{\mathcal{B}}$ (the Borel sets), is a subset of $\boldsymbol{\mathcal{A}}$. Thus, $f^{-1}(\mathcal{E}) \in \boldsymbol{\mathcal{X}}$ for all $\mathcal{E} \in \boldsymbol{\mathcal{B}}$. □

The following fact will be useful for the next proposition.

Fact 3.24. *A continuous function from $\mathbb{R}$ to $\mathbb{R}$ guarantees that for each open interval in its range there is a corresponding open interval in its domain. To see this apply the definition of continuity. For each $p \in \mathbb{R}$ and for $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x) \in (f(p)-\varepsilon, f(p)+\varepsilon)$ for each $x \in (p-\delta, p+\delta)$. Now since each Borel set can be written as the countable union of open intervals then for each Borel set in the range of a continuous function the inverse mapping of that set is a countable union of open intervals and thus a Borel set.*

Proposition 3.25. *If $(\mathcal{X}, \boldsymbol{\mathcal{X}})$ is a measurable space, f is a $\boldsymbol{\mathcal{X}}-$measurable function on $\mathcal{X}$ to $\mathbb{R}$ and φ is a continuous function on $\mathbb{R}$ to $\mathbb{R}$, then the composition $(\varphi \circ f)$ defined by $(\varphi \circ f)(x) = \varphi[f(x)]$, is $\mathcal{X}$-measurable.*

Proof. By assumption φ is continuous, thus $\varphi^{-1}(\mathcal{E}) \in \boldsymbol{\mathcal{B}}$ for each $\mathcal{E} \in \boldsymbol{\mathcal{B}}$ by Fact 3.24. Now f is measurable therefore Proposition 3.23 gives $f^{-1}[\mathcal{E}] \in \boldsymbol{\mathcal{X}}$ for all Borel sets $\mathcal{E}$. Since $\varphi^{-1}(\mathcal{E})$ is a Borel set for all Borel sets $\mathcal{E}$, $f^{-1}[\varphi^{-1}(x)] \in \boldsymbol{\mathcal{X}}$ for each Borel set, $\mathcal{E}$. Thus, by Proposition 3.23 $(\varphi \circ f)(x)$ is $\boldsymbol{\mathcal{X}}$-measurable . □

Lemma 3.26. *If f is a $\boldsymbol{\mathcal{X}}-$measurable and ψ is a Borel measurable function, then $(\psi \circ f)$ is $\boldsymbol{\mathcal{X}}-$ measurable.*

Proof. From assumption, ψ Borel measurable therefore $\psi^{-1}(\mathcal{E}) \in \boldsymbol{\mathcal{B}}$ for each $\mathcal{E} \in \boldsymbol{\mathcal{B}}$. By Proposition 3.23, f measurable implies that $f^{-1}[\psi^{-1}(\mathcal{E})] \in \boldsymbol{\mathcal{X}}$ and therefore $(\psi \circ f)$ is $\boldsymbol{\mathcal{X}}$-measurable since $f^{-1}[\psi^{-1}(\mathcal{E})] \in \boldsymbol{\mathcal{X}}$ for every Borel set $\mathcal{E}$. □

Proposition 3.27. *A function f on $\mathcal{X}$ to $\mathbb{R}$ is $\boldsymbol{\mathcal{X}}-$ measurable if and only if the set $\mathcal{A}_\alpha$ in Lemma 2.4(a)* [Bartle] *belongs to $\boldsymbol{\mathcal{X}}$ for each rational number α.*

Proof. If f is $\boldsymbol{\mathcal{X}}$-measurable then from the definition of an $\boldsymbol{\mathcal{X}}$-measurable function, $\mathcal{A}_\alpha = \{x \in \mathcal{X} : f(x) > \alpha\}$ belongs to $\boldsymbol{\mathcal{X}}$ for all $\alpha \in \mathbb{R}$.

If $\mathcal{A}_\alpha \in \boldsymbol{\mathcal{X}}$ for all $\alpha \in \mathbb{Q}$, then for each $\widetilde{\alpha} \in \mathbb{R}\backslash\mathbb{Q}$ there exists a $\{\alpha_n\} \in \mathbb{Q}$ such that

$$\mathcal{A}_{\widetilde{\alpha}} = \{x \in X : f(x) > \widetilde{\alpha}\} = \bigcup_{n=1}^{\infty} \{x \in X : f(x) > \alpha_n\}.$$

□

Definition 3.28. A nonempty collection $\boldsymbol{\mathcal{M}}$ of subsets of a set $\mathcal{X}$ is called a monotone class if, for each monotone increasing sequence $\{\mathcal{E}_n\}$ in $\boldsymbol{\mathcal{M}}$ and each monotone decreasing sequence $\{\mathcal{F}_n\}$ in $\boldsymbol{\mathcal{M}}$, the sets

$$\bigcup_{n=1}^{\infty} \mathcal{E}_n, \quad \bigcap_{n=1}^{\infty} \mathcal{F}_n$$

belong to $\boldsymbol{\mathcal{M}}$.

Proposition 3.29. *A σ-algebra is a monotone class.*

Proof. Let $\boldsymbol{\mathcal{X}}$ be a σ-algebra of $\mathcal{X}$ and let $\{\mathcal{E}_n\}$ be an increasing sequence of sets in $\boldsymbol{\mathcal{X}}$. By definition of $\boldsymbol{\mathcal{X}}$, $\bigcup_{n=1}^{\infty} \mathcal{E}_n \in \boldsymbol{\mathcal{X}}$ for all $n \in \mathbb{N}$. Now let $\{\mathcal{F}_n\}$ be a decreasing sequence of sets in $\boldsymbol{\mathcal{X}}$, thus $(\mathcal{F}_n)^c \in \boldsymbol{\mathcal{X}}$ for each $n \in \mathbb{N}$.

Therefore,

$$\bigcup_{n=1}^{\infty} (\mathcal{F}_n)^c \in \boldsymbol{\mathcal{X}}, \text{ by definition of } \boldsymbol{\mathcal{X}}.$$

Whence

$$\left[\bigcup_{n=1}^{\infty} (\mathcal{F}_n)^c\right]^c \in \boldsymbol{\mathcal{X}}, \text{ by definition of } \boldsymbol{\mathcal{X}}.$$

Applying DeMorgan's Laws,

$$\bigcap_{n=1}^{\infty} \mathcal{F}_n = \bigcap_{n=1}^{\infty} [(\mathcal{F}_n)^c]^c = \left[\bigcup_{n=1}^{\infty} (\mathcal{F}_n)^c\right]^c \in \boldsymbol{\mathcal{X}}.$$

Therefore, $\boldsymbol{\mathcal{X}}$ is a monotone class. □

Proposition 3.30. *If $\mathcal{A}$ is a nonempty collection of subsets of $\mathcal{X}$, then there is a smallest monotone class containing $\mathcal{A}$.*

Proof. Let $\mathfrak{F}$ be the family of all monotone classes of $\mathcal{X}$ that contain $\mathcal{A}$. Let $\boldsymbol{\mathcal{M}} = \bigcap\{\mathcal{B} : \mathcal{B} \in \mathfrak{F}\}$. By definition of $\boldsymbol{\mathcal{M}}$, $\boldsymbol{\mathcal{M}} \supseteq \mathcal{A}$. Now let $\{\mathcal{E}_n\}$ be an increasing sequence of sets in $\boldsymbol{\mathcal{M}}$. Therefore, $\{\mathcal{E}_n\} \in \mathcal{B}$ for each $\mathcal{B} \in \mathfrak{F}$. For all $\mathcal{B} \in \mathfrak{F}$, $\bigcup_{n=1}^{\infty} \mathcal{E}_n \in \mathcal{B}$ since for $\mathcal{B} \in \mathfrak{F}$, $\mathcal{B}$ is a monotone class. Thus $\bigcup_{n=1}^{\infty} \mathcal{E}_n \in \boldsymbol{\mathcal{M}}$. Similarly $\bigcap_{n=1}^{\infty} \mathcal{F}_n \in \boldsymbol{\mathcal{M}}$ for $\{\mathcal{F}_n\}$ a decreasing sequence in $\boldsymbol{\mathcal{M}}$. As a result, $\boldsymbol{\mathcal{M}}$ is a monotone class and if $\mathcal{B}$ is a monotone class containing $\mathcal{A}$ then $\mathcal{B} \supseteq \boldsymbol{\mathcal{M}}$. □

4. Measures

Definition 4.1. A **measure** is an extended real-valued function μ defined on a σ-algebra $\boldsymbol{\mathcal{X}}$ of subsets of $\mathcal{X}$ such that

(i) $\mu(\emptyset) = 0$

(ii) $\mu(\mathcal{E}) \geq 0$ for all $\mathcal{E} \in \boldsymbol{\mathcal{X}}$

(iii) μ is countably additive in the sense that if $\{\mathcal{E}_n\}$ is any disjoint sequence of sets in $\boldsymbol{\mathcal{X}}$, then

$$\mu\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n\right) = \sum_{n=1}^{\infty} \mu\left(\mathcal{E}_n\right). \tag{4.1}$$

Proposition 4.2. *If μ is a measure on $\boldsymbol{\mathcal{X}}$ and $\mathcal{A}$ is a fixed set in $\boldsymbol{\mathcal{X}}$, then the function λ, defined for $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ by $\lambda(\mathcal{E}) = \mu(\mathcal{A} \cap \mathcal{E})$ is a measure on $\boldsymbol{\mathcal{X}}$.*

Proof. First, $(\mathcal{A} \cap \emptyset) = \emptyset$. Therefore Condition (i) of Definition 4.1 is satisfied since,

$$\lambda(\emptyset) = \mu(\mathcal{A} \cap \emptyset) = \mu(\emptyset) = 0.$$

Next, for all $\mathcal{E} \in \boldsymbol{\mathcal{X}}$, $(\mathcal{A} \cap \mathcal{E}) \in \boldsymbol{\mathcal{X}}$ since $\boldsymbol{\mathcal{X}}$ is a σ-algebra . Thus, condition (ii) of Definition 4.1 is satisfied since

$$\lambda(\mathcal{E}) = \mu\left(\mathcal{A} \cap \mathcal{E}\right) \geq 0 \text{ for all } \mathcal{E} \in \boldsymbol{\mathcal{X}}.$$

Lastly, let $\{\mathcal{E}_n\}$ be a disjoint sequence in $\boldsymbol{\mathcal{X}}$. Then $(\mathcal{E}_i \cap \mathcal{A}) \cap (\mathcal{E}_j \cap \mathcal{A}) = \emptyset$ for all $i \neq j$. Also,

$$\left(\mathcal{A} \cap \bigcup_{n=1}^{\infty} \mathcal{E}_n\right) = \bigcup_{n=1}^{\infty} \left(\mathcal{A} \cap \mathcal{E}_n\right).$$

Therefore we can say,

$$\lambda\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n\right) = \mu\left(\bigcup_{n=1}^{\infty} \left[\mathcal{A} \cap \mathcal{E}_n\right]\right) = \sum_{n=1}^{\infty} \mu\left(\mathcal{A} \cap \mathcal{E}_n\right) = \sum_{n=1}^{\infty} \lambda\left(\mathcal{E}_n\right).$$

Thus λ satisfies Conditions (i), (ii) and (iii) of Definition 4.1. □

The following fact, from elementary Calculus, will be useful in the proof of the next two propositions.

Fact 4.3. *If* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent series of real numbers, then*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

Thus, if $\sum_{n=1}^{\infty} a_n^j$ *are a convergent series of real numbers for* $1 \leq j \leq N$, *then*

$$\sum_{n=1}^{\infty}\sum_{j=1}^{N} a_n^j = \sum_{j=1}^{N}\sum_{n=1}^{\infty} a_n^j.$$

Proposition 4.4. *If* $\mu_1, \cdots, \mu_n$ *are measures on* $\boldsymbol{\mathcal{X}}$ *and* $a_1, \cdots, a_n$ *are nonnegative real numbers, then the function* λ, *defined for* $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ *by*

$$\lambda(\mathcal{E}) = \sum_{j=1}^{n} a_j \mu_j(\mathcal{E}) \tag{4.2}$$

is a measure on $\boldsymbol{\mathcal{X}}$.

Proof. Conditions (i) and (ii) follow from the fact that $\mu_1, \cdots, \mu_n$ are measures on $\boldsymbol{\mathcal{X}}$ and that $a_1, \cdots, a_n$ are nonnegative real numbers. By plugging $\bigcup_{m=1}^{\infty} \mathcal{E}_m$ into (4.2) where $\{E_m\}$ are disjoint and by Fact 4.3 we get

$$\lambda\left(\bigcup_{m=1}^{\infty} \mathcal{E}_m\right) = \sum_{j=1}^{n} a_j \sum_{m=1}^{\infty} \mu_j(\mathcal{E}_m) = \sum_{j=1}^{n}\sum_{m=1}^{\infty} a_j\mu_j(\mathcal{E}_m) = \sum_{m=1}^{\infty}\sum_{j=1}^{n} a_j\mu_j(\mathcal{E}_m).$$

Therefore, λ is a measure. □

Proposition 4.5. *If* $\{\mu_n\}$ *is a sequence of measures on* $\boldsymbol{\mathcal{X}}$ *with* $\mu_n(\mathcal{X}) = 1$ *and if* λ *is defined by*

$$\lambda(\mathcal{E}) = \sum_{n=1}^{\infty} 2^{-n}\mu_n(\mathcal{E}), \mathcal{E} \in \boldsymbol{\mathcal{X}}. \tag{4.3}$$

then λ *is a measure on* $\boldsymbol{\mathcal{X}}$ *and* $\lambda(\mathcal{X}) = 1$.

Proof. Clearly (i) and (ii) hold for Definition 4.1. To show

$$\lambda\left(\bigcup_{j=1}^{\infty} \mathcal{E}_n\right) = \sum_{j=1}^{\infty} \lambda(\mathcal{E}_j)$$

where $\{\mathcal{E}_j\}$ are disjoint, we will show that the left hand side is greater than the right hand side then we will show that the right hand side is greater than the left hand side which implies equality.

Let $\{\mathcal{E}_j\}$ be a disjoint sequence from $\boldsymbol{\mathcal{X}}$. Fix $N \in \mathbb{N}$. Then by Fact 4.3

$$\sum_{j=1}^{N} \lambda\left(\mathcal{E}_j\right) = \sum_{j=1}^{N}\sum_{n=1}^{\infty} \frac{\mu_n\left(\mathcal{E}_j\right)}{2^n} = \sum_{n=1}^{\infty}\sum_{j=1}^{N} \frac{\mu_n\left(\mathcal{E}_j\right)}{2^n}.$$

Now for each n, μ_n is a measure, therefore,

$$\sum_{n=1}^{\infty}\sum_{j=1}^{N} \frac{\mu_n\left(\mathcal{E}_j\right)}{2^n} = \sum_{n=1}^{\infty} \frac{\mu_n\left(\bigcup_{j=1}^{N} \mathcal{E}_j\right)}{2^n} \leq \sum_{n=1}^{\infty} \frac{\mu_n\left(\bigcup_{j=1}^{\infty} \mathcal{E}_j\right)}{2^n} = \lambda\left(\bigcup_{j=1}^{\infty} \mathcal{E}_j\right).$$

Letting N go to infinity we see that

$$\sum_{j=1}^{\infty} \lambda\left(\mathcal{E}_j\right) \leq \lambda\left(\bigcup_{j=1}^{\infty} \mathcal{E}_j\right).$$

Now fix $\varepsilon > 0$. Since μ_n is a measure for each n and $\mu_n(\mathcal{X}) = 1$ for all n there exists a $N(\varepsilon) \in \mathbb{N}$ such that

$$\lambda\left(\bigcup_{j=1}^{\infty} \mathcal{E}_j\right) - \varepsilon = \sum_{n=1}^{\infty} \frac{\mu_n\left(\bigcup_{j=1}^{\infty} \mathcal{E}_j\right)}{2^n} - \varepsilon \leq \sum_{n=1}^{N(\varepsilon)} \frac{\mu_n\left(\bigcup_{j=1}^{\infty} \mathcal{E}_j\right)}{2^n}.$$

By Proposition 4.4,

$$\sum_{n=1}^{N(\varepsilon)} \frac{\mu_n\left(\bigcup_{j=1}^{\infty} \mathcal{E}_j\right)}{2^n} = \sum_{j=1}^{\infty}\sum_{n=1}^{N(\varepsilon)} \frac{\mu_n\left(\mathcal{E}_j\right)}{2^n} \leq \sum_{j=1}^{\infty}\sum_{n=1}^{\infty} \frac{\mu_n\left(\mathcal{E}_j\right)}{2^n} = \sum_{j=1}^{\infty} \lambda\left(\mathcal{E}_j\right).$$

Since ε was arbitrary

$$\lambda\left(\bigcup_{j=1}^{\infty} \mathcal{E}_j\right) \leq \sum_{j=1}^{\infty} \lambda\left(\mathcal{E}_j\right).$$

This concludes the proof. $\square$

Proposition 4.6. *Let $\mathcal{X} = \mathbb{N}$ and let $\boldsymbol{\mathcal{X}}$ be the σ-algebra of all subsets of $\mathbb{N}$. If $\{\alpha_n\}$ is a sequence of nonnegative real numbers and if we define $\mu : \mathcal{X} \to \bar{\mathbb{R}}$ by*

$$\mu(\emptyset) = 0; \;\; \mu(\mathcal{E}) = \sum_{n\in\mathcal{E}} \alpha_n, \;\; \mathcal{E} \neq \emptyset,$$

then μ is a measure.

Proof. Conditions (i) and (ii) are obvious. Let $\{\mathcal{E}_m\}$ be a disjoint sequence in $\boldsymbol{\mathcal{X}}$. If $\mu\left(\bigcup_{m=1}^{\infty}\mathcal{E}_m\right)=\infty$ then done. If this is not the case, then

$$\sum_{n\in\bigcup_{m=1}^{\infty}\mathcal{E}_m}\alpha_n$$

is unconditionally convergent since α_n is nonnegative for each n. (In other words, we can rearrange the order of the series). Therefore,

$$\begin{aligned}\mu\left(\bigcup_{m=1}^{\infty}\mathcal{E}_m\right) &= \sum_{n\in\bigcup_{m=1}^{\infty}\mathcal{E}_m}\alpha_n = \sum_{n\in\mathcal{E}_1\cup\mathcal{E}_2\cup\cdots}\alpha_n \\ &= \sum_{n\in\mathcal{E}_1}\alpha_n + \sum_{n\in\mathcal{E}_2}\alpha_n + \cdots = \sum_{m=1}^{\infty}\mu\left(\mathcal{E}_m\right).\end{aligned}$$

Thus Condition (iii) is satisfied. □

Proposition 4.7. *If $\mathcal{X}$ is an uncountable set and if $\boldsymbol{\mathcal{X}}$ is the family of all subsets of $\mathcal{X}$, then μ on $\mathcal{E}$ in $\boldsymbol{\mathcal{X}}$ defined by*

$$\mu(\mathcal{E}) = 0 \textit{ if } \mathcal{E} \textit{ is countable}$$

$$\mu(\mathcal{E}) = \infty \textit{ if } \mathcal{E} \textit{ is uncountable.}$$

is a measure on $\boldsymbol{\mathcal{X}}$.

Proof. Condition (i) holds since the $\emptyset$ is countable, therefore $\mu(\emptyset)=0$. Condition (ii) follows directly from the definition of μ. For Condition (iii), let $\{\mathcal{E}_n\}$ be a disjoint sequence in $\boldsymbol{\mathcal{X}}$. If for all $n\in\mathbb{N}$, $\mathcal{E}_n$ countable, then $\bigcup_{n=1}^{\infty}\mathcal{E}_n$ is countable and

$$\mu\left(\bigcup_{n=1}^{\infty}\mathcal{E}_n\right) = 0 = \sum_{n=1}^{\infty}\mu\left(\mathcal{E}_n\right).$$

If for any $n\in\mathbb{N}$, $\mathcal{E}_n$ is uncountable, then $\bigcup_{n=1}^{\infty}\mathcal{E}_n$ is uncountable and thus

$$\mu\left(\bigcup_{n=1}^{\infty}\mathcal{E}_n\right) = \infty = \sum_{n=1}^{\infty}\mu\left(\mathcal{E}_n\right).$$

□

Proposition 4.8. *Let $\mathcal{X} = \mathbb{N}$ and let $\boldsymbol{\mathcal{X}}$ be the family of all subsets of $\mathbb{N}$. If $\mathcal{E}$ is finite, let $\mu(\mathcal{E}) = 0$: if $\mathcal{E}$ is infinite, let $\mu(\mathcal{E}) = \infty$. Then, μ is not a measure on $\boldsymbol{\mathcal{X}}$.*

Proof. Let $\{\mathcal{E}_n\}$ be a disjoint sequence in $\boldsymbol{\mathcal{X}}$ where for all $n \in \mathbb{N}$, $\mathcal{E}_n$ is finite. This means that $\bigcup_{n=1}^{\infty} \mathcal{E}_n$ is infinite and therefore $\mu\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n\right) = \infty$. However $\mu\left(\mathcal{E}_n\right) = 0$ for all $n \in \mathbb{N}$ and thus $\sum_{n=1}^{\infty} \mu\left(\mathcal{E}_n\right) = 0$. □

Proposition 4.9. *Let $(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ be a measurable space and let $\{\mathcal{E}_n\}$ be a sequence in $\boldsymbol{\mathcal{X}}$. Then*

$$\mu\left(\liminf \mathcal{E}_n\right) \leq \liminf \mu\left(\mathcal{E}_n\right). \tag{4.4}$$

Proof. Let $\{\mathcal{E}_n\}$ be a sequence in $\boldsymbol{\mathcal{X}}$. It is clear that $\left\{\bigcap_{n=m}^{\infty} \mathcal{E}_n\right\}_{m=1}^{\infty}$ is an increasing sequence. Therefore by Lemma 3.4(a) [Bartle],

$$\mu\left(\bigcup_{m=1}^{\infty}\left[\bigcap_{n=m}^{\infty} \mathcal{E}_n\right]\right) = \lim_{m\to\infty} \mu\left(\bigcap_{n=m}^{\infty} \mathcal{E}_n\right). \tag{4.5}$$

Moreover, for all $n, m \in \mathbb{N}$ such that $n \geq m$, $\mathcal{E}_n \supseteq \bigcap_{n=m}^{\infty} \mathcal{E}_n$. Thus by Lemma (3.3) [Bartle] $\mu\left(\bigcap_{n=m}^{\infty} \mathcal{E}_n\right) \leq \mu\left(\mathcal{E}_n\right)$ for all $n \geq m$ which implies that

$$\lim_{m\to\infty} \mu\left(\bigcap_{n=m}^{\infty} \mathcal{E}_n\right) \leq \liminf \mu\left(\mathcal{E}_n\right). \tag{4.6}$$

Combining (4.5) and (4.6) gives (4.4). □

Proposition 4.10. *If $\mu\left(\bigcup \mathcal{E}_n\right) < \infty$, then*

$$\limsup \mu\left(\mathcal{E}_n\right) \leq \mu\left(\limsup \mathcal{E}_n\right). \tag{4.7}$$

Proof. Let $\{\mathcal{E}_n\}$ be a sequence such that $\bigcup_{n=1}^{\infty} \mathcal{E}_n < \infty$. Notice that $\{\bigcup_{n=m}^{\infty} \mathcal{E}_n\}_{m=1}^{\infty}$ is a decreasing sequence. Thus, by Lemma 3.4(b) EOI,

$$\mu\left(\bigcap_{m=1}^{\infty}\left[\bigcup_{n=m}^{\infty} \mathcal{E}_n\right]\right) = \lim_{m\to\infty} \mu\left(\bigcup_{n=m}^{\infty} \mathcal{E}_n\right). \tag{4.8}$$

Moreover, for all $n, m \in \mathbb{N}$ with $n \geq m$, $\mathcal{E}_n \subseteq \bigcup_{n=m}^{\infty} \mathcal{E}_n$ which implies that

$$\limsup \mu\left(\mathcal{E}_n\right) \leq \lim_{m\to\infty} \mu\left(\bigcup_{n=m}^{\infty} \mathcal{E}_n\right). \tag{4.9}$$

Combining (4.8) and (4.9) gives (4.7). □

Proposition 4.11. *If* $(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ *a measure space and* $\{\mathcal{E}_n\}$ *is a sequence in* $\boldsymbol{\mathcal{X}}$*, then*

$$\mu\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n\right) \leq \sum_{n=1}^{\infty} \mu\left(\mathcal{E}_n\right) \tag{4.10}$$

Proof. Let $\mathcal{F}_n = \mathcal{E}_n \backslash \bigcup_{j=1}^{n-1} \mathcal{E}_j$ when $n > 1$ and $\mathcal{E}_1 = \mathcal{F}_1$ (the disjunctification of $\{\mathcal{E}_n\}$). From Proposition 3.6, we know that $\bigcup_{n=1}^{\infty} \mathcal{E}_n = \bigcup_{n=1}^{\infty} \mathcal{F}_n$ therefore

$$\mu\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n\right) = \sum_{n=1}^{\infty} \mu\left(\mathcal{F}_n\right) \leq \sum_{n=1}^{\infty} \mu\left(\mathcal{E}_n\right),$$

since $\mathcal{F}_n \subseteq \mathcal{E}_n$ for each n. □

Proposition 4.12. *Let* $(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ *be a measure space and let*

$$\boldsymbol{\mathcal{Z}} = \{\mathcal{E} \in \boldsymbol{\mathcal{X}} : \mu(\mathcal{E}) = 0\}.$$

Then $\boldsymbol{\mathcal{Z}}$ *is not a* σ*-algebra .*

Proof. Let $\mathcal{E}$ be in $\boldsymbol{\mathcal{Z}}$, thus $\mathcal{X} \backslash \mathcal{E} \in \boldsymbol{\mathcal{X}}$. If $\mu(\mathcal{X}) = 0$, then $(\mathcal{E})^c \in \boldsymbol{\mathcal{Z}}$. However, in general $(\mathcal{E})^c \notin \boldsymbol{\mathcal{Z}}$ since $\mu(\mathcal{X} \backslash \mathcal{E}) = \mu(\mathcal{X}) - \mu(\mathcal{E})$. □

Though $\boldsymbol{\mathcal{Z}}$ is not a σ-algebra Proposition 4.13 will show that it does have some important properties that will be useful in the completion of $\boldsymbol{\mathcal{X}}$.

Proposition 4.13.

(i) *Let* $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ *and let* $\mathcal{F} \in \boldsymbol{\mathcal{Z}}$*, then* $\mathcal{E} \cap \mathcal{F} \in \boldsymbol{\mathcal{Z}}$

(ii) *Let* $\mathcal{E}_n$ *be in* $\boldsymbol{\mathcal{Z}}$ *for* $n \in \mathbb{N}$*, then* $(\bigcup_{n=1}^{\infty} \mathcal{E}_n) \in \boldsymbol{\mathcal{Z}}$*.*

Proof. To see (i) let $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ and let $\mathcal{F} \in \boldsymbol{\mathcal{Z}}$, then $\mathcal{E} \cap \mathcal{F} \in \boldsymbol{\mathcal{X}}$. Moreover, $\mu\left(\mathcal{E} \cap \mathcal{F}\right) \leq \mu\left(\mathcal{E}\right) = 0$. Therefore, $\mathcal{E} \cap \mathcal{F} \in \boldsymbol{\mathcal{Z}}$.

To see (ii) let $\mathcal{E}_n$ be in $\boldsymbol{\mathcal{Z}}$ for $n \in \mathbb{N}$ then $\mu(\mathcal{E}_n) = 0$ for each $n \in \mathbb{N}$. Thus $\sum_{n=1}^{\infty} \mu\left(\mathcal{E}_n\right) = 0$. But by Proposition 4.11,

$$\mu\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n\right) \leq \sum_{n=1}^{\infty} \mu\left(\mathcal{E}_n\right) = 0.$$

Consequently, $(\bigcup_{n=1}^{\infty} \mathcal{E}_n) \in \boldsymbol{\mathcal{Z}}$.

□

Proposition 4.14. *Let $\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu,$ and $\boldsymbol{\mathcal{Z}}$ be as in Proposition 4.12. Let $\boldsymbol{\mathcal{X}}'$ be the family of all subsets of $\mathcal{X}$ of the form*

$$(\mathcal{E} \cup Z_1) \setminus Z_2, \;\; \mathcal{E} \in \boldsymbol{\mathcal{X}} \tag{4.11}$$

where Z_1 and Z_2 are arbitrary subsets of sets belonging to $\boldsymbol{\mathcal{Z}}$. A set is in $\boldsymbol{\mathcal{X}}'$ if and only if it has the form $\widetilde{\mathcal{E}} \cup \widetilde{Z}$ where $\widetilde{\mathcal{E}} \in \boldsymbol{\mathcal{X}}$ and $\widetilde{Z}$ is a subset of a set in $\boldsymbol{\mathcal{Z}}$.

Proof. Let $\mathcal{F}$ be set in $\boldsymbol{\mathcal{X}}$'. Therefore, $\mathcal{F}$ has the form of (4.11). Let $\mathcal{A}$ be the set in $\boldsymbol{\mathcal{Z}}$ such that $Z_1 \subseteq \mathcal{A}$ and let $\mathcal{B}$ be the set in $\boldsymbol{\mathcal{Z}}$ such that $Z_2 \subseteq \mathcal{B}$. Thus,

$$\mathcal{F} = (\mathcal{E} \cup Z_1) \setminus Z_2 = \mathcal{E} \setminus Z_2 \cup Z_1 \setminus Z_2.$$

where $Z_1 \subseteq \mathcal{A} \in \boldsymbol{\mathcal{X}}$ and $Z_2 \subseteq \mathcal{B} \in \boldsymbol{\mathcal{X}}$. Now $\mathcal{E} \setminus Z_2 = (\mathcal{E} \setminus \mathcal{B}) \cup (\mathcal{E} \cap \mathcal{B} \setminus Z_2)$. Therefore

$$\mathcal{E} \setminus Z_2 \cup Z_1 \setminus Z_2 = \mathcal{E} \setminus \mathcal{B} \cup [\mathcal{E} \cap \mathcal{B} \setminus Z_2] \cup [Z_1 \setminus Z_2].$$

Notice that $\mathcal{E} \setminus \mathcal{B} \in \boldsymbol{\mathcal{X}}$, therefore let $\mathcal{E} \setminus \mathcal{B} = \widetilde{\mathcal{E}}$. Observing that $(\mathcal{E} \cap \mathcal{B} \setminus Z_2) \subseteq \mathcal{B} \in \boldsymbol{\mathcal{Z}}$ and $(Z_1 \setminus Z_2) \subseteq \mathcal{A} \in \boldsymbol{\mathcal{Z}}$, we let $\widetilde{\mathcal{Z}} = [\mathcal{E} \cap \mathcal{B} \setminus Z_2] \cup [Z_1 \setminus Z_2]$. Therefore, $\mathcal{F} = (\mathcal{E} \cup Z_1) \setminus Z_2 = \widetilde{\mathcal{E}} \cup \widetilde{\mathcal{Z}}$, where $\widetilde{\mathcal{E}} \in \boldsymbol{\mathcal{X}}$ and $\widetilde{\mathcal{Z}}$ is a subset of a set in $\boldsymbol{\mathcal{Z}}$.

To conclude, $\widetilde{\mathcal{E}} \cup \widetilde{\mathcal{Z}} = \left(\widetilde{\mathcal{E}} \cup \widetilde{\mathcal{Z}}\right) \setminus \emptyset$, which shows the other implication. □

Corollary 4.15. *As defined in Proposition 4.14, $\boldsymbol{\mathcal{X}}'$ is a σ-algebra .*

Proof. Obviously, $\emptyset$ is in $\boldsymbol{\mathcal{X}}'$. Let $\mathcal{F}$ be in $\boldsymbol{\mathcal{X}}'$. Therefore, $\mathcal{F} = (\mathcal{E} \cup Z_1) \setminus Z_2$ where $\mathcal{E} \in \boldsymbol{\mathcal{X}}$, $Z_1 \subseteq \mathcal{A} \in \boldsymbol{\mathcal{Z}}$ and $Z_2 \subseteq \mathcal{B} \in \boldsymbol{\mathcal{Z}}$. Now,

$$(\mathcal{F})^c = [(\mathcal{E} \cup Z_1) \setminus Z_2]^c = [(\mathcal{E})^c \cap (Z_1)^c] \cup Z_2$$

Notice that

$$(\mathcal{E})^c \cap (Z_1)^c = (\mathcal{E})^c \cap (\mathcal{A})^c \cup [\mathcal{A} \setminus Z_1 \cap (\mathcal{E})^c].$$

Therefore,

$$(\mathcal{F})^c = [(\mathcal{E})^c \cap (\mathcal{A})^c] \cup [\mathcal{A} \setminus Z_1 \cap (\mathcal{E})^c] \cup Z_2.$$

We see that $(\mathcal{E})^c \cap (\mathcal{A})^c \in \boldsymbol{\mathcal{X}}$ and $[\mathcal{A}\backslash Z_1 \cap (\mathcal{E})^c] \cup Z_2 \subseteq \mathcal{A} \cup \mathcal{B} \in \boldsymbol{\mathcal{Z}}$. Therefore, by Proposition 4.14, $(\mathcal{F})^c \in \boldsymbol{\mathcal{X}}$‘. Finally, let $\{\mathcal{F}_n\}$ be a sequence in $\boldsymbol{\mathcal{X}}$‘, therefore $\mathcal{F}_n = \mathcal{E}_n \cup Z_n$ where $\mathcal{E}_n \in \boldsymbol{\mathcal{X}}$ and $Z_n \subseteq \mathcal{A}_n \in \boldsymbol{\mathcal{Z}}$. Now

$$\bigcup_{n=1}^{\infty} (\mathcal{E}_n \cup Z_n) = \bigcup_{n=1}^{\infty} (\mathcal{E}_n) \cup \bigcup_{n=1}^{\infty} (Z_n).$$

It is clear that $\bigcup_{n=1}^{\infty} (\mathcal{E}_n) \in \boldsymbol{\mathcal{X}}$and $\bigcup_{n=1}^{\infty} (Z_n) \subseteq \bigcup_{n=1}^{\infty} (\mathcal{A}_n) \in \boldsymbol{\mathcal{Z}}$. Thus, $\boldsymbol{\mathcal{X}}$‘ is a σ-algebra . □

Proposition 4.16. *Let μ‘ be defined on $\boldsymbol{\mathcal{X}}$‘ by*

$$\mu‘ (\mathcal{E} \cup Z) = \mu (\mathcal{E}), \tag{4.12}$$

where $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ and Z is a subset of a set with μ-measure zero. Then μ‘ is well-defined and a measure on $\boldsymbol{\mathcal{X}}$‘ which agrees with μ on $\boldsymbol{\mathcal{X}}$. This measure μ‘ is called the completion of μ.

Proof. First, we claim that if $(\mathcal{E} \cup Z_1) = (\mathcal{F} \cup Z_2)$ where $\mathcal{E}, \mathcal{F} \in \boldsymbol{\mathcal{X}}$ and $Z_i \subseteq \mathcal{A}_i$ with $\mu(\mathcal{A}_i) = 0$ for $i = 1, 2$, then $\mu(\mathcal{E}) = \mu(\mathcal{F})$. To see this let $(\mathcal{E} \cup Z_1) = (\mathcal{F} \cup Z_2)$. Therefore $\mathcal{E} \subseteq (\mathcal{F} \cup \mathcal{A}_2)$ and thus by Lemma 3.3 [Bartle] and Proposition 4.14, $\mu(\mathcal{E}) \leq \mu(\mathcal{F}) + \mu(\mathcal{A}_2) = \mu(\mathcal{F})$. Similarly, we see that, $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. Therefore, $\mu(\mathcal{E}) = \mu(\mathcal{F})$ and our claim is demonstrated. Thus, μ‘ is well-defined.

Now, we demonstrate that $\mu‘(\mathcal{E} \cup Z) = \mu(\mathcal{E})$ is a measure by showing that it satisfies the three conditions of Definition 4.1. Condition (i) follows from the fact that $\mu‘(\emptyset \cup Z) = \mu(\emptyset) = 0$. Condition (ii) holds since $\mu‘(\mathcal{E} \cup Z) = \mu(\mathcal{E}) \geq 0$ for all $(\mathcal{E} \cup Z) \in \boldsymbol{\mathcal{X}}$‘. And finally, let $\{\mathcal{E}_n \cup Z_n\}$ be a sequence in $\boldsymbol{\mathcal{X}}$‘.

$$\begin{aligned} \mu‘ \left[\bigcup_{n=1}^{\infty} (\mathcal{E}_n \cup Z_n) \right] &= \mu‘ \left[\bigcup_{n=1}^{\infty} \mathcal{E}_n \cup \bigcup_{n=1}^{\infty} Z_n \right] \\ &= \sum_{n=1}^{\infty} \mu(\mathcal{E}_n) = \sum_{n=1}^{\infty} \mu‘ (\mathcal{E}_n \cup Z_n). \end{aligned}$$

□

Proposition 4.17. *Let $(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ be a measure space and let $(\mathcal{X}, \boldsymbol{\mathcal{X}}`, \mu`)$ be its completion in the sense of Proposition 4.16. Suppose that f is an $\boldsymbol{\mathcal{X}}`$-measurable function on $\mathcal{X}$ to the extended real line. Then, there exists a $\boldsymbol{\mathcal{X}}$-measurable function g on $\mathcal{X}$ to the extended real line which is μ-almost everywhere equal to f.*

Proof. For each rational number r, let $\mathcal{A}_r = \{x \in \mathcal{X} : f(x) > r\}$. Since f is $\boldsymbol{\mathcal{X}}`$-measurable, $\mathcal{A}_r \in \boldsymbol{\mathcal{X}}`$ for each rational number. Now we may write $\mathcal{A}_r = \mathcal{E}_r \cup Z_r$ where $\mathcal{E}_r \in \boldsymbol{\mathcal{X}}$ and Z_r a subset of a set in $\boldsymbol{\mathcal{Z}}$. Let Z be in $\boldsymbol{\mathcal{Z}}$ such that $\bigcup Z_r$ is a subset of Z. Furthermore, we define $g(x) = f(x)$ for all $x \notin Z$ and $g(x) = 0$ for all $x \in Z$. Therefore, $g(x)$ is μ-almost equal to $f(x)$.

To see that $g(x)$ is $\boldsymbol{\mathcal{X}}$-measurable, we show that $\{x \in \mathcal{X} : g(x) > r\} \in \boldsymbol{\mathcal{X}}$ for all $r \in \mathbb{Q}$ and then use Proposition 3.27.

If $r \geq 0$, then

$$\begin{aligned}\{x \in \mathcal{X} : g(x) > r\} &= \{x \in \mathcal{X} : f(x) > r\} \setminus Z \\ &= (\mathcal{E}_r \cup Z_r) \setminus Z = (\mathcal{E}_r \setminus Z \cup \emptyset) \in \boldsymbol{\mathcal{X}}.\end{aligned}$$

If $r < 0$, then

$$\begin{aligned}\{x \in \mathcal{X} : g(x) > r\} &= \{x \in \mathcal{X} : g(x) > r\} \cup Z \\ &= (\mathcal{E}_r \cup Z_r) \cup Z = (\mathcal{E}_r \cup Z) \in \boldsymbol{\mathcal{X}}.\end{aligned}$$

This completes the proof.

□

Proposition 4.18. *If μ is a charge on $\boldsymbol{\mathcal{X}}$, then Lemma 3.4* [Bartle] *holds.*

Proof. Lemma 3.4 does not depend on the fact that $\mu(\mathcal{E}_n) \geq 0$ for all $\mathcal{E}_n \in \boldsymbol{\mathcal{X}}$, therefore it holds for a charge. □

Proposition 4.19. *If μ is a charge on $\boldsymbol{\mathcal{X}}$ and π is defined for $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ as*

$$\pi(\mathcal{E}) = \sup\{\mu(A) : A \subseteq \mathcal{E},\ A \in \boldsymbol{\mathcal{X}}\}, \tag{4.13}$$

π is a measure on $\boldsymbol{\mathcal{X}}$.

Proof. Clearly, (i) and (ii) from Definition 4.1 hold. To show (iii) we will argue like Proposition 4.5. Let $\{\mathcal{E}_n\}$ be a pairwise disjoint sequence from $\boldsymbol{\mathcal{X}}$. Fix $\mathcal{A} \subseteq \bigcup_{n=1}^{\infty} \mathcal{E}_n$ with $\mathcal{A} \in \boldsymbol{\mathcal{X}}$. Therefore,

$$\mu(\mathcal{A}) = \mu\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n \cap \mathcal{A}\right) = \sum_{n=1}^{\infty} \mu(\mathcal{A} \cap \mathcal{E}_n) \leq \sum_{n=1}^{\infty} \pi\left(\mathcal{E}_n\right).$$

So

$$\mu(\mathcal{A}) \leq \sum_{n=1}^{\infty} \pi\left(\mathcal{E}_n\right). \tag{4.14}$$

In (4.14), taking the sup over all such $\mathcal{A}$'s gives

$$\pi\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n\right) \leq \sum_{n=1}^{\infty} \pi\left(\mathcal{E}_n\right). \tag{4.15}$$

From (4.15), if $\pi\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n\right) = \infty$ then the proof is done. So, assume $\pi\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n\right) < \infty$ and fix $\varepsilon > 0$. For each $j \in \mathbb{N}$,

$$\pi(\mathcal{E}_j) \leq \pi\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n\right) < \infty.$$

Therefore, there exists $\mathcal{A}_j \in \boldsymbol{\mathcal{X}}$ with $\mathcal{A}_j \subseteq \mathcal{E}_j$ and

$$\pi(\mathcal{E}_j) - \frac{\varepsilon}{2^j} \leq \mu(\mathcal{A}_j).$$

Thus,

$$\begin{aligned}\pi\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n\right) &\geq \mu\left(\bigcup_{n=1}^{\infty} \mathcal{A}_n\right) = \sum_{n=1}^{\infty} \mu\left(\mathcal{A}_n\right) \geq \sum_{n=1}^{\infty} \left[\pi(\mathcal{E}_n) - \frac{\varepsilon}{2^n}\right] \\ &= \sum_{n=1}^{\infty} \left[\pi(\mathcal{E}_n)\right] - \left[\sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}\right] = \sum_{n=1}^{\infty} \left[\pi(\mathcal{E}_n)\right] - \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary:

$$\pi\left(\bigcup_{n=1}^{\infty} \mathcal{E}_n\right) \geq \sum_{n=1}^{\infty} \pi(\mathcal{E}_n).$$

This finishes the proof. □

Proposition 4.20. *Let λ denote the Lebesgue measure defined on the Borel algebra, $\boldsymbol{\mathcal{B}}$ of $\mathbb{R}$.*

(i) *If $\mathcal{E}$ consists of a single point, then $\mathcal{E} \in \boldsymbol{\mathcal{B}}$ and $\lambda(\mathcal{E}) = 0$.*

(ii) *If $\mathcal{E}$ is countable, then $\mathcal{E} \in \boldsymbol{\mathcal{B}}$ and $\lambda(\mathcal{E}) = 0$.*

(iii) *The open interval (a, b), the half-open intervals $(a, b], [a, b)$, and $[a, b]$ all have Lebesgue measure $b - a$.*

Proof. Let $\mathcal{E} = \{x\}$ where $x \in \mathbb{R}$. Let $\mathcal{A}_n = (x - \frac{1}{n},\ x + \frac{1}{n})$ for each $n \in \mathbb{N}$. Clearly, $\mathcal{A}_n \in \boldsymbol{\mathcal{B}}$ for each $n \in \mathbb{N}$, thus $\mathcal{E} = \displaystyle\bigcap_{n=1}^{\infty}(x - \frac{1}{n},\ x + \frac{1}{n}) \in \boldsymbol{\mathcal{B}}$. Now, $\{\mathcal{A}_n\}$ is decreasing therefore from Lemma 3.4(b) [Bartle]

$$\lambda(\mathcal{E}) = \lambda\left(\bigcap_{n=1}^{\infty} \mathcal{A}_n\right) = \lim_{n\to\infty} \lambda(\mathcal{A}_n) = \lim_{n\to\infty} \frac{2}{n} = 0.$$

This shows (i).

Now, to show (ii) let $\mathcal{E} = \{x_n\}_{n\in\mathbb{N}}$ and $x_n \neq x_m$ for $n \neq m$. So by (i) and countable additivity of a measure we get (ii). For each $x_n \in \mathcal{E}$ define $\mathcal{A}_{n_m} = (x_n - \frac{1}{m},\ x_n + \frac{1}{m})$. Thus,

$$\bigcap_{m=1}^{\infty} \mathcal{A}_{n_m} = x_n \in \boldsymbol{\mathcal{B}} \text{ and therefore } \mathcal{E} = \bigcup_{n=1}^{\infty}\left(\bigcap_{m=1}^{\infty} \mathcal{A}_{n_m}\right) \in \boldsymbol{\mathcal{B}}.$$

Now, we see that since $\bigcap_{m=1}^{\infty} \mathcal{A}_{i_m} \cap \bigcap_{m=1}^{\infty} \mathcal{A}_{j_m} = \emptyset$ when $i \neq j$,

$$\lambda(\mathcal{E}) = \lambda\left[\bigcup_{n=1}^{\infty}\left(\bigcap_{m=1}^{\infty} \mathcal{A}_{n_m}\right)\right] = \sum_{n=1}^{\infty} \lambda\left(\bigcap_{m=1}^{\infty} \mathcal{A}_{n_m}\right) = 0.$$

This demonstrates (ii).

Lastly, by definition of λ, $\lambda(a,\ b) = b - a$. From Proposition 3.9, $(a,\ b] = \bigcap_{n=1}^{\infty}(a,\ b + \frac{1}{n}) \in \boldsymbol{\mathcal{B}}$. Therefore,

$$\begin{aligned}\lambda\,(a,\ b] &= \lambda\left[\bigcap_{n=1}^{\infty}(a,\ b + \frac{1}{n})\right] = \lim_{n\to\infty} \lambda(a,\ b + \frac{1}{n}) \\ &= \lim_{n\to\infty}(b + \frac{1}{n} - a) = b - a.\end{aligned}$$

Likewise, for $[a,\ b)$ and $[a,\ b]$. This shows (iii). □

Proposition 4.21. *If λ denotes the Lebesgue measure and $\mathcal{E} \neq \emptyset$ is an open subset of $\mathbb{R}$, then $\lambda(\mathcal{E}) > 0$. Also if $\mathcal{K}$ is compact subset of $\mathbb{R}$, then $\lambda(\mathcal{K}) < \infty$.*

Proof. From Theorem 3.1.13 [Stoll] there exits a finite or countable collection $\{\mathcal{I}_n\}$ of pairwise disjoint open intervals such that $\mathcal{E} = \bigcup_n \mathcal{I}_n$ for $\mathcal{E}$ an open subset of $\mathbb{R}$. Therefore, for $\mathcal{E}$ open $\mathcal{E} \in \boldsymbol{\mathcal{B}}$. Also since $\mathcal{E}$ open for every point p in $\mathcal{E}$ we can find an $\varepsilon > 0$ such that, $(p - \frac{\varepsilon}{2},\ p + \frac{\varepsilon}{2}) \subset \mathcal{E}$. Applying Lemma 3.3 [Bartle], we get,

$$0 < \lambda\left(p - \frac{\varepsilon}{2},\ p + \frac{\varepsilon}{2}\right) \leq \lambda(\mathcal{E}).$$

Therefore, $\lambda(\mathcal{E}) > 0$.

Let $\mathcal{K}$ be compact. Therefore, by Heine-Borel Theorem, $\mathcal{K}$ is bounded, i.e. there exists a positive constant M such that $\mathcal{K} \subset [-M,\ M]$. Thus

$$\lambda(\mathcal{K}) \leq \lambda([-M,\ M] = 2M < \infty.$$

This finishes the proof. □

In the following example, we vary the Cantor set such that we obtain a set of positive Lebesgue measure that contains no non-void open interval.

Example 4.22. The Fat Cantor Set, $\widetilde{P}$, is constructed like the Cantor set except the open intervals removed at the n^{th} step have length $\alpha 3^{-n}$, $0 < \alpha < 1$. Thus, $\widetilde{P} = \bigcap_{n=1}^{\infty} R_n^c$ where R_n is the part removed on the n^{th} step. Note that $\bigcap_{n=1}^{N} R_n^c$ is the disjoint union of 2^N closed intervals, each of length less than $\frac{1}{2^N}$. Therefore, if E (a non-void open interval) is in $\widetilde{P}$, then it sits in one of these intervals with length less than $\frac{1}{2^n}$ for all $n \in \mathbb{N}$. This implies that $\lambda(E) = 0$ which is a contradiction of Proposition 4.21. Finally, it is clear that

$$\lambda((\widetilde{P})^c) = \frac{1}{3}\alpha \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \alpha.$$

Therefore, $\lambda(\widetilde{P}) = 1 - \alpha > 0$.

In the following example the almost everywhere limit of a sequence of measurable functions is not measurable.

Example 4.23. Let $\mathcal{E}$ be a subset of a set $\mathcal{N} \in \boldsymbol{\mathcal{X}}$ with $\mu(\mathcal{N}) = 0$, but $\mathcal{E} \notin \boldsymbol{\mathcal{X}}$. Let $f_n = 0$ for all $n \in \mathbb{N}$. Thus, $\lim_{n\to\infty} f_n = \mathfrak{X}_{\mathcal{E}}$ (the characteristic equation of $\mathcal{E}$) almost everywhere. Therefore, the almost everywhere limit of f_n equals a non-measurable function.

5. The Integral

Definition 5.1. If f belongs to $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$, we define the integral of f with respect to μ to be the extended real number

$$\int f\, d\mu = \sup \int \phi\, d\mu,$$

where the supremum is extended over all simple functions ϕ in $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ satisfying $0 \leq \phi(x) \leq f(x)$ for all $x \in \mathcal{X}$. If f belongs to $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ and $\mathcal{E}$ belongs to $\boldsymbol{\mathcal{X}}$, then $f\mathbf{1}_{\mathcal{E}}$ belongs to $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ and we define the integral of f over $\mathcal{E}$ with respect to μ to be the extended real number

$$\int_{\mathcal{E}} f\, d\mu = \int f\mathbf{1}_{\mathcal{E}}\, d\mu.$$

Proposition 5.2. *If the simple function ϕ in $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ has the (not necessarily standard representation)*

$$\phi = \sum_{k=1}^{m} b_k \mathfrak{X}_{\mathcal{F}_k}, \tag{5.1}$$

where $b_k \in \mathbb{R}$ and $\mathcal{F}_k$, then

$$\int \phi\ d\mu = \sum_{k=1}^{m} b_k\ \mu\left(\mathcal{F}_k\right). \tag{5.2}$$

Proof. If ϕ has standard representation then done. Otherwise by rewriting (5.1), we get

$$\phi = b_1 \mathfrak{X}_{\mathcal{F}_1} + \cdots + b_m \mathfrak{X}_{\mathcal{F}_m}.$$

Therefore,

$$\begin{aligned}
\int \phi\ d\mu &= \int \left(b_1 \mathfrak{X}_{\mathcal{F}_1} + \cdots + b_m \mathfrak{X}_{\mathcal{F}_m}\right) d\mu \\
&= \int b_1 \mathfrak{X}_{\mathcal{F}_1} d\mu + \cdots + \int b_m \mathfrak{X}_{\mathcal{F}_m} d\mu \text{ (by Lemma 4.3(a) ([Bartle])} \\
&= b_1 \mu\left(\mathcal{F}_1\right) + \cdots + b_m \mu\left(\mathcal{F}_m\right) \text{ (since } b_k \mathfrak{X}_{\mathcal{F}_k} \text{ is a simple function.)} \\
&= \sum_{k=1}^{m} b_k\ \mu\left(\mathcal{F}_k\right)
\end{aligned}$$

□

Proposition 5.3. *If ϕ_1 and ϕ_2 are simple functions in $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$, then*

$$\psi = \sup\{\phi_1,\ \phi_2\}, \quad \omega = \inf\{\phi_1,\ \phi_2\}$$

are also simple functions in $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$.

Proof. Let $\psi = \sup\{\phi_1,\ \phi_2\}$. Therefore,

$$\psi = \phi_1 \mathbf{1}_{\{x\in\mathcal{X}:\phi_1\geq\phi_2\}} + \phi_2 \mathbf{1}_{\{x\in\mathcal{X}:\phi_2>\phi_1\}}$$

Thus ψ is a simple function since it is the sum of two simple functions. Likewise for ω. □

The following is a different proof of Corollary 4.7(a) from [Bartle]. We will attain the same result without using the Monotone Convergence Theorem.

Proposition 5.4. *If $f \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ and $c > 0$, then*

$$\int cf\, d\mu = c\int f\, d\mu. \tag{5.3}$$

Proof. Let $\phi \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ be a simple function such that $\phi \leq f$ and let $\psi \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ be a simple function such that $\phi \leq cf$. The map $\phi \to \psi = c\phi$ is clearly a one to one function since $c\phi_1 = c\phi_2$ implies that $\phi_1 = \phi_2$. Therefore,

$$\sup_{\psi\leq cf}\int \psi\, d\mu = \sup_{c\phi\leq cf}\int c\phi\, d\mu = c\sup_{\phi\leq f}\int \phi\, d\mu.$$

This gives (5.3). □

In problems, 4.E and 4.F from Elements of Integration, Bartle attempts to show another proof for 4.7 Corollary [Bartle] without using the Monotone Convergence Theorem. However, as stated Problem 4.E is false and therefore Problem 4.F cannot be completed. I will state 4.E, show that it is false by counterexample, then I will show 4.F assuming (incorrectly) that 4.E is true.

Proposition 5.5. *Let f, g belong to $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$, let ϕ be a simple function in $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ with $\phi \leq f$, and let ω be a simple function in $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ with $\omega \leq f + g$. Let $\theta_1 = \inf\{\omega, \phi\}$ and let $\theta_2 = \sup\{\omega - \phi, 0\}$. Then*

(i) $\omega = \theta_1 + \theta_2$

(ii) $\theta_1 \le f$

(iii) $\theta_2 \le g$.

Proof. First, conditions (i) and (ii) always hold.

(i) If $\theta_1 = \inf\{\omega, \phi\} = \omega$, then $\theta_2 = \sup\{\omega - \phi, 0\} = 0$. Therefore, $\theta_1 + \theta_2 = \omega$. If $\theta_1 = \inf\{\omega, \phi\} = \phi$, then $\theta_2 = \sup\{\omega - \phi, 0\} = \omega - \phi$. Therefore, $\theta_1 + \theta_2 = \phi + \omega - \phi = \omega$.

(ii) If $\theta_1 = \phi$, then $\theta_1 \le f$. If $\theta_1 = \omega$, then $\omega \le \phi \le f$.

Now, condition (iii) holds if $\inf\{\phi, \omega\} = \omega$ since $\inf\{\phi, \omega\} = \omega$ implies that $\sup\{\omega - \phi, 0\} = 0$ and $0 \le g$. However if $\inf\{\phi, \omega\} = \phi$ the conclusion need not be true. Let $\phi = 1\mathbf{1}_{[0,\ 1]}, f = 2\mathbf{1}_{[0,\ 1]}, \omega = 4\mathbf{1}_{[0,\ 1]}$ and $f + g = 4.001\mathbf{1}_{[0,\ 1]}$. Therefore, $\phi \le f$ and $\omega \le f + g$. But

$$\theta_2 = \omega - \phi = 3\chi_{[0,\ 1]} \ge 2.001\mathbf{1}_{[0,\ 1]} = g.$$

□

Assuming Proposition 5.5 is true, I'll show the following proposition.

Proposition 5.6. *Using Proposition 5.5, if f, g belong to $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$, then*

(i) $f + g \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$

(ii) $\int (f + g)\, d\mu = \int f\, d\mu + \int g\, d\mu$.

Proof. From 2.6 Lemma [Bartle] $f + g \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ Furthermore, by the definition of the integral with respect to μ,

$$\sup_{\omega} \int \omega\, d\mu = \int (f + g)\, d\mu \qquad \text{where the simple function } \omega \le f + g.$$

Moreover, from Proposition 5.5, 4.5 Lemma [Bartle] and 4.3 Lemma [Bartle],

$$\int f\, d\mu + \int g\, d\mu \ge \int \theta_1\, d\mu + \int \theta_2\, d\mu = \int (\theta_1 + \theta_2)\, d\mu = \int \omega\, d\mu.$$

Therefore, from the definition of the integral and by taking the supremum over all simple functions $\omega \le f + g$.

$$\int f + g\, d\mu = \int f\, d\mu + \int g\, d\mu.$$

□

Proposition 5.7. *Let $\mathcal{X} = \mathbb{N}$, let $\boldsymbol{\mathcal{X}}$ be all subsets of $\mathbb{N}$, and let μ be the counting measure on $\boldsymbol{\mathcal{X}}$. If f is a nonnegative function on $\mathbb{N}$, then $f \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ and*

$$\int f \, d\mu = \sum_{n=1}^{\infty} f(n).$$

Proof. Let $f_N = \sum_{n=1}^{N} f(n)\mathbf{1}_{[n,n+1)}$. Clearly, f_N monotone increasing to f a function on $\mathbb{N}$. Applying the MCT we get,

$$\int f \, d\mu = \lim_{N\to\infty} \int \sum_{n=1}^{\infty} f(n)\mathbf{1}_{[n,n+1)} = \lim_{N\to\infty} \sum_{n=1}^{N} f(n) = \sum_{n=1}^{\infty} f(n).$$

□

Example 5.8. Let $\mathcal{X} = \mathbb{R}$, $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{B}}$, and let λ be the Lebesgue measure on $\boldsymbol{\mathcal{X}}$. Let $f_n = \mathbf{1}_{[0,\ n]}$. Clearly, f_n converges to $f = \mathbf{1}_{[0,\ \infty]}$. Therefore, by Monotone Convergence we have, $\int f d\lambda = \int f_n d\lambda = \infty$. Note the MCT can be applied here since $f_n \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ which means that $\int f_n d\lambda$ will always have a limit in the extended real line.

Example 5.9. Let $\mathcal{X} = \mathbb{R}$, $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{B}}$, and let λ be the Lebesgue measure on $\boldsymbol{\mathcal{X}}$. Let $f_n = \frac{1}{n}\mathbf{1}_{[n,\ \infty]}$. Obviously f_n is monotone decreasing to $f = 0$. In fact, f_n converges uniformly to $f = 0$ since given $\varepsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that,

$$|f_n(x) - f(x)| \leq \left|\frac{1}{n}\right| = \frac{1}{n} < \varepsilon$$

for all $n \geq n_0$ and for all $x \in \mathcal{X}$. However,

$$\int f d\lambda = 0 \neq \infty = \lim_{n\to\infty} \int f_n \, d\lambda$$

This example does not contradict the MCT because f_n is a decreasing sequence.

Example 5.10. Let $\mathcal{X} = \mathbb{R}$, $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{B}}$, and let λ be the measure on $\boldsymbol{\mathcal{X}}$. Let $f_n = \frac{1}{n}\mathbf{1}_{[0,\ n]}$. Like Example 5.9, f_n converges uniformly to $f = 0$ and

$$\int f d\lambda = 0 \neq 1 = \lim_{n\to\infty} \int f_n \, d\lambda.$$

Again, this example does not contradict the MCT since f_n is not monotone increasing. But we can apply Fatou's Theorem which gives

$$\int (\liminf f_n) \; d\lambda = 0 \leq 1 = \liminf \int f_n \, d\lambda.$$

Example 5.11. Let $g_n = n\mathbf{1}_{[\frac{1}{n},\ \frac{2}{n}]}$ and let $g = 0$. Now, $\lim g_n(x) = 0$ for each $x \in \mathcal{X}$. Therefore, g_n converges everywhere to g but not uniformly. However, MTC does not apply since g_n is not monotone increasing. Fatou's Theorem though does apply since $g_n \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$.

Proposition 5.12. *If $(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ is a finite measure space, and if $\{f_n\}$ is a sequence of real-valued functions in $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ which converges uniformly to a function f, then f belongs to $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ and*

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu. \tag{5.4}$$

Proof. By Corollary 2.10 from [Bartle], $f \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ since f_n converges uniformly to f. Also, f_n converges uniformly to f and $(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ finite implies that given $\varepsilon > 0$ there exists an n_o such that

$$f(x) - \frac{\varepsilon}{\mu(\mathcal{X})} < f_n(x) < f(x) + \frac{\varepsilon}{\mu(\mathcal{X})}.$$

for all $n \geq n_o$ and for all $x \in \mathcal{X}$. Thus, by 4.5 Lemma [Bartle] and 4.7 Corollary [Bartle],

$$\int f(x) \, d\mu - \int \frac{\varepsilon}{\mu(\mathcal{X})} \, d\mu < \int f_n(x) \, d\mu < \int f(x) \, d\mu + \int \frac{\varepsilon}{\mu(\mathcal{X})} \, d\mu.$$

Whence,

$$\left| \int f_n(x) \, d\mu - \int f(x) \, d\mu \right| < \frac{\varepsilon \mu(\mathcal{X})}{\mu(\mathcal{X})} = \varepsilon.$$

This gives (5.4). □

Proposition 5.13. *Let $\mathcal{X}$ be a closed interval $[a,\ b]$ in $\mathbb{R}$, let λ be the Lebesgue measure on $\boldsymbol{\mathcal{X}}$. If f is a nonnegative continuous function on $\mathcal{X}$, then*

$$\int f\, d\lambda = \int_a^b f(x)\, dx, \tag{5.5}$$

where the right side denotes the Riemann integral of f.

Proof. Let ϕ be a nonnegative step function on $[a,\ b]$. Therefore, there exists, $\mathcal{P}$, a partition of $[a,\ b]$ where $a = x_0 < x_1 < \cdots < x_n = b$ such that for $i \in \mathbb{N}_n$ ϕ takes on one value in the interval $[x_{i-1},\ x_i)$. Let $a_i \geq 0$ denote this value. Thus,

$$\phi(x) = a_1\mathbf{1}_{[x_0,\ x_1)} + a_2\mathbf{1}_{[x_1,\ x_2)} + \cdots + a_n\mathbf{1}_{[x_{n-1},\ x_n)}$$

is a simple function. Therefore,

$$\begin{aligned}\int \phi(x)\, d\lambda &= a_1\lambda([x_0,\ x_1) + \cdots + a_n\lambda([x_{n-1},\ x_n) \\ &= \sum_{i=1}^{n} a_i(x_i - x_{i-1}) = \int_a^b \phi(x)\, dx.\end{aligned}$$

Thus, we have shown the proposition for nonnegative step functions.

Now let f be continuous and nonnegative. Thus,

$$\int_a^b f(x)\, dx = \sup_{\mathcal{P}} \{\mathcal{L}(\mathcal{P}, f) : \mathcal{P} \text{ a partition of } [a,\ b]\}.$$

By definition,

$$\mathcal{L}(\mathcal{P}, f) = \sum_{i=1}^{n} m_i \Delta x_i,$$

where $m_i = \inf\{f(t) : f(x_{i-1}) \leq f(t) \leq f(x_i)\}$ and $\mathcal{P}$ a partition of $[a,\, b]$. Therefore,

$$\mathcal{L}(\mathcal{P}, f) = \int \phi\, d\lambda$$

where $\phi(x) \leq f(x)$ and $\phi(x)$ a simple function. Thus

$$\int_a^b f(x)\, dx = \sup_{\phi} \int \phi\, d\lambda = \int f\, d\lambda.$$

□

Proposition 5.14. *Let $\mathcal{X} = [0,\ \infty)$, let $\boldsymbol{\mathcal{X}}$ be the Borel subsets and let λ be the Lebesgue measure on $\boldsymbol{\mathcal{X}}$. If f is a nonnegative continuous function on $\mathcal{X}$, show that*

$$\int_{\mathcal{X}} f\, d\lambda = \lim_{b\to\infty} \int_0^b f(x)\, dx.$$

Proof. By Proposition 5.13,

$$\int_{[0,\ b]} f\, d\lambda = \int_0^b f(x)\, dx.$$

Therefore,

$$\lim_{b\to\infty} \int_0^b f(x)\, dx = \lim_{b\to\infty} \int_{[0,\ b]} f\, d\lambda.$$

We conclude the proof by showing,

$$\lim_{b\to\infty} \int_{[0,\ b]} f\, d\lambda = \int_{\mathcal{X}} f\, d\lambda.$$

The above is true however by the MCT since $f\mathbf{1}_{[0,b]}$ is monotone increasing to f. □

Proposition 5.15. *If $f \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ and*

$$\int f\, d\mu < \infty,$$

then $\mu\{x \in \mathcal{X} : f(x) = \infty\} = 0$.

Proof. Let $\mathcal{E}_n = \{x \in \mathcal{X} : f(x) \geq n\}$. Therefore, $\mu\{x \in \mathcal{X} : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \mathcal{E}_n$. Clearly, $\{\mathcal{E}_n\}$ is decreasing. Thus, by 3.4(b) Lemma [Bartle],

$$\mu\{x \in \mathcal{X} : f(x) = \infty\} = \mu\left(\bigcap_{n=1}^{\infty} \mathcal{E}_n\right) = \lim_{n\to\infty} \mu(\mathcal{E}_n)$$

provided $\mu(\mathcal{E}_1) < \infty$. For each $n \in \mathbb{N}$, $n\mathbf{1}_{\mathcal{E}_n} \leq f$. By 4.5(a) Lemma [Bartle] $\int n\mathbf{1}_{\mathcal{E}_n} \leq \int f\, d\mu < \infty$ for each $n \in \mathbb{N}$. Now $n\mu(\mathcal{E}_n) < \infty$ for each n. Thus, $\mu(\mathcal{E}_1) < \infty$. To finish the proof, it will be shown that $\lim \mu(\mathcal{E}_n) = 0$. It has already been established that $n\mu(\mathcal{E}_n) \leq \int f\, d\mu$ for all n. Therefore, $\mu(\mathcal{E}_n) \leq \frac{1}{n} \int f\, d\mu$ for all n. This implies that

$$\lim \mu(\mathcal{E}_n) \leq \left(\int f\, d\mu\right) \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore, $\mu\left(\bigcap_{n=1}^{\infty}\mathcal{E}_n\right)=0.$

$\square$

Proposition 5.16. *If $f \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ and*

$$\int f\,d\mu < \infty,$$

then the set $\mathcal{N} = \{x \in \mathcal{X} : f(x) > 0\}$ is σ-finite (that is, there exists a sequence, $\{\mathcal{F}_n\}_{n\in\mathbb{N}} \in \boldsymbol{\mathcal{X}}$, such that $\mathcal{N} \subseteq \bigcup \mathcal{F}_n$ and $\mu(\mathcal{F}_n) < \infty$).

Proof. Let $\mathcal{F}_n = \left\{x \in \mathcal{X} : f(x) > \frac{1}{n}\right\}$. For each n, $\mathcal{F}_n \in \boldsymbol{\mathcal{X}}$, since $f \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$. Therefore, $\bigcup_{n=1}^{\infty}\{\mathcal{F}_n\} \in \boldsymbol{\mathcal{X}}$. Now, if $x \in \mathcal{N}$, then $f(x) > 0$. But, there exists a n such that $f(x) > \frac{1}{n}$. Thus, $x \in \bigcup_{n=1}^{\infty}\mathcal{F}_n$. Therefore, $\mathcal{N} \subseteq \bigcup_{n=1}^{\infty}\mathcal{F}_n$. By construction, $\frac{1}{n}\mathbf{1}_{\mathcal{F}_n} < f$ for each n. Thus,

$$\int \frac{1}{n}\mathbf{1}_{\mathcal{F}_n}\,d\mu < \int f\,d\mu < \infty.$$

Clearly, $\frac{1}{n}\mu(\mathcal{F}_n) < \infty$ and therefore $\mu(\mathcal{F}_n) < \infty$. $\square$

Proposition 5.17. *If $f \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ and*

$$\int f\,d\mu < \infty,$$

then for any $\varepsilon > 0$ there exists a set $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ such that $\mu(\mathcal{E}) < \infty$ and

$$\int f\,d\mu \leq \int_{\mathcal{E}} f\,d\mu + \varepsilon.$$

Proof. Let $\{\mathcal{E}_n\}$ be a disjoint sequence in $\boldsymbol{\mathcal{X}}$. Therefore

$$\int_{\bigcup_{n=1}^{\infty}\mathcal{E}_n} f\,d\mu = \sum_{n=1}^{\infty}\int_{\mathcal{E}_n} f\,d\mu = \lim_{m\to\infty}\sum_{n=1}^{m}\int_{\mathcal{E}_n} f\,d\mu \leq \int f\,d\mu.$$

The series $\sum_{n=1}^{\infty}\int_{\mathcal{E}_n} f\,d\mu$ is increasing. Therefore given an $\varepsilon > 0$ there exist a n_0 such that

$$\int f\,d\mu - \varepsilon \leq \sum_{n=1}^{n_0}\int_{\mathcal{E}_n} f\,d\mu = \int_{\bigcup_{n=1}^{n_0}\mathcal{E}_n} f\,d\mu.$$

for all $n \geq n_o$. By letting $\mathcal{E} = \bigcup_{n=1}^{n_0}\mathcal{E}_n$ the proposition is proved. $\square$

Proposition 5.18. *If* $\{f_n\} \subset M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$, $\{f_n\}$ *converges to* f *almost everywhere, and*

$$\int f \, d\mu = \lim \int f_n \, d\mu < \infty,$$

then

$$\int_{\mathcal{E}} f \, d\mu = \lim \int_{\mathcal{E}} f_n \, d\mu$$

for each $\mathcal{E} \in \mathcal{X}$.

Proof. Since f_n converges to f, $f_n \mathbf{1}_{\mathcal{E}}$ converges to $f \mathbf{1}_{\mathcal{E}}$. By Fatou's Theorem,

$$\int_{\mathcal{E}} (\liminf f_n) \, d\mu = \int_{\mathcal{E}} f \, d\mu \leq \liminf \int_{\mathcal{E}} f_n \, d\mu. \tag{5.6}$$

By applying Fatou's Theorem to $f_n - f_n \mathbf{1}_{\mathcal{E}}$ it is seen that,

$$\int \big(\liminf (f_n - f_n \mathbf{1}_{\mathcal{E}}) \big) \, d\mu \leq \liminf \int (f_n - f_n \mathbf{1}_{\mathcal{E}}) \, d\mu.$$

Since $\lim f_n$ and $\lim \int f_n$ exist, the limits can be "pushed through the parenthesis" to give,

$$\int (f - f \mathbf{1}_{\mathcal{E}}) \, d\mu \leq \int f \, d\mu + \liminf \left(- \int f_n \mathbf{1}_{\mathcal{E}} \, d\mu \right).$$

By subtracting $\int f \, d\mu$ and applying a property of $\liminf$,

$$- \int f \mathbf{1}_{\mathcal{E}} \, d\mu \leq - \limsup \left(\int f_n \mathbf{1}_{\mathcal{E}} \, d\mu \right).$$

Therefore,

$$\int f \mathbf{1}_{\mathcal{E}} \, d\mu \geq \limsup \left(\int f_n \mathbf{1}_{\mathcal{E}} \, d\mu \right). \tag{5.7}$$

By combining (5.6) and (5.7), the proof is concluded. □

6. Integrable Functions

In Chapter 5 from [Bartle], the General Lebesgue Integral is established for functions with negative values. More specifically, $\mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ is defined as the collection of all $\boldsymbol{\mathcal{X}}$-measurable real valued functions such that both the positive and negative parts have finite integrals.

Proposition 6.1.

(a) *If $f \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ and $a > 0$, then the set $\{x \in \mathcal{X} : |f(x)| \geq a\}$ has finite measure.*

(b) *If $f \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ and $a > 0$, then the set $\{x \in \mathcal{X} : f(x) \neq 0\}$ has σ-finite measure (that is, the union of measurable sets with finite measure).*

Proof. Let $\mathcal{P}_a = \{x \in \mathcal{X} : f^+(x) \geq a\}$ and let $\mathcal{N}_a = \{x \in \mathcal{X} : f^-(x) \geq a\}$. Therefore, $\{x \in \mathcal{X} : |f(x)| \geq a\} = \mathcal{P}_a \cup \mathcal{N}_a$. Now $a\mathbf{1}_{\mathcal{P}_a} \leq f^+$. Thus by Lemma 4.5 (a) and since $f \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$,

$$a\mu(\mathcal{P}_a) \leq \int f^+ \, d\mu < \infty.$$

Likewise $\mu(\mathcal{N}_a) < \infty$. Therefore, for $a > 0$

$$\mu(\{x \in \mathcal{X} : |f(x)| \geq a\}) = \mu(\mathcal{P}_a \cup \mathcal{N}_a) < \infty.$$

For (b), notice that

$$\{x \in \mathcal{X} : f(x) \neq 0\} = \left\{x \in \mathcal{X} : f^+(x) > 0\right\} \cup \left\{x \in \mathcal{X} : f^-(x) > 0\right\}.$$

Since $f \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$, then (b) follows from Proposition 5.16 □

Proposition 6.2. *If f is $\boldsymbol{\mathcal{X}}$-measurable function and if $f(x) = 0$ for μ-almost all $x \in \boldsymbol{\mathcal{X}}$, then $f \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ and*

$$\int f \, d\mu = 0. \tag{6.1}$$

Proof. Since $f = 0$ μ-almost everywhere, then $f^+ = 0$, $f^- = 0$ μ-almost everywhere. From Corollary 4.19 [Bartle], $\int f^+ \, d\mu = 0$, $\int f^- \, d\mu = 0$. Therefore, $f \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ and (6.1) holds. □

Proposition 6.3. *If $f \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ and g is an $\boldsymbol{\mathcal{X}}$-measurable real valued function such that $f(x) = g(x)$ almost everywhere on $\mathcal{X}$, then $g \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ and*

$$\int f \, d\mu = \int g \, d\mu.$$

Proof. Let $\mathcal{N} = \{x \in \mathcal{X} : f(x) \neq g(x)\}$. Therefore, by assumption $\mu(\mathcal{N}) = 0$. Thus

$$\begin{aligned}\int |g| \, d\mu &= \int_{\mathcal{X}\setminus\mathcal{N}} |g| \, d\mu + \int_{\mathcal{N}} |g| \, d\mu \\ &= \int_{\mathcal{X}\setminus\mathcal{N}} |f| \, d\mu + \int_{\mathcal{N}} |g| \, d\mu = \int_{\mathcal{X}\setminus\mathcal{N}} |f| \, d\mu < \infty.\end{aligned}$$

Thus, by 5.3 Theorem [Bartle] $g \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$. Finally, arguing as above,

$$\begin{aligned}\int_{\mathcal{X}} g \, d\mu &= \int_{\mathcal{X}\setminus\mathcal{N}} g \, d\mu + \int_{\mathcal{N}} g \, d\mu = \int_{\mathcal{X}\setminus\mathcal{N}} f \, d\mu + \int_{\mathcal{N}} g \, d\mu \\ &= \int_{\mathcal{X}} f \, d\mu - \int_{\mathcal{N}} f \, d\mu + \int_{\mathcal{N}} g \, d\mu = \int_{\mathcal{X}} f \, d\mu.\end{aligned}$$

This completes the proof. □

Proposition 6.4. *If $f \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ and $\varepsilon > 0$, then there exists a $\boldsymbol{\mathcal{X}}$ – measurable simple function ϕ such that*

$$\int |f - \phi| \, d\mu < \varepsilon. \tag{6.2}$$

Proof. By Lemma 2.11 [Bartle] and MCT there exists a ϕ^+ and ϕ^- in $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ such that

$$\int \left|f^+ - \phi^+\right| \, d\mu < \varepsilon \text{ and } \int \left|f^- - \phi^-\right| \, d\mu < \varepsilon.$$

Define $\mathcal{P} = \{x \in \mathcal{X} : f(x) \geq 0\}$ and $\mathcal{N} = \{x \in \mathcal{X} : f(x) < 0\}$. Certainly, $\mathcal{P} \cup \mathcal{N} = \mathcal{X}$ and $\mathcal{P} \cap \mathcal{N} = \emptyset$. Therefore,

$$\begin{aligned}\int |f - \phi| \, d\mu &= \int_{\mathcal{P}} |f - \phi| \, d\mu + \int_{\mathcal{N}} |f - \phi| \, d\mu \\ &= \int \left|f^+ - \phi^+\right| \mathbf{1}_{\mathcal{P}} \, d\mu + \int \left|f^- - \phi^-\right| \mathbf{1}_{\mathcal{N}} \, d\mu < \varepsilon.\end{aligned}$$

□

Proposition 6.5. *If $f \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ and g is a bounded measurable function, then the product fg also belongs to $\mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$.*

Proof. By assumption g bounded therefore there exists a constant K such that $|g| \leq K$. By 5.5 Theorem [Bartle], Kf is integrable. By 2.6 Lemma [Bartle], fg is measurable. Clearly, $|fg| \leq |Kf|$, therefore by 5.4 Corollary, fg is in $\mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$. □

Proposition 6.6. *Suppose that f is in $\mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ and that its indefinite integral is*

$$\lambda(\mathcal{E}) = \int_{\mathcal{E}} f \, d\mu, \qquad \mathcal{E} \in \mathcal{X}.$$

Then

(a) $\lambda(\mathcal{E}) \geq 0$ *for all* $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ *if and only if* $f(x) \geq 0$ *for almost all* $x \in \mathcal{X}$.

(b) $\lambda(\mathcal{E}) = 0$ *for all* $\mathcal{E}$ *if and only if* $f(x) = 0$ *for almost all* $x \in \mathcal{X}$.

Proof.

(a) Assume $\lambda(\mathcal{E}) \geq 0$ for all $\mathcal{E} \in \boldsymbol{\mathcal{X}}$. Let $\mathcal{N} = \{x \in \mathcal{X} : f(x) < 0\}$. Therefore, $\lambda(\mathcal{N}) = \int_{\mathcal{N}} f^+ \, d\mu - \int_{\mathcal{N}} f^- \, d\mu$. Clearly $\int_{\mathcal{N}} f^+ \, d\mu = 0$, thus $\int_{\mathcal{N}} f^- \, d\mu = 0$ for all $x \in \mathcal{N}$. Since $\int_{\mathcal{N}} f^- \, d\mu = 0$ by 4.10 Corollary, $f^- = 0$ for almost all $x \in \boldsymbol{\mathcal{X}}$. Whence, $f = f^+$ for almost all $x \in \mathcal{X}$ and thus $f \geq 0$ for almost all $x \in \mathcal{X}$.

Now suppose $f \geq 0$ for almost all $x \in \mathcal{X}$. Then $f^- = 0$ almost everywhere. By 4.10 Corollary [Bartle], $\int f^- \, d\mu = 0$ For all $\mathcal{E} \in \boldsymbol{\mathcal{X}}$,

$$\lambda(\mathcal{E}) = \int f \mathbf{1}_{\mathcal{E}} \, d\mu = \int f^+ \mathbf{1}_{\mathcal{E}} \, d\mu \geq 0.$$

(b) Define $\mathcal{P} = \{x \in \mathcal{X} : f(x) \geq 0\}$. Let $\mathcal{N}$ be as in (a). Suppose $\lambda(\mathcal{E}) = 0$ for all $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ then,

$$0 = \lambda(\mathcal{P}) = \int f^+ \, d\mu$$

Therefore by 4.10 Corollary [Bartle], $f^+ = 0$ almost everywhere. Likewise

$$0 = \lambda(\mathcal{N}) = \int f^- \, d\mu.$$

Therefore, $f^- = 0$ almost everywhere. So then $f = 0$ almost everywhere.

Assume $f = 0$ for almost $x \in \boldsymbol{\mathcal{X}}$. Then by Proposition 6.2 for all $\mathcal{E} \in \boldsymbol{\mathcal{X}}$

$$\lambda(\mathcal{E}) = \int_{\mathcal{E}} f \, d\mu = 0.$$

□

Proposition 6.7. *Suppose that f_1, f_2 are in $\mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ and let λ_1, λ_2 be their indefinite integrals, then $\lambda_1(\mathcal{E}) = \lambda_2(\mathcal{E})$ for all $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ if and only if $f_1(x) = f_2(x)$ for almost all x in $\boldsymbol{\mathcal{X}}$.*

Proof. Define $f = f_1 - f_2$ and $\lambda(\mathcal{E}) = \int_{\mathcal{E}} f \, d\mu$. Clearly $\int f \, d\mu = \int f_1 \, d\mu - \int f_2 \, d\mu$. Assume $\lambda_1(\mathcal{E}) = \lambda_2(\mathcal{E})$ for all $\mathcal{E} \in \boldsymbol{\mathcal{X}}$. Then

$$\lambda(\mathcal{E}) = \int_{\mathcal{E}} f \, d\mu = \int_{\mathcal{E}} f_1 \, d\mu - \int_{\mathcal{E}} f_2 \, d\mu = 0.$$

By Proposition 6.6, $f = 0$ for almost all x in $\mathcal{X}$ which means that $f_1 = f_2$ for almost all x in $\mathcal{X}$.

Let $f_1 = f_2$ for almost all x on $\mathcal{X}$. Then $f = 0$ almost everywhere. Again by Proposition 6.2 for all $\mathcal{E} \in \boldsymbol{\mathcal{X}}$,

$$0 = \lambda(\mathcal{E}) = \lambda_1(\mathcal{E}) - \lambda_2(\mathcal{E}).$$

This completes the proof. □

Proposition 6.8. *Let $\mathcal{X} = \mathbb{N}$, let $\boldsymbol{\mathcal{X}}$ be all subsets of $\mathbb{N}$, and let μ be the counting measure on $\boldsymbol{\mathcal{X}}$, then f belongs to $\mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ if and only if the series $\sum_{n=1}^{\infty} f(n)$ is absolutely convergent, in which case*

$$\int f \, d\mu = \sum_{n=1}^{\infty} f(n). \tag{6.3}$$

Proof. From 5.3 Theorem [Bartle] and the definition of $\mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$, $f \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ if and only if $\int |f|^+ \, d\mu$, $\int |f|^- \, d\mu$ have finite values and $|f|^+$ and $|f|^-$ are in $M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$. By Proposition 5.7,

$$\int |f|^+ \, d\mu = \sum_{n=1}^{\infty} |f(n)|^+ < \infty \text{ and } \int |f|^- \, d\mu = \sum_{n=1}^{\infty} |f(n)|^- < \infty$$

Therefore, combining the above equations, it is clear that $\sum f(n)$ is absolutely convergent, provided $f \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$.

If $\sum_{n=1}^{\infty} |f(n)| < \infty$, then $\sum_{n=1}^{\infty} f^+(n)$ and $\sum_{n=1}^{\infty} f^-(n)$ have finite values. Now f^+ and f^- have nonnegative values, therefore Proposition 5.7 may be applied to give $f \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$. Lastly,

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu = \sum_{n=1}^{\infty} f^+(n) - \sum_{n=1}^{\infty} f^-(n) = \sum_{n=1}^{\infty} f(n).$$

□

Proposition 6.9. *If $\{f_n\}$ is a sequence in $\mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ which converges uniformly on $\mathcal{X}$ to a function f, and $\mu(\mathcal{X}) < \infty$, then*

$$\int f \, d\mu = \lim_{n \to \infty} \int f_n \, d\mu.$$

Proof. Let $f = f^+ - f^-$ and $f_n = f_n^+ - f_n^-$. By Proposition 5.12,

$$\lim_{n \to \infty} \int f_n^+ \, d\mu = \int f^+ \, d\mu \text{ and } \lim_{n \to \infty} \int f_n^- \, d\mu = \int f^- \, d\mu.$$

Therefore,

$$\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu = \lim \int f_n^+ \, d\mu - \lim \int f_n^- \, d\mu = \lim \int f_n \, d\mu.$$

□

Remark 2. In general, the conclusion in Proposition 6.9 is false if the condition $\mu(\mathcal{X}) < \infty$ is dropped. Let $f_n = \frac{1}{n}\mathbf{1}_{[0,\ n]}$, let $\mathcal{X} = \mathbb{R}$ and $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{B}}$ and let μ be the Lebesgue measure. Clearly, f_n converges uniformly to $f = 0$. Moreover, $\int f_n \, d\mu = 1$. Thus,

$$\lim \int f_n \, d\mu = 1 \neq \int f \, d\mu = 0.$$

Remark 3. In general, the Lebesgue Dominated Convergence Theorem fails if the condition $|f_n| \le g$ for all n and g integrable is dropped. Let $f_n = n\mathbf{1}_{[0,\ \frac{1}{n}]}$, let $\mathcal{X} = \mathbb{R}$ and $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{B}}$ and let λ be the Lebesgue measure. Now $\lim f_n$ converges almost everywhere to $f = 0$. However, there does not exist an integrable function g such that $|f_n| \le g$ for all n. Clearly, $\int f_n \, d\mu = 1$. Thus,

$$\int f \, d\lambda = 0 \neq 1 = \lim \int f_n \, d\lambda.$$

Proposition 6.10. *If $f_n \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$, and if*

$$\sum_{n=1}^{\infty} \int |f_n| \; d\mu < \infty,$$

then the series $\sum f_n(x)$ converges almost everywhere to a function f in $\mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$. Moreover,

$$\int f \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.$$

Proof. By 4.13 Corollary [Bartle] and since $f_n^+ \le |f_n|$ for all $n \in \mathbb{N}$,

$$\int \sum_{n=1}^{\infty} f_n^+ \, d\mu \le \int \sum_{n=1}^{\infty} |f_n| \; d\mu < \infty.$$

Therefore, $\sum_{n=1}^{\infty} f_n^+$ converges almost everywhere to some function in $\mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$. Call this function f^+. Likewise, $\sum_{n=1}^{\infty} f_n^-$ converges almost everywhere to some function in $\mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$. Call this function f^-. Combining $\sum_{n=1}^{\infty} f_n^+$ and $\sum_{n=1}^{\infty} f_n^+$, it is clear that $\sum_{n=1}^{\infty} f_n(x)$ converges almost everywhere to f in $\mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$. Moreover,

$$\begin{aligned}
\int f \, d\mu = \int \sum_{n=1}^{\infty} f_n \, d\mu &= \int \left[\sum_{n=1}^{\infty} f_n^+ - \sum_{n=1}^{\infty} f_n^- \right] d\mu \\
&= \int \sum_{n=1}^{\infty} f_n^+ \, d\mu - \int \sum_{n=1}^{\infty} f_n^- \, d\mu \\
&= \sum_{n=1}^{\infty} \left[\int f_n^+ - \int f_n^- \, d\mu \right] = \sum_{n=1}^{\infty} \int f_n \, d\mu.
\end{aligned}$$

□

Proposition 6.11. *Let $f_n \in \mathcal{L}(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$, and suppose that $\{f_n\}$ converges to a function f. If*

$$\lim_{n\to\infty} \int |f_n - f| \, d\mu = 0, \qquad \textit{then} \int |f| \, d\mu = \lim_{n\to\infty} \int |f_n| \, d\mu.$$

Proof. By hypothesis and by the triangle inequality, for $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ so that for each $n \geq N_\varepsilon$

$$\int \Big| |f_n| - |f| \Big| \, d\mu \leq \int |f_n - f| \, d\mu < \varepsilon.$$

By 5.3 Theorem [Bartle] for each $n \geq N_\varepsilon$

$$\left| \int (|f_n| - |f|) \, d\mu \right| \leq \int \Big| |f_n| - |f| \Big| \, d\mu < \varepsilon.$$

Therefore,

$$\lim_{n\to\infty} \int |f_n| \, d\mu = \int |f| \, d\mu.$$

□

Proposition 6.12. *Let f be an $\boldsymbol{\mathcal{X}}$ – measurable function on $\mathcal{X}$ to $\mathbb{R}$. For $n \in \mathbb{N}$, let $\{f_n\}$ be the sequence of truncates of f (see Proposition 3.20. If f is integrable with respect to μ, then*

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu.$$

Conversely, if

$$\sup_{n\in\mathbb{N}} \int |f_n| \, d\mu < \infty,$$

then f is integrable.

Proof. Assume f is integrable with respect to μ. By Proposition 3.20 f_n measurable for each $n \in \mathbb{N}$. Since $|f_n| \leq |f|$ for each $n \in \mathbb{N}$ and f integrable, by 5.4 Corollary f_n is integrable for each $n \in \mathbb{N}$. Clearly f_n converges almost everywhere to f, therefore by the Lebesgue Dominated Convergence Theorem (LDCT),

$$\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu.$$

Now assume $\sup_n \int |f_n| \, d\mu < \infty$. Then $|f_n|$ is integrable for all n and $|f_n|$ is monotone increasing to $|f|$. So by (MCT)

$$\int |f| \, d\mu = \lim_{n \to \infty} \int |f_n| \, d\mu \leq \sup_n \int |f_n| \, d\mu < \infty.$$

Thus f is integrable.

□

7. The Lebesgue Spaces L_p

Definition 7.1. If $\mathcal{V}$ is a real linear (=vector) space, then a real valued function N on $\mathcal{V}$ is said to be a norm for $\mathcal{V}$ in case it satisfies

(i) $N(v) \geq 0$ for all $v \in \mathcal{V}$
(ii) $N(v) = 0$ if and only if $v = 0$
(iii) $N(\alpha v) = |\alpha|\, N(v)$ for all $v \in \mathcal{V}$ and real α;
(iv) $N(u + v) \leq N(u) + N(v)$ for all $u, v \in \mathcal{V}$

If condition (ii) is dropped, the function N is said to be a **semi-norm** or a **pseudo-norm** for $\mathcal{V}$.

Example 7.2. Let $C[0,1]$ be the linear space of continuous functions on $[0,1]$ to $\mathbb{R}$. Define N_0 for f in $C[0,1]$ by $N_0(f) = |f(0)|$.Clearly, $N_0(f) = |f(0)| \geq 0$ for all $f \in C[0,1]$. Moreover, $N_0(\alpha f) = |\alpha|\,|f(0)| = |\alpha|\, N_0(f)$. Lastly, if $f, g \in C[0,1]$, then

$$N_0(f + g) = |f(0) + g(0)| \leq |f(0)| + |g(0)| = N_0(f) + N_0(g).$$

Therefore, $N_0(f) = |f(0)|$ is a semi-norm since Conditions (i), (iii), and (iv) of Definition 7.1 are satisfied.

Example 7.3. Let $C[0,1]$ be as before and define N_1 for f in $C[0,1]$ to be the Riemann integral of $|f|$ over $[0,1]$. Obviously, by the properties of the Riemann integral N_1 satisfies the conditions of a semi-norm.

Proposition 7.4. *If $\{f_n\}$ is defined for $n \geq 1$ to be equal to 0 for $0 \leq x \leq (1 - 1/n)/2$, to be equal to 1 for $\frac{1}{2} \leq x \leq 1$, and to be linear for $(1 - 1/n)/2 \leq x \leq \frac{1}{2}$, then $\{f_n\}$ is a Cauchy sequence, but $\{f_n\}$ does not converge relative in N_1 (as defined in Example 7.3) to an element of $C[0,1]$.*

Proof. Clearly, $\lim f_n = f$ where $f = 0$ if $0 < x \leq \frac{1}{2}$ and $f = 1$ if $\frac{1}{2} \leq x \leq 1$. Certainly, f is not in $C[0,1]$. Now, let $m \geq n$ where $m, n \in \mathbb{N}$. From the definition of N_1,

$$N_1(f_m - f_n) = \int_0^1 |f_m - f_n| \, dx = \frac{1}{4n}\left[f_n\left(\frac{1}{2} - \frac{1}{2m}\right)\right].$$

However $f_n\left(\frac{1}{2} - \frac{1}{2m}\right) \leq 1$. Therefore,

$$\lim_{n\to\infty} \frac{1}{4n}\left[f_n\left(\frac{1}{2} - \frac{1}{2m}\right)\right] \leq \lim_{n\to\infty} \frac{1}{4n} = 0.$$

Hence, for $\varepsilon > 0$, there exists an $M(\varepsilon) \in \mathbb{N}$ such that

$$N_1(f_m - f_n) = \int_0^1 |f_m - f_n| \, dx = \frac{1}{4n}\left[f_n\left(\frac{1}{2} - \frac{1}{2m}\right)\right] < \varepsilon$$

for all $n, m \geq M(\varepsilon)$. Therefore $\{f_n\}$ Cauchy. □

Proposition 7.5. *Let N be a norm on a linear space $\mathcal{V}$ and let d be defined for $u, v \in \mathcal{V}$ by $d(u,v) = N(u-v)$, then d is a metric on N; that is*

(i) $d(u,v) \geq 0$ *for all* $u, v \in \mathcal{V}$

(ii) $d(u,v) = 0$ *if and only if* $u = v$

(iii) $d(u,v) = d(v,u)$

(iv) $d(u,v) \leq d(u,w) + d(w,v)$.

Proof. Condition (i) follows immediately, since N is a norm and $u - v \in \mathcal{V}$. Condition (ii) is satisfied since $d(u,v) = N(u-v)$ if and only if $u - v = 0$ if and only if $u = v$. Condition (iii) follows since

$$d(u,v) = N(u-v) = N\big((-1)(v-u)\big) = |-1| \, N(v-u) = d(v,u).$$

Finally let $w, u, v \in \mathcal{V}$

$$\begin{aligned} d(u,v) = N(u-v) &= N(u - w + w - v) \\ &\leq N(u,w) + N(w,v) = d(u,w) + d(w,v). \end{aligned}$$

Thus, Condition (iv) is satisfied. □

Proposition 7.6. *Let $1 \leq p < \infty$. If $f \in L_p$ and $\varepsilon > 0$, then there exists a simple $\boldsymbol{\mathcal{X}}$ − measurable function ϕ such that $\|f - \phi\|_p < \varepsilon$.*

Proof. The case $p = 1$ follows from Proposition 6.4. Let $1 < p < \infty$. From 2.11 Lemma [Bartle] there exists $\{\phi_n\}$ of $\boldsymbol{\mathcal{X}}$-measurable functions such that

$|f^+ - \phi_n^+|$ converges almost everywhere to 0 and $\phi_n^+ \leq f^+$. Certainly, then $|f^+ - \phi_n^+|^p$ converges almost everywhere to 0. Since $|f^+ - \phi_n^+|^p \leq 2^p\,|f|$ the Lebesgue Dominated Convergence Theorem implies that

$$\int |f^+ - \phi_n^+|^p \, d\mu = \lim_{n\to\infty} \int |f^+ - \phi_n^+|^p \, d\mu$$

Therefore, given $\varepsilon > 0$, $\|f^+ - \phi_n^+\|_p < \varepsilon$ for n sufficiently large. Likewise, $\|f^- - \phi_n^-\|_p < \varepsilon$ for n sufficiently large. Let $\lim \phi_n^+ = \phi^+$ and let $\lim \phi_n^- = \phi^-$. Moreover, let $\mathcal{P} = \{x \in \mathcal{X} : f(x) \geq 0\}$ and let $\mathcal{N} = \{x \in \mathcal{X} : f(x) < 0\}$. Then,

$$\left[\int |f - \phi|^p \, d\mu\right]^{\frac{1}{p}} = \left[\int |f^+ - \phi^+|^p \, \mathbf{1}_{\mathcal{P}} \, d\mu + \int |f^- - \phi^-|^p \, \mathbf{1}_{\mathcal{N}} \, d\mu\right]^{\frac{1}{p}} < \varepsilon$$

for n sufficiently large enough. □

Remark 4. Proposition 7.6 holds if $f \in L_\infty$.

Proof. For $p = \infty$, $\|f(x) - \phi(x)\|_\infty = \inf\{S(\mathcal{N}) : \mathcal{N} \in \boldsymbol{\mathcal{X}}, \mu(\mathcal{N}) = 0\}$ where $S(\mathcal{N}) = \sup\{|f(x) - \phi(x)| : x \notin \mathcal{N}\}$. As seen before, for $\varepsilon > 0$ there exists a simple $\boldsymbol{\mathcal{X}}$-measurable function such that $|f(x) - \phi(x)| < \varepsilon$ for all $x \notin \mathcal{N}$. Therefore,

$$S(\mathcal{N}) = \sup\{|f(x) - \phi(x)| : x \notin \mathcal{N}\} < \varepsilon.$$

Thus

$$\|f(x) - \phi(x)\|_\infty = \inf\{S(\mathcal{N}) : \mathcal{N} \in \boldsymbol{\mathcal{X}}, \mu(\mathcal{N}) = 0\} < \varepsilon.$$

□

Proposition 7.7. *If $f \in L_p$, $1 \leq p < \infty$, and if $\mathcal{E} = \{x \in \mathcal{X} : |f(x)| \neq 0\}$, then $\mathcal{E}$ is σ-finite.*

Proof. Let $\mathcal{E}_n = \left\{x \in \mathcal{X} : |f(x)| \geq \frac{1}{n}\right\}$. Then $\mathcal{E} = \bigcup_{n=1}^\infty \mathcal{E}_n$. The proof is complete if $\mu(\mathcal{E}_n) < \infty$ for each $n \in \mathbb{N}$. Clearly, $|f| \geq \frac{1}{n}\mathbf{1}_{\mathcal{E}_n}$ for each n. Thus, $|f|^p \geq \left(\frac{1}{n}\mathbf{1}_{\mathcal{E}_n}\right)^p$ for each n. By 4.5 Lemma, $\int |f|^p \, d\mu \geq \int \left(\frac{1}{n}\right)^p \mu(\mathcal{E}_n)$ for each n. Therefore,

$$\infty > \left[\int |f|^p \, d\mu\right]^{\frac{1}{p}} \geq \frac{1}{n}\left[\mu(\mathcal{E}_n)\right]^{\frac{1}{p}} \qquad \text{for each } n.$$

It can be concluded that $\mu(\mathcal{E}_n) < \infty$ for each $n \in \mathbb{N}$. □

Proposition 7.8. *If $f \in L_p$, and if $\mathcal{E}_n = \{x \in \mathcal{X} : |f(x)| \geq n\}$, then $\mu(\mathcal{E}_n) \to 0$ as $n \to \infty$.*

Proof. From the definition of $\mathcal{E}_n$, $|f(x)| \geq n\mathbf{1}_{\mathcal{E}_n}(x)$ for all $x \in \mathcal{X}$. Whence, $|f(x)|^p \geq n^p \mathbf{1}_{\mathcal{E}_n}(x)$. By 4.5 Lemma and $f \in L_p$

$$\lim_{n\to\infty} \mu(\mathcal{E}_n) \leq \lim_{n\to\infty} \frac{1}{n^p} \int |f|^p \, d\mu = 0.$$

This finishes the proof. □

Proposition 7.9. *Let $\mathcal{X} = \mathbb{N}$ and let μ be the counting measure on $\mathbb{N}$. If f is defined on $\mathbb{N}$ by $f(n) = \frac{1}{n}$, then f does not belong to L_1, but it does belong to L_p for $1 < p \leq \infty$.*

Proof. Let $p = 1$ therefore,

$$\int |f(n)| \, d\mu = \sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Let $p > 1$ therefore,

$$\int |f(n)|^p \, d\mu = \sum_{n=1}^{\infty} \frac{1}{n^p} < \infty.$$

□

An alternative way to look at Proposition 7.9 is to let $\mathcal{X} = \mathbb{R}$, $\boldsymbol{\mathcal{X}} = \boldsymbol{\mathcal{B}}$, and μ be the Lebesgue measure and define $g(x) = 0$ for $x < 1$ and $g(x) = \frac{1}{x}$ for $x \geq 1$. For $p = 1$,

$$\int |g(x)| \, d\mu = \int_{[1,\infty]} |g(x)| \, d\mu = \lim_{n\to\infty} \int_1^n \frac{1}{x} \, dx = \lim_{n\to\infty} \ln(n) = \infty.$$

For $p > 1$,

$$\int |g(x)|^p \, d\mu = \int_{[1,\infty]} |g(x)|^p \, d\mu = \lim_{n\to\infty} \left[\frac{1}{p-1} + \frac{1}{(-p+1)n^{p-1}} \right] = \frac{1}{p-1}.$$

Proposition 7.10. *Let $\mathcal{X} = \mathbb{N}$, and let λ be the measure on $\mathbb{N}$ which has measure $\frac{1}{n^2}$ at the point n. (More precisely $\lambda(\mathcal{E}) = \sum \left\{ \frac{1}{n^2} : n \in \mathcal{E} \right\}$.) Then*

(i) $\lambda(\mathcal{X}) < \infty$

(ii) *for f defined on $\mathcal{X}$ as $f(n) = \sqrt{n}$, $f \in L_p$ if and only if $1 \leq p < 2$.*

Proof. Clearly,

$$\lambda(\mathcal{X}) = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Also,

$$\int \left|\sqrt{n}\right|^p \, d\mu = \sum_{n=1}^{\infty} n^{\frac{p-4}{2}}.$$

For $1 \leq p < 2$, $-\frac{3}{2} \leq \frac{p-4}{2} < -1$. The result follows from the fact that $\sum \frac{1}{n^p} < \infty$ when $p > 1$. □

By letting $\lambda(\mathcal{E}) = \sum \left\{\frac{1}{n^{2.1}} : n \in \mathcal{E}\right\}$ and letting $f(n) = n^{\frac{1}{p_0}}$ then $f \in L_p$ if and only if $1 \leq p \leq (1.1)p_0$. It is seen that

$$\int |f(n)|^p \, d\mu = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{2.1 - \frac{p}{p_0}}.$$

Obviously $2.1 - \frac{p}{p_0} > 1$ for all $1 \leq p \leq (1.1)p_0$.

Proposition 7.11. *Let $(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ be a finite measure space. If f is $\boldsymbol{\mathcal{X}}$-measurable function let $\mathcal{E}_n = \{x \in \mathcal{X} : (n-1) \leq |f(x)| < n\}$. Show that $f \in L_1$ if and only if*

$$\sum_{n=1}^{\infty} n \, \mu(\mathcal{E}_n) < \infty. \tag{7.1}$$

More generally, $f \in L_p$ for $1 \leq p < \infty$, if and only if

$$\sum_{n=1}^{\infty} n^p \mu(\mathcal{E}_n) < \infty. \tag{7.2}$$

Proof. Assume (7.1) holds. From the definition of $\mathcal{E}_n$, $n \, \mathbf{1}_{\mathcal{E}_n}(x) \geq |f(x)| \, \mathbf{1}_{\mathcal{E}_n}(x)$ for each $n \in \mathbb{N}$ and $x \in \mathcal{X}$. By a now familiar argument,

$$\int_{\mathcal{E}_n} |f(x)| \, d\mu \leq n \, \mu(\mathcal{E}_n) \text{ for each } n \in \mathbb{N}.$$

Since $\mathcal{E}_n \cap \mathcal{E}_m = \emptyset$ when $n \neq m$ and $\bigcup \mathcal{E}_n = \mathcal{X}$,

$$\sum_{n=1}^{\infty} \int_{\mathcal{E}_n} |f(x)| \, d\mu = \int_{\mathcal{X}} |f(x)| \, d\mu \leq \sum_{n=1}^{\infty} n \, \mu(\mathcal{E}_n) < \infty.$$

Therefore, $f \in L_1$.

Assume $f \in L_1$. For each $n \in \mathbb{N}$ and $x \in \mathcal{X}$, $(n-1)\mathbf{1}_{\mathcal{E}_n}(x) \leq |f(x)|\, \mathbf{1}_{\mathcal{E}_n}(x)$ Arguing as above,

$$\sum_{n=1}^{\infty}(n-1)\,\mu(\mathcal{E}_n) \leq \int_{\mathcal{X}} |f(x)|\; d\mu < \infty.$$

Now

$$\sum_{n=1}^{\infty}(n-1)\,\mu(\mathcal{E}_n) = \sum_{n=1}^{\infty}(n)\,\mu(\mathcal{E}_n) - \sum_{n=1}^{\infty}\mu(\mathcal{E}_n) < \infty.$$

Since $(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ is a finite measure space, $\sum_{n=1}^{\infty}\mu(\mathcal{E}_n) < \infty$, therefore (7.1) holds.

As above, it is quickly seen that

$$\int_{\mathcal{X}} |f(x)|^p\; d\mu \leq \sum_{n=1}^{\infty} n^p\,\mu(\mathcal{E}_n) < \infty$$

Thus, if (7.2) holds, then $f \in L_p$. Now suppose $f \in L_p$. For all $x \in \mathcal{E}_n$, $n-1 \leq |f(x)|$. Clearly then, $n^p \leq (|f(x)|+1)^p \leq 2^p|f(x)|^p + 1m$ for all $x \in \mathcal{E}_n$. From this inequality, we get (7.2).

□

Proposition 7.12. *If $(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ is a finite measure space and $f \in L_p$, then $f \in L_r$ for $1 \leq r \leq p$.*

Proof. From Proposition 7.11, $f \in L_p$ implies that $\sum n^p\mu(\mathcal{E}_n) < \infty$. For $1 \leq r \leq p$, $n^r\mu(\mathcal{E}_n) \leq n^p\mu(\mathcal{E}_n)$ for each $n \in \mathbb{N}$. Therefore,

$$\sum_{n=1}^{\infty} n^r\mu(\mathcal{E}_n) \leq \sum_{n=1}^{\infty} n^p\mu(\mathcal{E}_n) < \infty.$$

Employing Proposition 7.11 again, gives the conclusion. □

Proposition 7.13. *Suppose that $\mathcal{X} = \mathbb{N}$ and μ is the counting measure on $\mathbb{N}$. If $f \in L_p$, then $f \in L_s$ with $1 \leq p \leq s < \infty$, and $\|f\|_s \leq \|f\|_p$.*

Proof. Since μ is the counting measure on $\mathbb{N}$, f can be viewed as a sequence, $\{a_n\}$ of real numbers. Therefore,

$$\|f\|_p = \left[\sum_{n=1}^{\infty}|a_n|^p\right]^{\frac{1}{p}} < \infty$$

Since $\sum |a_n|^p < \infty$, the $\lim |a_n|^p = 0$. Thus for some $n_\circ \in \mathbb{N}$, $|a_n| < 1$ for all $n \geq n_\circ$. Since $1 \leq p \leq s$, $|a_n|^s < |a_n|^p$ for all $n \geq n_\circ$. It can be concluded then that

$$\infty > \sum_{n=n_0}^{\infty} |a_n|^p > \sum_{n=n_\circ}^{\infty} |a_n|^s .$$

Clearly,

$$\sum_{n=1}^{n_\circ - 1} |a_n|^s < \infty.$$

Therefore, $f \in L_s$. □

Proposition 7.14. *Let $(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ be any measure space and let f belong to both L_{p_1} and L_{p_2}, with $1 \leq p_1 \leq p_2 < \infty$, then $f \in L_p$ for any value of p such that $p_1 \leq p \leq p_2$.*

Proof. Let $\mathcal{A} = \{x \in \mathcal{X} : |f(x)| \geq 1\}$. Let $\mathcal{B} = \{x \in \mathcal{X} : |f(x)| < 1\}$. Now,

$$\int |f(x)|^p \mathbf{1}_{\mathcal{A}}\, d\mu \leq \int |f(x)|^{p_2} \mathbf{1}_{\mathcal{A}}\, d\mu < \infty.$$

Furthermore,

$$\int |f(x)|^p \mathbf{1}_{\mathcal{B}}\, d\mu \leq \int |f(x)|^{p_1} \mathbf{1}_{\mathcal{B}}\, d\mu < \infty.$$

Therefore,

$$\int |f(x)|^p \mathbf{1}_{\mathcal{A}}\, d\mu + \int |f(x)|^p \mathbf{1}_{\mathcal{B}}\, d\mu = \int |f(x)|^p \, d\mu < \infty.$$

□

From Hölder's inequality, if $1 < p < \infty$, $\left(\frac{1}{p}\right) + \left(\frac{1}{q}\right) = 1$ and $f \in L_p$, then

$$\left| \int fg \, d\mu \right| \leq \|f\|_p$$

for all $g \in L_q$ such that $\|g\|_q \leq 1$. This leads to the following proposition.

Proposition 7.15. *If $f \neq 0$, $f \in L_p$, and $g_\circ(x) = c[signum f(x)] f(x)^{p-1}$ for x on $\mathcal{X}$ where $c = (\|f\|_p)^{-p/q}$, then $g_\circ \in L_q$, $\|g_\circ\|_q = 1$ and*

$$\left| \int f g_\circ \, d\mu \right| = \|f\|_p.$$

Proof. First, note that if $\frac{1}{p}+\frac{1}{q}=1$, then $pq-q=p$. Now

$$\int |g_\circ|^q \, d\mu = |c|^q \int |f|^{pq-q} \, d\mu = |c|^q \int |f|^p \, d\mu < \infty.$$

Thus, $g_\circ \in L_q$. Secondly,

$$\|g_\circ\|_q = \left[\int |g_\circ|^q \, d\mu\right]^{\frac{1}{q}} = |c| \left[\int |f|^p \, d\mu\right]^{\frac{1}{q}} = |c| \left(\|f\|_p\right)^{\frac{p}{q}} = \frac{\left(\|f\|_p\right)^{\frac{p}{q}}}{\left(\|f\|_p\right)^{\frac{p}{q}}} = 1.$$

Finally

$$\left|\int f g_\circ \, d\mu\right| = c \int [signum f] \, f^p \, d\mu = c \left(\|f\|_p\right)^p = \frac{\left(\|f\|_p\right)^p}{\left(\|f\|_p\right)^{p/q}} = \|f\|_p,$$

since $p - p/q = 1$. □

Proposition 7.16. *If $f \in L_p$, $1 \le p < \infty$ and $\varepsilon > 0$ then there exists a set $\mathcal{E}_\varepsilon \in \boldsymbol{\mathcal{X}}$ with $\mu(\mathcal{E}_\varepsilon) < \infty$ such that if $\mathcal{F} \in \boldsymbol{\mathcal{X}}$ and $\mathcal{F} \cap \mathcal{E}_\varepsilon = \emptyset$, then $\|f\mathbf{1}_{\mathcal{F}}\|_p < \varepsilon$.*

Proof. Let $\varepsilon > 0$. If $f \in L_p$, then $|f| \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$. Consequently, $|f|^p \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ since for $\alpha \in \mathbb{R}$,

$$\{x \in \mathcal{X} : |f|^p < \alpha\} = \left\{x \in \mathcal{X} : |f| < \alpha^{\frac{1}{p}}\right\} \in \boldsymbol{\mathcal{X}}.$$

From Proposition 5.17 there exists a set in $\boldsymbol{\mathcal{X}}$, call it $\mathcal{E}_\varepsilon$, such that $\mu(\mathcal{E}_\varepsilon) < \infty$ and

$$\int |f|^p \, d\mu < \int_{\mathcal{E}_\varepsilon} |f|^p \, d\mu + \varepsilon^p.$$

Let $\mathcal{F}$ be such that $\mathcal{F} \cap \mathcal{E}_\varepsilon = \emptyset$. It is clear then that

$$\int_{\mathcal{E}_\varepsilon} |f|^p \, d\mu + \int_{\mathcal{F}} |f|^p \, d\mu \le \int |f|^p \, d\mu < \int_{\mathcal{E}_\varepsilon} |f|^p \, d\mu + \varepsilon^p.$$

Therefore, $\|f\mathbf{1}_{\mathcal{F}}\|_p < \varepsilon$. □

Proposition 7.17. *Let $f_n \in L_p\left(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu\right)$, $1 \le p < \infty$, and let β_n be defined for $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ by*

$$\beta_n(\mathcal{E}) = \left[\int_{\mathcal{E}} |f_n|^p \, d\mu\right]^{\frac{1}{p}}.$$

Then $|\beta_n(\mathcal{E}) - \beta_m(\mathcal{E})| \le \|f_n - f_m\|_p$

Proof. Clearly it is sufficient to show that,

$$| \, \|f_n\|_p - \|f_m\|_p| \leq \|f_n - f_m\|_p.$$

Now

$$\|f_n\|_p = \|f_n\|_p - \|f_m\|_p + \|f_m\|_p \leq \|f_n - f_m\|_p + \|f_m\|_p.$$

Therefore, $\|f_n\|_p - \|f_m\|_p \leq \|f_n - f_m\|_p$ Arguing in the same fashion, it is seen that

$$\|f_m\|_p - \|f_n\|_p \leq \|f_m - f_n\|_p = \|f_n - f_m\|_p.$$

Finally,

$$-\|f_n - f_m\|_p \leq \|f_n\|_p - \|f_m\| \leq \|f_n - f_m\|_p$$

which proves the proposition. □

As a consequence of Proposition 7.17 if $\{f_n\}$ Cauchy sequence in L_p, then $\lim \beta_n$ exists for each $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ since every Cauchy sequence in $\mathbb{R}$ converges.

Proposition 7.18. *Let f_n, β_n be as in Proposition 7.17. If $\{f_n\}$ is a Cauchy sequence and $\varepsilon > 0$, then there exists a set $\mathcal{E}_\varepsilon \in \mathcal{X}$ with $\mu(\mathcal{E}_\varepsilon) < \infty$ such that if $\mathcal{F} \in \boldsymbol{\mathcal{X}}$ and $\mathcal{F} \cap \mathcal{E}_\varepsilon = \emptyset$, then $\beta_n(\mathcal{F}) < \varepsilon$ for all $n \in \mathbb{N}$.*

Proof. Let $\{f_n\}$ be a Cauchy sequence and let $\varepsilon > 0$, therefore we can find an $N(\varepsilon)$ such that for $n, m \geq N(\varepsilon)$, $\|f_n - f_m\|_p \leq \frac{\varepsilon}{2}$. Employing Proposition 7.16 there exist $\mathcal{E}_1, \cdots, \mathcal{E}_{N(\varepsilon)}$ such that

(i) $\mu\left([\mathcal{E}_i]^c\right) < \infty$ where $1 \leq i \leq N(\varepsilon)$

(ii) $\|f_i \mathbf{1}_{\mathcal{E}_i}\|_p < \frac{\varepsilon}{2}$ for $1 \leq i \leq N(\varepsilon)$.

Let $\mathcal{E}_\varepsilon = \displaystyle\bigcup_{i=1}^{N(\varepsilon)} \mathcal{E}_i^c$ and $\mathcal{F} \in \boldsymbol{\mathcal{X}}$ with $\mathcal{F} \cap \mathcal{E}_\varepsilon = \emptyset$. Clearly, $\mu(\mathcal{E}_\varepsilon) < \infty$. If $1 \leq n \leq N(\varepsilon)$, then $\mathcal{F} \subseteq \mathcal{E}_n$. Thus, by (ii)

$$\|f_n \mathbf{1}_{\mathcal{F}}\|_p \leq \|f_n \mathbf{1}_{\mathcal{E}_n}\|_p < \frac{\varepsilon}{2} < \varepsilon.$$

If $N(\varepsilon) < n$, then

$$\begin{aligned}\|f_n\mathbf{1}_{\mathcal{F}}\|_p &\leq \|f_n\mathbf{1}_{\mathcal{F}} - f_{N(\varepsilon)}\mathbf{1}_{\mathcal{F}}\|_p + \|f_{N(\varepsilon)}\mathbf{1}_{\mathcal{F}}\|_p < \|f_n\mathbf{1}_{\mathcal{F}} - f_{N(\varepsilon)}\mathbf{1}_{\mathcal{F}}\|_p + \frac{\varepsilon}{2} \\ &\leq \|f_n - f_{N(\varepsilon)}\|_p + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This concludes the proof. □

Proposition 7.19. *Let f_n, β_n be as in Proposition 7.18, and suppose $\{f_n\}$ is Cauchy. If $\varepsilon > 0$, then there exists a $\delta(\varepsilon)$ such that if $\mathcal{E} \in \mathcal{X}$ and $\mu(\mathcal{E}) < \delta(\varepsilon)$, then $\beta_n < \varepsilon$ for all $n \in \mathbb{N}$.*

Proof. Let $\lambda_n(\mathcal{E}) = \int |f_n|^p \, d\mu$. Clearly, $|f_n|^p \in M^+(\mathcal{X}, \boldsymbol{\mathcal{X}})$ for each $n \in \mathbb{N}$. By 4.9 Corollary [Bartle], λ_n is a measure for each n. By 4.11 Corollary [Bartle], λ_n is absolutely continuous with respect to μ for each n. Fix $\varepsilon > 0$. Since $\{f_n\}$ is Cauchy there exists an $N(\varepsilon)$ such that for $\|f_n - f_m\|_p < \frac{\varepsilon}{2}$ for $n, m \geq N(\varepsilon)$. Since λ_n is absolutely continuous with respect to μ, for each $n \in \mathbb{N}$ there exists an $\delta_n < 0$ such that if $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ and $\mu(\mathcal{E}) < \delta_n$, then $\lambda_n(\mathcal{E}) < \left(\frac{\varepsilon}{2}\right)^p$. Let $\delta_\circ(\varepsilon) = \inf\left\{\delta_1, \cdots, \delta_{N(\varepsilon)}\right\}$. Therefore for all n such that $1 \leq n \leq N(\varepsilon)$, if $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ and $\mu(\mathcal{E}) < \delta_\circ$, then $\lambda_n(\mathcal{E}) < \left(\frac{\varepsilon}{2}\right)^p \leq (\varepsilon)^p$, (i.e) for $1 \leq n \leq N(\varepsilon)$, $\|f_n\mathbf{1}_{\mathcal{E}}\| < \varepsilon$. Since $\{f_n\}$ is Cauchy and $\|f_{N(\varepsilon)}\mathbf{1}_{\mathcal{E}}\|_p < \frac{\varepsilon}{2}$, then for $n \geq N(\varepsilon)$,

$$\|f_n\mathbf{1}_{\mathcal{E}}\|_p \leq \|f_n\mathbf{1}_{\mathcal{E}} - f_{N(\varepsilon)}\mathbf{1}_{\mathcal{E}}\|_p + \|f_{N(\varepsilon)}\mathbf{1}_{\mathcal{E}}\|_p < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, if $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ and $\mu(\mathcal{E}) < \delta_\circ(\varepsilon)$, then

$$\beta_n(\mathcal{E}) = \|f_n\mathbf{1}_{\mathcal{E}}\|_p < \varepsilon,$$

for all $n \in \mathbb{N}$. □

The following remark will be useful in the next proposition.

Remark 5. If $f \in L_p(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$, $p = \infty$, then from the definition of $\|f\|_\infty$, $|f(x)| \leq \|f\|_\infty$ for almost all x

Proposition 7.20. *If $A < \|f\|_\infty$, then there exists a set $\mathcal{E} \in \boldsymbol{\mathcal{X}}$ with $\mu(\mathcal{E}) > 0$ such that $|f(x)| > A$ for all $x \in \mathcal{E}$.*

Proof. Without loss of generality, let $0 \leq A < \|f\|_\infty$. Define $f_A(x)$ for $x \in \mathcal{X}$ as $f_A(x) = A$ for $x \in \mathcal{E} = \{x \in \mathcal{X} : |f(x)| > A\}$ and $f_A(x) = |f(x)|$ if $|f(x)| \leq A$. From Proposition 3.20, $\mathcal{E} \in \boldsymbol{\mathcal{X}}$. Moreover, $|f(x)| > A$ for all $x \in \mathcal{E}$. Now, let $\varepsilon > 0$ be such that $A+\varepsilon < \|f\|_\infty$. Clearly $(A+\varepsilon)\,\mathbf{1}_{\mathcal{E}} > A\mathbf{1}_{\mathcal{E}}$. Thus $A\mu(\mathcal{E}) + \varepsilon\mu(\mathcal{E}) > A\mu(\mathcal{E})$. Therefore $\mu(\mathcal{E}) > 0$. This completes the proof. □

Proposition 7.21. *If $f \in L_p$, $1 \leq p \leq \infty$, and $g \in L_\infty$, then the product $fg \in L_p$ and $\|fg\|_p \leq \|f\|_p \|g\|_\infty$.*

Proof. From the definition of $\|g\|_\infty$, there is an $M \in \mathbb{R}$ such that $|g| \leq M$ almost everywhere. Therefore $|fg|^p \leq M^p\,|f|^p$. By a now familiar argument,

$$\int |fg|^p \; d\mu \leq M^p \int |f|^p \; d\mu < \infty.$$

Therefore, $fg \in L_p$. Moreover from above

$$\left[\int |fg|^p \; d\mu\right]^{\frac{1}{p}} \leq \|g\|_\infty \|f\|_p.$$

□

Proposition 7.22. *The space L_∞ is contained in L_1 if and only if $\mu(\mathcal{X}) < \infty$. Moreover, if $\mu(\mathcal{X}) = 1$ and $f \in L_\infty$, then*

$$\|f\|_\infty = \lim_{p\to\infty} \|f\|_p.$$

Proof. From the definition of L_∞ and 4.7 Corollary [Bartle],

$$\int |f| \; d\mu \leq \|f\|_\infty \, \mu(\mathcal{X}) \qquad \text{where } \|f\|_\infty = \inf\{M \in \mathbb{R} : |f| \leq M \ \mu\text{-a.e}\}.$$

Clearly then $\mu(\mathcal{X}) < \infty$ implies $f \in L_1$. To show the other implication, we will show the contrapositive (i.e. if $\mu(\mathcal{X}) = \infty$, then L_∞ is not contained in L_1). To that end, assume $\mu(\mathcal{X}) = \infty$ and let $f = \mathbf{1}_{\mathcal{X}}$ then $f \in L_\infty$ but $f \notin L_1$.

Moreover if $f \in L_p$ and $\mu(\mathcal{X}) = 1$, then again employing the definition of $\|f\|_\infty$, it is evident that $|f|^p \leq \|f\|_\infty^p$ almost everywhere. Therefore

$$\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty.$$

Now let $0 < \varepsilon < \|f\|_\infty$. By Proposition 7.20, the measurable set

$$\mathcal{E} = \{x \in \mathcal{X} : |f(x)| \geq \|f\|_\infty - \varepsilon\}$$

satisfies $\mu(\mathcal{E}) > 0$. For $1 < p < \infty$, $(\|f\|_\infty - \varepsilon)^p \leq |f|^p$ on the set $\mathcal{E}$. Thus

$$(\|f\|_\infty - \varepsilon)\, \mu(\mathcal{E})^{\frac{1}{p}} \leq \|f\|_p$$

for $1 < p < \infty$. As $p \to \infty$, $\mu(\mathcal{E})^{\frac{1}{p}} = 1$ therefore

$$\|f\|_\infty - \varepsilon \leq \liminf_{p\to\infty} \|f\|_p.$$

Thus,

$$\limsup_{p\to\infty} \|f\|_p \leq \|f\|_\infty \leq \liminf_{p\to\infty} \|f\|_p.$$

This completes the proof.

□

8. Modes of Convergence

In following examples $(\mathbb{R}, \boldsymbol{\mathcal{B}}, \lambda)$ denotes the real line with Lebesgue measure defined on the Borel subsets of $\mathbb{R}$. Also, $1 \leq p \leq \infty$.

Example 8.1. Let $f_n = n^{-\frac{1}{p}} \mathbf{1}_{[0,n]}$ Then, the sequence $\{f_n\}$ converges uniformly to the 0-function but it does not converge in $L_p(\mathbb{R}, \boldsymbol{\mathcal{B}}, \lambda)$.

Proof. Clearly $n^{\frac{1}{p}} > 0$ for $1 \leq p < \infty$. By Theorem 2.2.6 [Stoll]

$$\lim_{n\to\infty} \frac{1}{n^{\frac{1}{p}}} = 0.$$

Thus given $\varepsilon > 0$ there exists $n_\circ \in \mathbb{N}$ such that $|f_n(x) - 0| = \frac{1}{n^{\frac{1}{p}}} < \varepsilon$ for all $x \in \mathbb{R}$ and for all $n \geq n_\circ$. This shows that $f_n \to 0$ uniformly. However,

$$\|f_n - 0\|_p = \left[\int_{[0,n]} \left|\frac{1}{n^{\frac{1}{p}}}\right|^p d\mu\right]^{\frac{1}{p}} = \left[\frac{n-0}{n}\right]^{\frac{1}{p}} = 1$$

Letting $\varepsilon = 1$ it is seen that f does not converge to 0 in L_p. □

Example 8.2. Let $f_n = n\mathbf{1}_{[\frac{1}{n}, \frac{2}{n}]}$. Then the sequence $\{f_n\}$ converges everywhere to the 0-function, but it does not converge in $L_p(\mathbb{R}, \boldsymbol{\mathcal{B}}, \lambda)$.

Proof. Clearly for each $x \in \mathbb{R}$ and for $\varepsilon > 0$ there exists an $N(\varepsilon, x) \in \mathbb{N}$ such that $|f_n(x)| < \varepsilon$ for all $n \geq N(\varepsilon, x)$ therefore f_n converges everywhere to the 0-function. However

$$\|f_n - 0\|_p = \left[\int_{\left[\frac{1}{n}, \frac{2}{n}\right]} |n|^p \, d\mu\right]^{\frac{1}{p}} = \left[n^p \frac{1}{n}\right]^{\frac{1}{p}} = n^{1-\frac{1}{p}}.$$

Since $1 - \frac{1}{p} \geq 0$ when $p \geq 1$,

$$\lim_{n\to\infty} \|f_n\|_p \neq 0.$$

So $\{f_n\}$ does not converge in L_p to the 0-function. □

Example 8.2 shows that convergence in measure does not imply L_p convergence even for a finite measure space since the integral of f_n vanishes outside of the interval $\left[\frac{1}{n}, \frac{2}{n}\right]$ and it equals $n^{1-\frac{1}{p}}$ on the interval $\left[\frac{1}{n}, \frac{2}{n}\right]$.

Proposition 8.3. *Both sequences in Examples 8.1 and 8.2 converge in measure to their limits.*

Proof. Let $\alpha > 0$. For Example 8.1, it has been shown that there exists an $N \in \mathbb{N}$ such that $|f_n(x) - 0| < \alpha$ for all $x \in \mathbb{R}$. For $n \geq N$,

$$\{x \in \mathcal{X} : |f_n - 0| \geq \alpha\} = \emptyset.$$

Therefore,

$$\lim_{n\to\infty} \lambda\left(\{x \in \mathcal{X} : |f_n - 0| \geq \alpha\}\right) = \lambda(\emptyset) = 0.$$

For Example 8.2, provided $0 < \alpha \leq 1$,

$$\lim \lambda\left(\{x \in \mathcal{X} : |f_n - 0| \geq \alpha\}\right) = \lim \lambda\left(\left[\frac{1}{n}, \frac{2}{n}\right]\right) = \lim \lambda\left(\frac{1}{n}\right) = 0.$$

Therefore, each sequence of functions converges in measure. □

Example 8.4. Let $f_n = \mathbf{1}_{[n,n+1]}$. The sequence $\{f_n\}$ converges everywhere to the 0-function, but it does not converge in measure.

Proof. Clearly for each $x \in \mathbb{R}$ there exists an $N(x) \in \mathbb{N}$ such that $f_n(x) = 0$ for all $n \geq N(x)$. Let $0 < \alpha \leq 1$. Now for all $n \in \mathbb{N}$,

$$\lambda\left(\{x \in \mathcal{X} : |f_n(x) - 0| \geq \alpha\}\right) = 1.$$

Therefore, $\{f_n\}$ does not converge in measure to the 0-function. □

Looking at 7.4 Example [Bartle], it is seen that the sequence converges in L_p to the 0-function and thus converges in measure to the 0-function, but does not converge at any point of $[0,1]$. However, from 7.6 Theorem [Bartle] there exists a subsequence of $\{f_n\}$ which converges almost everywhere to the 0-function. Clearly, if $f_n = \mathbf{1}_{[0,\frac{1}{n}]}$ then for $\varepsilon > 0$ and for each $x \in [0,1]$ there exists an $N(\varepsilon, x) \in \mathbb{N}$ such that $|f_n(x)| \leq \varepsilon$ except for $x = 0$ but $\lambda\left(\{0\}\right) = 0$. However there is not a subsequence from 7.4 Example that converges everywhere since each f_n maps a rational to 1.

Proposition 8.5. *If a sequence $\{f_n\}$ converges in measure to a function f, then every subsequence of $\{f_n\}$ converges in measure to f. More generally, if $\{f_n\}$ is Cauchy in measure, then every subsequence is Cauchy in measure.*

Proof. Let $\{f_{n_k}\}$ be a subsequence of $\{f_n\}$. From assumption, for $\alpha > 0$

$$\lim_{n\to\infty} \mu\left(\{x \in \mathcal{X} : |f(x) - f_n(x)| \geq \alpha\}\right) = 0.$$

Therefore for $\varepsilon > 0$ there exists an N such that

$$\mu\left(\{x \in \mathcal{X} : |f(x) - f_n(x)| \geq \alpha\}\right) < \varepsilon$$

for all $n \geq N$. Since $\{n_k\}$ strictly increasing, there exists $K \in \mathbb{N}$ so that $n_k \geq N$ for all $k \geq K$. Therefore

$$\mu\left(\{x \in \mathcal{X} : |f(x) - f_{n_k}(x)| \geq \alpha\}\right) < \varepsilon$$

for $k \geq K$. Thus,

$$\lim_{k\to\infty} \mu\left(\{x \in \mathcal{X} : |f(x) - f_{n_k}(x)| \geq \alpha\}\right) = 0.$$

Similarly, if $\{f_n\}$ is Cauchy, then

$$\lim_{m,n\to\infty} \mu\left(\{x \in \mathcal{X} : |f_m(x) - f_n(x)| \geq \alpha\}\right) = 0.$$

For $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\mu\left(\{x \in \mathcal{X} : |f_m(x) - f_n(x)| \geq \alpha\}\right) < \varepsilon$$

for all $m \geq n \geq N$. Since $\{n_k\}$ strictly increasing, there exists $K \in \mathbb{N}$ so that $n_l \geq n_k \geq N$ for all $l \geq k \geq K$. Therefore

$$\lim_{l,k\to\infty} \mu\left(\{x \in \mathcal{X} : |f_{n_l}(x) - f_{n_k}(x)| \geq \alpha\}\right) = 0.$$

□

Proposition 8.6. *If a sequence $\{f_n\}$ converges in L_p to a function f, and a subsequence of $\{f_n\}$ converges in L_p to g, then $f = g$ almost everywhere.*

Proof. Since convergence in L_p implies convergence in measure, $\{f_n\}$ converges to f in measure and $\{f_{n_k}\}$ converges to g in measure. Moreover, from Proposition 8.5 $\{f_{n_k}\}$ converges in measure to f. By 7.7 Corollary

[Bartle] f is uniquely determined almost everywhere therefore $f = g$ almost everywhere. $\square$

Proposition 8.7. *If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence of characteristic functions of sets in $\boldsymbol{\mathcal{X}}$, and if $\{f_n\}_{n\in\mathbb{N}}$ converges to f in L_p, then f is almost everywhere equal to the characteristic function of a set in $\boldsymbol{\mathcal{X}}$.*

Proof. Let $f_n = \mathbf{1}_{\mathcal{A}_n}$ where $\mathcal{A}_n \in \boldsymbol{\mathcal{X}}$ for all $n \in \mathbb{N}$ and let $f_n(x) \to f(x)$ in L_p. Since $f_n(x) \to f(x)$ in L_p there is a subsequence $\left\{f_{n_j}\right\}_{j\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ such that there exists $\mathcal{E} \in \mathcal{X}$ where $\mu(\mathcal{E}^c) = 0$ and $f_{n_j}(x) \to f(x)$ for all $x \in \mathcal{E}$. Since $f_{n_j}(x) = 0$ or 1 on $\mathcal{E}$, then $f(x) = 0$ or 1 on $\mathcal{E}$. This finishes the proof. $\square$

Proposition 8.8. *Let $\{f_n\}$ be as in Example 8.2. If $\delta > 0$, then $\{f_n\}$ converges uniformly to the 0-function on the complement of $[0, \delta]$.*

Proof. Let $\delta > 0$. Clearly, there exists an $N \in \mathbb{N}$ such that $\frac{2}{n} < \delta$ for all $n \geq N$. Therefore, $f_n(x) = 0$ for all $x \in [0, 1] \cap ([0, \delta])^c$ and for all $n \geq N$. $\square$

Remark 6. There does not, however, exist a set of measure zero, on the complement of which the sequence from Example 8.2 is uniformly convergent to the 0-function.

Proof. Without loss of generality we can restrict ourselves to the interval $[0, 2]$ on the real line since for points outside of this interval, $f_n = 0$ for all $n \in \mathbb{N}$. Let $\mathcal{E} \in \boldsymbol{B}$ be such that $\mathcal{E} \subset [0, 2]$ and $\lambda(\mathcal{E}) = 0$. It is enough then to show that $\mathcal{E}^c \cap [\frac{1}{n}, \frac{2}{n}] \neq \emptyset$ for all $n \in \mathbb{N}$. Clearly, $\lambda(\mathcal{E}^c) = 2$. Therefore, if $\mathcal{E}^c$ and $\left[\frac{1}{n}, \frac{2}{n}\right]$ are disjoint, then $\lambda(\left[\frac{1}{n}, \frac{2}{n}\right]) + \lambda(\mathcal{E}^c) > 2$ which is a contradiction. $\square$

The following propositions demonstrate that in Fatou's Lemma and the Lebesgue Dominated Convergence Theorem almost everywhere convergence can be replaced by convergence in measure. In fact once it is shown that

Fatou's Lemma holds for convergence in measure, the Lebesgue Dominated Convergence Theorem for convergence in measure is immediate.

First the following fact will be useful in the proof of Fatou's Lemma for convergence in measure.

Fact 8.9. *A sequence $\{a_n\}_{n=1}^{\infty}$ converges to a if and only if every subsequence of $\{a_n\}_{n=1}^{\infty}$ has a subsequence that converges to a.*

Proposition 8.10. *If $\{f_n\}$ is a sequence of nonnegative measurable functions and $\{f_n(x)\}$ converges to $f(x)$ in measure, then*

$$\int f\, d\mu \leq \liminf \int f\, d\mu.$$

Proof. By Fact 8.9, it will suffice to show that each subsequence of $\{f_n\}$ has a subsequence that converges a.e. to f. By assumption f_n converges to f in measure therefore from Proposition 8.5 every subsequence of $\{f_n\}$ converges in measure to f. By 7.6 Theorem [Bartle] each subsequence of $\{f_n\}$ has a subsequence which converges a.e. to f. So f_n converges a.e. to f. Thus by Fatou's Theorem for a.e. convergence we are done. □

Proposition 8.11. *The Lebesgue Dominated Convergence holds for convergence in measure.*

Proof. Follows from Proposition 8.10. □

Proposition 8.12. *Let $(\mathcal{X}, \boldsymbol{\mathcal{X}}, \mu)$ be a finite measure space. If f is an $\boldsymbol{\mathcal{X}}$-measurable function, let*

$$r(f) = \int \frac{|f|}{1+|f|}\, d\mu.$$

A sequence $\{f_n\}$ of $\boldsymbol{\mathcal{X}}$-measurable functions converges in measure to f if and only if $r(f_n - f) \to 0$.

Proof. Assume that $r(f_n - f) \to 0$. Notice

$$0 \leq \frac{|f_n - f|}{1 + |f_n - f|} < 1 \tag{8.1}$$

Therefore, the limit of the left hand side of (8.1) exists. By Fatou's Theorem,

$$0 \leq \int \liminf \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu \leq \liminf \int \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu = 0.$$

By 4.10 Corollary,

$$\lim \frac{|f_n - f|}{1 + |f_n - f|} = 0$$

almost everywhere. Thus $|f_n - f| = 0$ almost everywhere. Applying Egoroff's Theorem, it is concluded that $\{f_n\}$ converges to f in measure.

Fix $\varepsilon > 0$ and assume that $\{f_n\} \to f$ in measure. Then there exists N such that for all $n \geq N$

$$\mu\left(\left\{x \in \mathcal{X} : |f_n - f| \geq \frac{\varepsilon}{2\mu(\mathcal{X})}\right\}\right) < \frac{\varepsilon}{2}.$$

Let $\mathcal{B}_{n_\varepsilon} = \left\{x \in \mathcal{X} : |f_n - f| \geq \frac{\varepsilon}{2\mu(\mathcal{X})}\right\}$. So for all $n \geq N$,

$$\begin{aligned} r(f_n - f) &= \int_{\mathcal{B}_{n_\varepsilon}} \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu + \int_{\mathcal{X} \backslash \mathcal{B}_{n_\varepsilon}} \frac{|f_n - f|}{1 + |f_n - f|}\, d\mu \\ &\leq \int_{\mathcal{B}_{n_\varepsilon}} 1\, d\mu + \int_{\mathcal{X} \backslash \mathcal{B}_{n_\varepsilon}} |f_n - f|\, d\mu \\ &\leq \mu(\mathcal{B}_{n_\varepsilon}) + \frac{\varepsilon}{2}\left(\frac{\mu(\mathcal{X} \backslash \mathcal{B}_{n_\varepsilon})}{\mu(\mathcal{X})}\right) < \frac{e}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $r(f_n - f) \to 0$. □

Proposition 8.13. *If the sequence $\{f_n\}$ of measurable functions converges almost everywhere to a measurable function f and ϕ is continuous on $\mathbb{R}$ to $\mathbb{R}$, then the sequence $\{\phi \circ f_n\}$ converges almost everywhere to $\phi \circ f$.*

Proof. By Proposition 3.26 $\phi \circ f$ is measurable and $\phi \circ f_n$ is measurable for each $n \in \mathbb{N}$. Since ϕ is continuous, then for $\varepsilon > 0$ and for each $x \in \mathcal{X}$ there exists a $\delta > 0$ such that $|\phi(f_n(x)) - \phi(f(x))| < \varepsilon$ where $|f_n(x) - f(x)| < \delta$. Moreover $f_n \to f$ almost everywhere therefore, there exists a set $\mathcal{M} \in \boldsymbol{\mathcal{X}}$ such that for every $\delta > 0$ and $x \in \mathcal{M}^c$ there exists $N(\varepsilon, x) \in \mathbb{N}$ such that if $n \geq N(\varepsilon, x)$ then $|f_n(x) - f(x)| < \delta$. Thus for $\varepsilon > 0$ there exists a set $\mathcal{M} \in \boldsymbol{\mathcal{X}}$ such that for each $\boldsymbol{\mathcal{X}} \in \mathcal{M}^c$ there exists $N(\varepsilon, x)$ such that if $n \geq$

$N(\varepsilon, x)$ then $|\phi(f_n(x) - \phi(f(x))| < \varepsilon$. In other words, $\{\phi \circ f_n\}$ converges almost everywhere to $\phi \circ f$ □

In the following example, it will be shown that if ϕ has a point of discontinuity, then there exists a sequence $\{f_n\}$ which converges almost everywhere to f but $\{\phi \circ f\}$ does not converge almost everywhere to $\phi \circ f$.

Example 8.14. Let $\phi = 1$ for $x \geq 0$ and let $\phi = -1$ for $x < 0$. Let $\{f_n\}$ be a sequence of negative-valued functions which converges almost everywhere to $f = 0$. Therefore $\phi(f(x)) = 1$. Now let $\mathcal{M} \in \boldsymbol{\mathcal{X}}$ be such that for all $x \notin \mathcal{M}$ f_n converges everywhere to the 0-function and let $\varepsilon = 2$. Thus for all $x \notin \mathcal{M}$ and for all $n \in \mathbb{N}$, $|\phi(f_n(x)) - \phi(f(x))| = 2$. Therefore $\{\phi \circ f_n\}$ does not converge almost everywhere to $\phi \circ f$.

9. Conclusion

To write a conclusion to this paper is difficult and inappropriate since the paper, in a sense, opens a can of worms. It has discussed the foundations of the Lebesgue Integral and some of the important theorems that emerge from the topic. Certainly, sections seven and eight provide a glimpse at some of the higher theory that uses this integral. To be sure, the problems presented in this paper merely scratch the surface of the theory behind the General Lebesgue Integral, however they suggest the depth and power of this mathematical technique.

References

[CAOB] Charalambos D. Aliprantis and Owen Burkinshaw, *Principles of real analysis*, third ed., Academic Press Inc., San Diego, CA, 1998.

[CAOB] Charalambos D. Aliprantis and Owen Burkinshaw, *Problems in real analysis*, second ed., Academic Press Inc., San Diego, CA, 1999, A workbook with solutions.

[Bartle] Robert G. Bartle, *The elements of integration*, John Wiley & Sons Inc., New York, 1966.

[Lang] Serge Lang, *Real and functional analysis*, third ed., Springer-Verlag, New York, 1993.

[JMNW] John N. McDonald and Neil A. Weiss, *A course in real analysis*, Academic Press Inc., San Diego, CA, 1999, Biographies by Carol A. Weiss.

[Royden] H. L. Royden, *Real analysis*, third ed., Macmillan Publishing Company, New York, 1988.

[Rudin] Walter Rudin, *Principles of mathematical analysis*, third ed., McGraw-Hill Book Co., New York, 1976, International Series in Pure and Applied Mathematics.

[Stoll] Manfred Stoll, *Introduction to real analysis*, Addison-Wesley, Reading Massachusetts, 1997.