OPERATOR-VALUED FOURIER HAAR MULTIPLIERS

MARIA GIRARDI

ABSTRACT. Criteria are given to ensure the boundedness of Fourier Haar multiplier operators from $L_p([0,1], X)$ to $L_q([0,1], Y)$ where the Fourier Haar multiplier sequences come not from $\mathbb{R}$, as in the classical setting, but rather from the space of bounded linear operators from a Banach space X into a Banach space Y.

1. INTRODUCTION

It is well known that the Haar system $\{h_j\}_{j\in\mathbb{N}}$ forms an unconditional basis in $L_p([0,1], \mathbb{R})$ for $1 < p < \infty$. Thus the Fourier Haar multiplier operator T, generated by the Fourier Haar multiplier sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ from $\mathbb{R}$, defined on the span of the Haar system $\{h_j\}_{j\in\mathbb{N}}$ by

$$T\left(\sum_{j=1}^{m} c_j h_j\right) = \sum_{j=1}^{m} \lambda_j c_j h_j \qquad \text{where } c_j \in \mathbb{R} \text{ and } m \in \mathbb{N}\ , \tag{1.1}$$

extends (uniquely) to a bounded linear operator on the whole of $L_p([0,1], \mathbb{R})$ provided the multiplier sequence is bounded, in which case,

$$\|T\|_{L_p([0,1],\mathbb{R})\to L_p([0,1],\mathbb{R})} \ \le\ C_p \sup_{j\in\mathbb{N}} |\lambda_j|$$

for some constant C_p for $1 < p < \infty$. Much is known (cf. e.g. [19] and the references therein) about the boundedness of such Fourier Haar multiplier operator from $L_p([0,1], \mathbb{R})$ to $L_q([0,1], \mathbb{R})$. If $1 < p \le q < \infty$, then

$$\|T\|_{L_p([0,1],\mathbb{R})\to L_q([0,1],\mathbb{R})} \ \approx \sup_{(n,k)\in\Delta} 2^{n\left(\frac{1}{p}-\frac{1}{q}\right)} |\lambda_{2^n+k}| \tag{1.2}$$

where $\{h_k^n\}_{(n,k)\in\Delta}$ is the dyadic enumeration of the Haar system. While if $1 < q \le p < \infty$, then

$$\|T\|_{L_p([0,1],\mathbb{R})\to L_q([0,1],\mathbb{R})} \ \approx\ \left\| \sup\nolimits_{j\in\mathbb{N}} \ |\ \lambda_j\ h_j\ | \right\|_{L_r([0,1],\mathbb{R})} \tag{1.3}$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. In both cases, the equivalence constants depend only on p and q.

In (1.1), one can replace $c_j \in \mathbb{R}$ by x_j in some Banach space X and then consider the boundedness of T on $L_p([0,1], X)$. Here UMD (unconditionality property for martingale differences) spaces play

Date: 11 July 2005.
2000 *Mathematics Subject Classification.* 46E40, 46B20, 60G42.
Key words and phrases. Fourier Haar multipliers, UMD, R-boundedness.
Girardi is supported in part by the National Science Foundation Grant DMS-0306750.

a key role. Indeed, each T generated by a Fourier Haar multiplier sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ from $\{\pm 1\}$ is bounded (by some constant depending only on X and p) on $L_p([0,1],X)$ for some (or equivalently, for each) $p \in (1,\infty)$ if and only if X is a UMD space.

This paper considers Fourier Haar multiplier operators from $L_p([0,1],X)$ to $L_q([0,1],Y)$ where the Fourier Haar multiplier sequence comes not from $\mathbb{R}$ but rather from the space $\mathcal{B}(X,Y)$ of bounded linear operators from a Banach space X into a Banach space Y. Not surprisely, UMD plays a role. However, an R-boundedness assumption on the multiplier sequence is also used. R-boundedness was introduced by Berkson and Gillespie in [2]. This notion grew out of work of J. Bourgain on vector-valued Fourier transform [3] and has been central to recent results on operator-valued Fourier multipliers and singular integrals with operator-valued kernels on Bochner spaces (e.g. [1, 11, 13, 23]). Through these tools, R-boundedness became important for maximal regularity of parabolic differential equations (e.g. [8, 16, 23]) and the holomorphic functional calculus of sectorial operators (e.g. [14, 15, 16]). It is a key notion in the study [12] of martingales transforms by operator-valued multiplier, which is especially useful for the theory of stochastic integration on Banach spaces which recently was developed in [21] and [22]. For more information on R-boundedness and its properties, see [7, 10, 16].

Theorem 3.3, which covers the case that $1 < q \le p < \infty$, generalizes (1.3). Its simple short proof, which uses the notions of UMD and R-boundedness, is very different from the usual proof for scalar-valued multiplier sequences, which uses interpolation and is much longer. Theorem 3.4, which covers the case that $1 \le p < q < \infty$, generalizes (1.2). In this case, the usual proof of the scalar-valued case can be generalized and so no UMD nor R-boundedness assumptions are necessary. It is interesting that in one case UMD and R-boundedness need to be used but in the other case they do not. This work was motivated by a recent paper [12] on martingale transforms where the multiplier sequence is $\mathcal{B}(X,Y)$-valued.

This paper is organized as follows. Section 2 collects the needed definitions and notation. Section 3 contains the main results. Closing examples and remarks are in Section 4.

2. DEFINITIONS AND NOTATION

Throughout this paper, the Banach spaces that appear are over the fixed scalar field of either the real or complex numbers. X, Y, and Z are Banach spaces. $B(X)$ is the closed unit ball of X. The space $\mathcal{B}(X,Y)$ of bounded linear operators from X into Y is endowed with the usual operator norm topology. For a measure space $(\Omega,\mathcal{F},\mu)$, the Bochner-Lebesgue space $L_p(\Omega,X)$ consists of the measurable functions from Ω into X with finite $L_p(\Omega,X)$-norm where $1 \le p \le \infty$. The weak-L_p

space $L_p^{\mathrm{wk}}(\Omega, X)$, for $1 \leq p < \infty$, consists of the measurable functions from Ω into X that satisfy

$$\|f\|_{L_p^{\mathrm{wk}}(\Omega,X)} := \sup_{\lambda>0} \lambda \ [\mu(\{\omega \in \Omega \colon \|f(\omega)\|_X > \lambda\})]^{\frac{1}{p}} < \infty .$$

It is well-known that the above expression $\|\cdot\|_{L_p^{\mathrm{wk}}(\Omega,X)}$ is a quasi-norm on $L_p^{\mathrm{wk}}(\Omega, X)$ with

$$\|f+g\|_{L_p^{\mathrm{wk}}(\Omega,X)} \leq 2 \left[\|f\|_{L_p^{\mathrm{wk}}(\Omega,X)} + \|g\|_{L_p^{\mathrm{wk}}(\Omega,X)}\right] .$$

The balls with respect to $\|\cdot\|_{L_p^{\mathrm{wk}}(\Omega,X)}$ define a linear topology on $L_p^{\mathrm{wk}}(\Omega, X)$ and $L_p^{\mathrm{wk}}(\Omega, X)$, endowed with this topology, is a quasi-Banach space.

$\mathbb{N}$ is the set of natural numbers while $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Nonnumerical subscripts on constants indicate dependency.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space with a *filtration* $\{\mathcal{F}_n\}_{n=1}^m$ (i.e., $\{\mathcal{F}_n\}_{n=1}^m$ is a nondecreasing sequence of sub-σ-fields of $\mathcal{F}$) where $m \in \mathbb{N}$. A sequence $\{d_n\}_{n=1}^m$ of functions from Ω into X is a (stationary) *martingale difference sequence* with respect to $\{\mathcal{F}_n\}_{n=1}^m$ provided $d_n \in L_1((\Omega, \mathcal{F}_n, \mu), X)$ and $\mathbb{E}(d_{n+1} \mid \mathcal{F}_n) = 0$ for each admissible n. There is a one-to-one correspondence between martingales $\{f_n\}_{n=1}^m$ and martingale difference sequence $\{d_n\}_{n=1}^m$ given by $f_n = \sum_{k=1}^n d_k$. A sequence $\{v_n\}_{n=1}^m$ of functions from Ω into Z is $\{\mathcal{F}_n\}_{n=1}^m$*-predictable* provided v_n is $\mathcal{F}_{n-1}$-measurable for each $n \in \{1, 2, \ldots m\}$ (where $\mathcal{F}_0 := \mathcal{F}_1$). The *martingale transform* of an X-valued martingale $\{\sum_{k=1}^n d_k\}_{n=1}^m$ with respect to $\{\mathcal{F}_n\}_{n=1}^m$ by a $\mathcal{B}(X, Y)$-valued $\{\mathcal{F}_n\}_{n=1}^m$-predictable sequence $\{v_n\}_{n=1}^m$ is the Y-valued martingale $\{\sum_{k=1}^n v_k d_k\}_{n=1}^m$ with respect to $\{\mathcal{F}_n\}_{n=1}^m$. Burkholder [4] introduced UMD Banach spaces.

Definition 2.1. The UMD constant of X is the smallest $\beta_p(X) \in [1, \infty]$ so that

$$\|\varepsilon_1 d_1 + \ldots + \varepsilon_m d_m\|_{L_p(\Omega,X)} \leq \beta_p(X) \ \|d_1 + \ldots + d_m\|_{L_p(\Omega,X)}$$

for each X-valued martingale difference sequence $\{d_n\}_{n=1}^m$ with respect to some filtration $\{\mathcal{F}_n\}_{n=1}^m$, choice $\{\varepsilon_n\}_{n=1}^m$ of signs from $\{\pm 1\}$, and $m \in \mathbb{N}$. A Banach space X is UMD provided that its UMD constant is finite for some (or equivalently, by Pisier [18], for each) $p \in (1, \infty)$.

One obtains an equivalent formulation of UMD spaces (with the same UMD constant) if, in Definition 2.1, one replaces *choice* $\{\varepsilon_n\}_{n=1}^m$ *of signs from* $\{\pm 1\}$ with $[-1, 1]$*-valued* $\{\mathcal{F}_n\}_{n=1}^m$*-predictable sequence* $\{v_n\}_{n=1}^m$.

Notation 2.2. Henceforth, $(\Omega, \mathcal{F}, \mu)$ denotes the usual Lebesgue measure space on $[0, 1]$.

Let

$$\Delta_1 = \{(n, k) \in \mathbb{N}_0 \times \mathbb{N} \colon 1 \leq k \leq 2^n\}$$

$$\Delta_0 \;=\; \{(0,0)\} \quad \text{and} \quad \Delta \;=\; \Delta_0 \cup \Delta_1 \ .$$

There is a bijection from Δ onto $\mathbb{N}$ given by $(n,k) \to 2^n + k$, which provides a linear ordering on Δ. Thus any sequence $\{\Theta_j\}_{j\in\mathbb{N}}$ of objects can also be denoted by $\{\Theta_{(n,k)}\}_{(n,k)\in\Delta}$ where $\Theta_{(n,k)} = \Theta_{2^n+k}$. This identification will be used freely throughout this paper.

The dyadic intervals $\{I_k^n \colon (n,k) \in \Delta_1\}$ are given by

$$I_1^n \;=\; \left[0, \frac{1}{2^n}\right] \qquad \text{and} \qquad I_k^n \;=\; \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right] \quad \text{for } k > 1 \ .$$

The Haar system $\{h_k^n\}_{(n,k)\in\Delta}$ is given by $h_0^0 \;=\; 1_{I_1^0}$ and, for $(n,k) \in \Delta_1$,

$$h_k^n \;=\; 1_{I_{2k-1}^{n+1}} \;-\; 1_{I_{2k}^{n+1}} \ .$$

The Haar filtration $\{\mathcal{H}_j\}_{j\in\mathbb{N}}$ is defined by

$$\mathcal{H}_j \;=\; \sigma\left\{h_1, \ldots, h_j\right\} \ .$$

The Rademacher functions $\{r_n\}_{n\in\mathbb{N}_0}$ take the form $r_0 = h_0^0$ and, for $n \in \mathbb{N}$,

$$r_n \;=\; \sum_{k=1}^{2^{n-1}} h_k^{n-1} \ .$$

Let

$$E(\Omega, X) \;:=\; \left\{ f \colon \Omega \to X \mid f = \sum\nolimits_{j=1}^{n} x_j h_j \text{ for some } n \in \mathbb{N} \ , \ x_j \in X \right\} \ .$$

$E(\Omega, X)$ is norm dense in $L_p(\Omega, X)$ for $1 \le p < \infty$; indeed, X-valued simple functions are dense in $L_p(\Omega, X)$ and the Haar system is a basis for $L_p(\Omega, \mathbb{R})$. Also, the representation of functions in $E(\Omega, X)$ is unique: if $f = \sum_{j=1}^{n} x_j h_j$ then $x_j = \|h_j\|_{L_1}^{-1} \int_\Omega f(\omega)\, h_j(\omega)\; d\omega$.

Definition 2.3. The *Fourier Haar multiplier operator* T, generated by a *Fourier Haar multiplier sequence* $\{T_j\}_{j\in\mathbb{N}}$ from $\mathcal{B}(X,Y)$, is the linear mapping from $E(\Omega, X)$ to $E(\Omega, Y)$ given by

$$T\left(\sum\nolimits_{j\in\mathbb{N}} x_j h_j\right) \;=\; \sum\nolimits_{j\in\mathbb{N}} T_j x_j h_j \qquad \text{for} \qquad \sum_{j\in\mathbb{N}} x_j h_j \in E(\Omega, X) \ .$$

For $1 \le p, q < \infty$ define

$$\|T\|_{L_p(\Omega,X)\to L_q(\Omega,Y)} \;:=\; \sup_{\substack{f\in E(\Omega,X)\\ f\neq 0}} \frac{\|Tf\|_{L_q(\Omega,Y)}}{\|f\|_{L_p(\Omega,X)}} \ . \tag{2.1}$$

If the supremum in (2.1) is finite, the T is called a *bounded* Fourier Haar multiplier operator (from $L_p(\Omega, X)$ to $L_q(\Omega, Y)$).

In Definition 2.3, if T is a *bounded* Fourier Haar multiplier operator, then $T\colon E(\Omega, X) \to E(\Omega, Y)$ extends uniquely to a bounded linear operator from $L_p(\Omega, X)$ to $L_q(\Omega, Y)$, with norm the supremum in (2.1). In Definition 2.3, one can replace $L_q(\Omega, Y)$ with $L_q^{\mathrm{wk}}(\Omega, Y)$ for $1 \le q < \infty$. All remains valid except, in the bounded case, the norm of the extension is at most twice the supremum in (2.1).

Loosely speaking, a set τ of operators is R-bounded provided Kahane's Contraction Principle holds for *operator coefficients* from τ. The precise definition is as follows.

Definition 2.4. Let τ be a subset of $\mathcal{B}(X, Y)$ and $p \in [1, \infty)$. Let $R_p(\tau)$ be the smallest constant $R \in [0, \infty]$ with the property that for each $n \in \mathbb{N}$ and subset $\{T_j\}_{j=1}^n$ of τ and subset $\{x_j\}_{j=1}^n$ of X,

$$\left\| \sum_{j=1}^n r_j(\cdot) T_j(x_j) \right\|_{L_p([0,1],Y)} \le R \left\| \sum_{j=1}^n r_j(\cdot) x_j \right\|_{L_p([0,1],X)} .$$

The set τ is *R-bounded* provided $R_p(\tau)$ is finite for some (and thus then, by Kahane's inequality, for each) $p \in [1, \infty)$.

Pisier [1] showed that each (norm) bounded subset of $\mathcal{B}(X, Y)$ is R-bounded if and only if X has cotype 2 and Y has type 2 (cf. e.g. [17] for needed definitions). Note that if X and Y are q-concave Banach lattices for some finite q (e.g. $X = Y = L_q(\Omega, \mathbb{C})$ where $1 \le q < \infty$) then R-boundedness is equivalent to the square function estimate

$$\left\| \left(\sum_{j=1}^m |T_j x_j|^2 \right)^{1/2} \right\|_Y \le R \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_X$$

known from harmonic analysis (cf. [17, Thm. II.1.d.6]). For basic properties of R-bounded sets and further references, see [7, 10, 16, 23].

All notation and terminology, not otherwise explained, are as in [6, 9, 17].

3. MAIN RESULTS

Consider a Fourier Haar multiplier operator T generated by $\{T_j\}_{j\in\mathbb{N}}$ from $\mathcal{B}(X, Y)$. This section gives conditions on $\{T_j\}_{j\in\mathbb{N}}$ that guarantee that T is bounded from $L_p(\Omega, X)$ to $L_q(\Omega, Y)$. Remark 3.1 relates the boundedness of T to the boundedness of certain martingale transforms.

Remark 3.1. Note that $\{d_n\}_{n=1}^m$ is an X-valued martingale difference sequence with respect to the Haar filtration $\{\mathcal{H}_n\}_{n=1}^m$ if and only if it takes the form $d_n = x_n h_n$ for some $x_n \in X$. Let

$$v_j(\cdot) := T_j \; |h_j(\cdot)| \; .$$

Then $\{v_n\}_{n=1}^m$ is a $\mathcal{B}(X,Y)$-valued $\{\mathcal{H}_n\}_{n=1}^m$-predictable sequence. Furthermore, the martingale transform of $\{\sum_{k=1}^n d_k\}_{n=1}^m$ by $\{v_n\}_{n=1}^m$ has the form

$$\sum_{n=1}^m v_n(\cdot)\, d_n(\cdot) \;=\; \sum_{n=1}^m T_n \;\; |h_n(\cdot)| \;\; x_n h_n(\cdot) \;=\; \sum_{n=1}^m T_n \;\, x_n h_n(\cdot) \;\; .$$

Thus T is bounded (by some constanst C_{XYpq}) if and only if

$$\left\| \sum_{n=1}^m v_n d_n \right\|_{L_q(\Omega,Y)} \;\leq\; C_{XYpq} \; \left\| \sum_{n=1}^m d_n \right\|_{L_p(\Omega,X)}$$

for each X-valued *Haar* martingale difference sequence $\{d_n\}_{n=1}^m$.

Motivated by Remark 3.1, define $u_p \colon \Omega \to [0,\infty]$ by

$$u_p(\cdot) \;:=\; R_p\left(\{T_j \;\; |h_j(\cdot)| \;\; : j \in \mathbb{N}\}\right) \tag{3.1}$$

for $1 \leq p < \infty$. Thus

$$u_p(\omega) \;=\; R_p\left(\left\{T_0^0\right\} \;\cup\; \{T_k^n \;:\, (n,k) \in \Delta_1 \;,\; \omega \in I_k^n\}\right) \;\; .$$

Clearly u_p is measurable since it is the pointwise limit of the sequence $\{s_n\}_{n\in\mathbb{N}}$ where

$$s_n(\cdot) \;:=\; \sum_{k=1}^{2^n} \; R_p\left(\left\{T_0^0\right\} \;\cup\; \left\{T_j^m \colon\; (m,j) \in \Delta_1 \;,\; I_k^n \subset I_j^m\right\}\right) \; 1_{I_k^n}(\cdot) \;\; .$$

The case $p = q$ is a direct consequence of results in [12].

Theorem 3.2. *Let T be the Fourier Haar multiplier operator generated by $\{T_j\}_{j\in\mathbb{N}}$ from $\mathcal{B}(X,Y)$. Let X and Y be UMD spaces. Let u_p be as in* (3.1)*.*

(a) *If $1 < p < \infty$ then*

$$\|T\|_{L_p(\Omega,X)\to L_p(\Omega,Y)} \;\leq\; \beta_p(X) \;\; \beta_p(Y) \;\; \|u_p\|_{L_\infty(\Omega,[0,\infty])} \;\; .$$

(b) *There exists a constant A_{XY} so that*

$$\|T\|_{L_1(\Omega,X)\to L_1^{wk}(\Omega,Y)} \;\leq\; A_{XY} \;\; \|u_1\|_{L_\infty(\Omega,[0,\infty])} \;\; .$$

Proof. Theorem 3.2 follows easily from Remark 3.1 and [12, Theorem 3.2 and Fact 5.1]). □

The next theorems covers the case $q \leq p$. Its rather simple proof is quite different from the usual proof for scalar-valued multiplier sequences (see [19, Theorem 12.2]), which uses interpolation.

Theorem 3.3. *Let $1 < q \leq p < \infty$. Let X and Y be UMD spaces. Let T be the Fourier Haar multiplier operator generated by $\{T_j\}_{j\in\mathbb{N}}$ from $\mathcal{B}(X,Y)$. Then*

$$\|T\|_{L_p(\Omega,X)\to L_q(\Omega,Y)} \;\leq\; \beta_p(X) \;\; \beta_q(Y) \;\; \|u_q\|_{L_r(\Omega,[0,\infty])}$$

where $r \in (1, \infty]$ *is given by* $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

Recall that u_q is defined in (3.1).

Proof. Fix $\sum_{j\in\mathbb{N}} x_j h_j \in E(\Omega, X)$. Note that for each fixed $t \in [0,1]$,

$$\begin{aligned} \left\| \sum_{j\in\mathbb{N}} r_j(t)\, x_j h_j \right\|_{L_p(\Omega,X)}^q &\le \beta_p^q(X) \left\| \sum_{j\in\mathbb{N}} x_j h_j \right\|_{L_p(\Omega,X)}^q \\ \left\| \sum_{j\in\mathbb{N}} T_j x_j h_j \right\|_{L_q(\Omega,Y)}^q &\le \beta_q^q(Y) \left\| \sum_{j\in\mathbb{N}} r_j(t)\, T_j x_j h_j \right\|_{L_q(\Omega,Y)}^q . \end{aligned}$$

Thus

$$\begin{aligned} \left\| \sum_{j\in\mathbb{N}} T_j x_j h_j \right\|_{L_q(\Omega,Y)}^q &\le \beta_q^q(Y) \int_{[0,1]} \left\| \sum_{j\in\mathbb{N}} r_j(t)\, T_j x_j h_j(\cdot) \right\|_{L_q(\Omega,Y)}^q dt \\ &= \beta_q^q(Y) \int_\Omega \left\| \sum_{j\in\mathbb{N}} r_j(\cdot)\left(T_j \left|h_j(\omega)\right|\right) x_j h_j(\omega) \right\|_{L_q([0,1],Y)}^q d\omega \\ &\le \beta_q^q(Y) \int_\Omega R_q^q\left(\{T_j \left|h_j(\omega)\right| : j \in \mathbb{N}\}\right) \left\| \sum_{j\in\mathbb{N}} r_j(\cdot)\, x_j h_j(\omega) \right\|_{L_q([0,1],X)}^q d\omega \\ &= \beta_q^q(Y) \int_{[0,1]} \left\| u_q(\cdot) \left(\sum_{j\in\mathbb{N}} r_j(t)\, x_j h_j(\cdot) \right) \right\|_{L_q(\Omega,X)}^q dt \\ &\le \beta_q^q(Y) \int_{[0,1]} \|u_q\|_{L_r(\Omega,[0,\infty])}^q \left\| \sum_{j\in\mathbb{N}} r_j(t)\, x_j h_j(\cdot) \right\|_{L_p(\Omega,X)}^q dt \\ &\le \beta_q^q(Y)\, \|u_q\|_{L_r(\Omega,[0,\infty])}^q\, \beta_p^q(X) \left\| \sum_{j\in\mathbb{N}} x_j h_j \right\|_{L_p(\Omega,X)}^q . \end{aligned}$$

This completes the proof. □

The next theorem covers the case $p < q$. It gives a vector-valued analogue of (1.2). In this case, the usual proof of the scalar-valued case can be generalized and so no UMD nor R-boundedness assumptions are necessary.

Theorem 3.4. *Let* $1 \le p < q < \infty$. *Let* T *be the Fourier Haar multiplier operator generated by* $\{T_j\}_{j\in\mathbb{N}}$ *from* $\mathcal{B}(X,Y)$. *Define*

$$A_{pq} := \sup_{(n,k)\in\Delta} 2^{n\left(\frac{1}{p}-\frac{1}{q}\right)} \|T_k^n\|_{\mathcal{B}(X,Y)} .$$

(a) *If* $1 < p$ *then*

$$A_{pq} \le \|T\|_{L_p(\Omega,X)\to L_q(\Omega,Y)} \le C_{pq}\, A_{pq} .$$

(b) *If* $1 = p$ *then*

$$\|T\|_{L_1(\Omega,X)\to L_q^{wk}(\Omega,Y)} \le C_q\, A_{1q} .$$

Proof. The lower bound in part (a) follows from Remark 4.1.

Set

$$\alpha \;=\; \frac{1}{p} - \frac{1}{q}\ ;$$

thus, $0 < \alpha < 1$. Define $J\colon \mathbb{R} \to \mathbb{R}$ via

$$J\left(t\right) \;:=\; \begin{cases} \left|t\right|^{\alpha-1} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0\ . \end{cases}$$

By the Hardy-Littlewood-Sobolev theorem (cf. [20, page 119]), for each $g \in L_p\left(\mathbb{R},\mathbb{R}\right)$, the integral

$$\left(Sg\right)\left(t\right) \;:=\; \int_{\mathbb{R}} \frac{g\left(s\right)}{\left|t-s\right|^{1-\alpha}}\, ds \;=\; \left(J * g\right)\left(t\right)$$

converges absolutely for a.e. $t \in \mathbb{R}$ and the operator S satisfies

$$\begin{aligned} \left\|Sg\right\|_{L_q\left(\mathbb{R},\mathbb{R}\right)} &\;\leq\; C_{pq}\ \left\|g\right\|_{L_p\left(\mathbb{R},\mathbb{R}\right)} && \text{if } p > 1 \\ \left\|Sg\right\|_{L_q^{\mathrm{wk}}\left(\mathbb{R},\mathbb{R}\right)} &\;\leq\; C_{pq}\ \left\|g\right\|_{L_1\left(\mathbb{R},\mathbb{R}\right)} && \text{if } p = 1 \end{aligned}$$

for some constants C_{pq}.

Define $K\colon \Omega \times \Omega \to \mathcal{B}\left(X,Y\right)$ via

$$K\left(t,s\right) \;=\; \begin{cases} \sum_{(n,k)\in\Delta}\ 2^n T_k^n h_k^n\left(t\right) h_k^n\left(s\right) & \text{if } t \neq s \\ 0 & \text{if } t = s\ . \end{cases} \tag{3.2}$$

Note that, for each fixed t and s with $t \neq s$, only a finite number of terms in the summand in (3.2) are nonzero. Fix $f = \sum_{(n,k)\in\Delta} x_k^n h_k^n \in E\left(\Omega, X\right)$. Thus, for each $t \in \Omega$,

$$\begin{aligned} \int_{\Omega} K\left(t,s\right) f\left(s\right)\, ds \;&=\; \int_{\Omega\setminus\{t\}} \left[\sum_{(m,j)\in\Delta} 2^m T_j^m h_j^m\left(t\right) h_j^m\left(s\right)\right] \left[\sum_{(n,k)\in\Delta} x_k^n h_k^n\left(s\right)\right] ds \\ &=\; \sum_{(n,k)\in\Delta} \int_{\Omega} \sum_{(m,j)\in\Delta} \left[T_j^m x_k^n h_j^m\left(t\right)\right]\ \left[2^m h_j^m\left(s\right) h_k^n\left(s\right)\right]\, ds \\ &=\; \sum_{(n,k)\in\Delta} T_k^n x_k^n h_k^n\left(t\right) \;=\; \left(Tf\right)\left(t\right)\ . \end{aligned}$$

Fix $t, s \in \Omega$ with $t \neq s$. Find the unique $m \in \mathbb{N}$ so that

$$2^{-m} < \left|t-s\right| \leq 2^{-m+1}\ .$$

So $h_k^n(t)\, h_k^n(s) = 0$ if $n \geq m$ and $(n,k) \in \Delta_1$. Thus

$$\begin{aligned}
\|K(t,s)\|_{\mathcal{B}(X,Y)} &\leq \sum_{(n,k)\in\Delta} 2^n \|T_k^n\|_{\mathcal{B}(X,Y)} \left|h_k^n(t)\, h_k^n(s)\right| \\
&\leq \sum_{(n,k)\in\Delta} 2^n A_{pq} 2^{-n\alpha} \left|h_k^n(t)\, h_k^n(s)\right| \\
&\leq A_{pq} \left[1 + \sum_{n=0}^{m-1} \left(2^{1-\alpha}\right)^n\right] \\
&= A_{pq} \left[1 + \frac{2^{(1-\alpha)m} - 1}{2^{1-\alpha} - 1}\right] \frac{2^{\alpha-1}}{2^{\alpha-1}} \\
&\leq \frac{A_{pq}}{1-2^{\alpha-1}}\ \frac{1}{\left(2^{-m+1}\right)^{1-\alpha}} \\
&\leq \frac{A_{pq}}{1-2^{\alpha-1}}\ \frac{1}{|t-s|^{1-\alpha}} \\
&= \frac{A_{pq}}{1-2^{\alpha-1}}\ J(t-s)\ .
\end{aligned} \tag{3.3}$$

Fix $f \in E(\Omega, X)$. Define $g \in L_\infty(\mathbb{R},\mathbb{R})$ via

$$g(t) \ := \begin{cases} \|f(t)\|_X & \text{if } t \in \Omega \\ 0 & \text{if } t \notin \Omega\ . \end{cases}$$

Towards part (b), now let $1 = p < q < \infty$. For each $t \in \Omega$

$$\|(Tf)(t)\|_Y \ = \ \left\|\int_\Omega K(t,s)\, f(s)\ ds\right\|_Y \ \leq \ \frac{A_{1q}}{1-2^{\alpha-1}} \int_\Omega J(t-s)\, g(s)\ ds$$

by (3.3). Thus, for each $\lambda > 0$,

$$\begin{aligned}
&\lambda \mu^{1/q}\left(\left\{t \in \Omega\colon\ \|(Tf)(t)\|_Y > \lambda\right\}\right) \\
&\quad\leq \frac{A_{1q}}{1-2^{\alpha-1}}\ \frac{\lambda\left(1-2^{\alpha-1}\right)}{A_{1q}}\ \mu^{1/q}\left(\left\{t \in \mathbb{R}\colon\ |(J * g)(t)| > \frac{\lambda\left(1-2^{\alpha-1}\right)}{A_{1q}}\right\}\right) \\
&\quad\leq \frac{A_{1q}}{1-2^{\alpha-1}}\ \|J * g\|_{L_q^{\mathrm{wk}}(\mathbb{R},\mathbb{R})} \\
&\quad\leq \frac{C_{1q}}{1-2^{\alpha-1}}\ A_{1q}\ \|g\|_{L_1(\mathbb{R},\mathbb{R})} \ = \ \frac{C_{1q}}{1-2^{\alpha-1}}\ A_{1q}\ \|f\|_{L_1(\Omega,X)}\ .
\end{aligned}$$

Thus part (b) holds.

Towards part (a), now let $1 < p < q < \infty$. By (3.3)

$$\begin{aligned}
\|Tf\|_{L_q(\Omega,Y)}^q &= \int_\Omega \left\|\int_\Omega K(t,s)\, f(s)\ ds\right\|_Y^q dt \\
&\leq \int_\Omega \left[\int_\Omega \|K(t,s)\|_{\mathcal{B}(X,Y)} \|f(s)\|_X\ ds\right]^q dt
\end{aligned}$$

$$
\begin{aligned}
&\leq \left[\frac{A_{pq}}{1-2^{\alpha-1}}\right]^q \int_\Omega \left[\int_{\mathbb{R}} J(t-s)\, g(s)\, ds\right]^q dt \\
&= \left[\frac{A_{pq}}{1-2^{\alpha-1}}\right]^q \int_\Omega |(Sg)(t)|^q \, dt \\
&\leq \left[\frac{A_{pq}}{1-2^{\alpha-1}}\right]^q \|Sg\|^q_{L_q(\mathbb{R},\mathbb{R})} \leq \left[\frac{A_{pq}}{1-2^{\alpha-1}}\right]^q C^q_{pq} \; \|g\|^q_{L_p(\mathbb{R},\mathbb{R})} \\
&= \left[\frac{C_{pq}}{1-2^{\alpha-1}} \, A_{pq} \; \|f\|_{L_p(\Omega,X)}\right]^q .
\end{aligned}
$$

Thus part (a) holds. □

4. EXAMPLES AND REMARKS

A lower bound on the norm of a Fourier Haar multiplier operator is easy.

Remark 4.1. Let T be the Fourier Haar multiplier operator generated by $\{T_j\}_{j\in\mathbb{N}}$ from $\mathcal{B}(X,Y)$. Then

$$
\|T\|_{L_p(\Omega,X)\to L_q(\Omega,Y)} \geq \sup_{(n,k)\in\Delta} 2^{n\left(\frac{1}{p}-\frac{1}{q}\right)} \; \|T^n_k\|_{\mathcal{B}(X,Y)}
$$

for each $1 \leq p, q < \infty$.

Proof. Fix $(n,k) \in \Delta$. Then

$$
\begin{aligned}
\|T\|_{L_p(\Omega,X)\to L_q(\Omega,Y)} &\geq \sup_{x\in B(X)} \frac{\|T^n_k x h^n_k\|_{L_q(\Omega,Y)}}{\left\|x h^n_k\right\|_{L_p(\Omega,X)}} \\
&= \sup_{x\in B(X)} \frac{\|T^n_k x\|_Y}{\|x\|_X} \frac{\|h^n_k\|_{L_q(\Omega,Y)}}{\left\|h^n_k\right\|_{L_p(\Omega,X)}} = \|T^n_k\|_{\mathcal{B}(X,Y)} \; \frac{(2^{-n})^{1/q}}{(2^{-n})^{1/p}} .
\end{aligned}
$$

This finishes the proof. □

Example 4.2 shows that R-bounded is a natural assumption in Section 3.

Example 4.2. Consider a sequence $\{S_n\}_{n\in\mathbb{N}_0}$ from $\mathcal{B}(X,Y)$. Define $\{T^n_k\}_{(n,k)\in\Delta}$ by $T^0_0 = S_0$ and $T^n_k = S_{n+1}$ for $(n,k) \in \Delta_1$. Then

$$
C_{pq} \; R_q\left(\{S_n \colon n \in \mathbb{N}_0\}\right) \leq \|T\|_{L_p(\Omega,X)\to L_q(\Omega,Y)}
$$

for $1 \leq p, q < \infty$. Indeed

$$
\|T\|_{L_p(\Omega,X)\to L_q(\Omega,Y)} \geq \sup_{\substack{N\in\mathbb{N}\\ x_n\in X\\ x_n\neq 0}} \frac{\left\|T^0_0 x_0 h^0_0 + \sum_{n=0}^N \sum_{k=1}^{2^n} T^n_k x_{n+1} h^n_k\right\|_{L_q(\Omega,Y)}}{\left\|x_0 h^0_0 + \sum_{n=0}^N \sum_{k=1}^{2^n} x_{n+1} h^n_k\right\|_{L_p(\Omega,X)}}
$$

$$
\begin{aligned}
&= \sup_{\substack{N\in\mathbb{N}\\ x_n\in X\\ x_n\neq 0}} \frac{\left\| S_0 x_0 r_0 + \sum_{n=0}^{N} S_{n+1} x_{n+1} r_{n+1} \right\|_{L_q(\Omega,Y)}}{\left\| x_0 r_0 + \sum_{n=0}^{N} x_{n+1} r_{n+1} \right\|_{L_p(\Omega,X)}} \\
&= \sup_{\substack{N\in\mathbb{N}\\ x_n\in X\\ x_n\neq 0}} \frac{\left\| \sum_{n=0}^{N} S_n x_n r_n \right\|_{L_q(\Omega,Y)}}{\left\| \sum_{n=0}^{N} x_n r_n \right\|_{L_q(\Omega,X)}} \, \frac{\left\| \sum_{n=0}^{N} x_n r_n \right\|_{L_q(\Omega,X)}}{\left\| \sum_{n=0}^{N} x_n r_n \right\|_{L_p(\Omega,X)}} \\
&\geq R_q\left(\{S_n \colon n\in\mathbb{N}_0\}\right) \inf_{\substack{N\in\mathbb{N}\\ x_n\in X\\ x_n\neq 0}} \frac{\left\| \sum_{n=0}^{N} x_n r_n \right\|_{L_q(\Omega,X)}}{\left\| \sum_{n=0}^{N} x_n r_n \right\|_{L_p(\Omega,X)}} \\
&\geq C_{pq}\, R_q\left(\{S_n \colon n\in\mathbb{N}_0\}\right)
\end{aligned}
$$

for some constant $C_{pq} \in (0,\infty)$.

Example 4.2 also sheds light on the proper generalization of (1.3).

Example 4.3. Now let X and Y be UMD spaces and $1 < q \leq p < \infty$.

Theorem 3.3 generalizes (1.3) via the function

$$u_q(\cdot) \;=\; R_q\left(\{T_j \ \ |h_j(\cdot)| \ : j\in\mathbb{N}\}\right) \ .$$

Also consider the function

$$\widetilde{u}(\cdot) \;=\; \sup_{j\in\mathbb{N}} \|T_j \ \ |h_j(\cdot)| \ \|_{\mathcal{B}(X,Y)} \ .$$

Clearly, $\widetilde{u} \leq u_q$. If X has cotype 2 and Y has type 2, then $u_q \leq C_{XYq}\widetilde{u}$ for some constant $C_{XYq} \in (0,\infty)$.

Note that, in Example 4.2, the functions u_q and $\widetilde{u}$ are constant:

$$u_q(\omega) \;=\; R_q\left(\{S_n \colon n\in\mathbb{N}_0\}\right) \quad \text{and} \quad \widetilde{u}(\omega) \;=\; \sup_{n\in\mathbb{N}_0} \|S_n\|_{\mathcal{B}(X,Y)}$$

for each $\omega \in \Omega$. Thus, for this example, the bounds in Theorems 3.2 and 3.3 are of the proper order; that is,

$$\|T\|_{L_p(\Omega,X)\to L_q(\Omega,Y)} \;\approx\; \|u_q\|_{L_r(\Omega,[0,\infty])}$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$.

However, if X does not have cotype 2 or Y does not have type 2, then there exists a sequence $\{S_n\}_{n\in\mathbb{N}_0}$ from the unit sphere of $\mathcal{B}(X,Y)$ that is not R-bounded. Consider the corresponding Fourier Haar multiplier operator T as defined in Example 4.2. Then T is not bounded, indeed, $u_q(\omega) = \infty$ for each $\omega \in \Omega$. However, $\widetilde{u}(\omega) = 1$ for each $\omega \in \Omega$.

Thus, in generalizing (1.3), R-boundedness is needed.

Remark 4.4. In Theorems 3.2 and 3.3, it is essential that X and Y be UMD spaces. Indeed, one obtains ([18], see [5]) an equivalent formulation of UMD spaces (with the same constant) if, in Definition 2.1, one replaces the arbitrary filtration $\{\mathcal{F}_n\}_{n=1}^m$ with the Haar filtration $\{\mathcal{H}_n\}_{n=1}^m$. Thus (see Remark 3.1), X is a UMD space if and only if each Fourier Haar multiplier operator T generated by a multiplier sequence of the form $\{\varepsilon_j 1_X\}_{j\in\mathbb{N}}$ for some choice $\{\varepsilon_j\}_{j\in\mathbb{N}}$ of signs $\{\pm 1\}$ is bounded from $L_p(\Omega, X)$ to $L_p(\Omega, X)$ by a constant depending only on X and p for some (or equivalently, for each) $p \in (1, \infty)$. Note that for such an operator T

$$u_p(\omega) \ = \ 1$$

for each $\omega \in \Omega$.

Remark 4.5. Theorem 3.2 part (a) fails for $p = 1$ (or equivalently: Theorem 3.2 part (b) fails if $L_1^{\mathrm{wk}}(\Omega, Y)$ is replaces with $L_1(\Omega, Y)$). Indeed, let $X = Y = \mathbb{R}$ and assume that there is a constant C such that each Fourier Haar multiplier operator T generated by $\{T_k^n\}_{(n,k)\in\Delta}$ satisfies

$$\|T\|_{L_1(\Omega,\mathbb{R})\to L_1(\Omega,\mathbb{R})} \ \le \ C \ \|u_1\|_{L_\infty(\Omega,[0,\infty])} \ . \tag{4.1}$$

By considering $\{T_k^n\}_{(n,k)\in\Delta}$ of the form $T_k^n = \varepsilon_k^n$ where $\varepsilon_k^n \in \{\pm 1\}$, equation (4.1) would imply that the $\{h_n^k\}_{(n,k)\in\Delta}$ is an *unconditional* basis for $L_1(\Omega, \mathbb{R})$, which is not true (cf. [17]).

Remark 4.6. Theorem 3.4 part (a) fails for $1 = p < q < \infty$ (or equivalently: Theorem 3.4 part (b) fails if $L_q^{\mathrm{wk}}(\Omega, Y)$ is replaced with $L_q(\Omega, Y)$). Indeed, let $X = Y = \mathbb{R}$ and consider the Fourier Haar multiplier operator T generated by $\{\lambda_k^n\}_{(n,k)\in\Delta}$ where $\lambda_k^n := 2^{-\frac{n}{q'}}$ with $\frac{1}{q} + \frac{1}{q'} = 1$. Clearly

$$A_{pq} \ := \ \sup_{(n,k)\in\Delta} \ 2^{n\left(\frac{1}{p}-\frac{1}{q}\right)} \ |\lambda_k^n| \ = \ 1 \ .$$

If T where bounded from $L_1(\Omega, \mathbb{R})$ to $L_q(\Omega, \mathbb{R})$, then its adjoint T^* would be bounded from $L_{q'}(\Omega, \mathbb{R})$ to $L_\infty(\Omega, \mathbb{R})$. But

$$\left\| \sum_{n=1}^m 2^{\frac{n}{q'}} \ \frac{1}{n} \ h_1^n \right\|_{L_{q'}(\Omega,\mathbb{R})} \ \approx \ \left\| \left\{ \frac{1}{n} \right\}_{n=1}^m \right\|_{\ell_{q'}}$$

$$\left\| T^* \left(\sum_{n=1}^m 2^{\frac{n}{q'}} \ \frac{1}{n} \ h_1^n \right) \right\|_{L_\infty(\Omega,\mathbb{R})} \ = \ \left\| \left\{ \frac{1}{n} \right\}_{n=1}^m \right\|_{\ell_1} \ .$$

for each $m \in \mathbb{N}$.

Remark 4.7. Theorem 3.4 part (a), it is essential that $p \neq q$. Indeed, let $1 < p = q < \infty$. Let X and Y be UMD spaces such that X does not have cotype 2 or Y does not have type 2. Then there exists a sequence $\{S_n\}_{n\in\mathbb{N}_0}$ from the unit sphere of $\mathcal{B}(X,Y)$ that is not R-bounded. Consider the corresponding Fourier Haar multiplier operator T as defined in Example 4.2. Clearly,

$$A_{pq} \ := \sup_{(n,k)\in\Delta} \ 2^{n\left(\frac{1}{p}-\frac{1}{q}\right)} \ \|T_k^n\|_{\mathcal{B}(X,Y)} \ = \ 1 \ .$$

However, as noted in Example 4.3, T is not bounded from $L_p(\Omega,X)$ to $L_p(\Omega,Y)$.

Remark 4.8. Let $\{v_j\}_{j\in\mathbb{N}}$ be a $\mathcal{B}(X,Y)$-valued $\{\mathcal{H}_j\}_{j\in\mathbb{N}}$-predictable sequence and $1\le p,q<\infty$.

Let's consider the following natural question.

When does there exist a constant C_{XYpq} so that

$$\left\|\sum_{n=1}^{m} v_n d_n\right\|_{L_q(\Omega,Y)} \ \le \ C_{XYpq} \ \left\|\sum_{n=1}^{m} d_n\right\|_{L_p(\Omega,X)} \tag{4.2}$$

for each X-valued Haar martingale difference sequence $\{d_n\}_{n=1}^m$?

This question reduces to Fourier Haar multipliers.

Indeed, by predictability, each v_j is constant on the support of h_j and so there exists a *unique* sequence $\{T_j\}_{j\in\mathbb{N}}$ from $\mathcal{B}(X,Y)$ so that

$$v_j(\cdot)\,|h_j(\cdot)| \ = \ T_j\,|h_j(\cdot)|$$

for each $j\in\mathbb{N}$.

It now follows from Definition 2.3 and Remark 3.1 that question (4.2) is *true* if and only if the Fourier Haar multiplier generated by the Fourier Haar multiplier sequence $\{T_j\}_{j\in\mathbb{N}}$ is *bounded* from $L_p(\Omega,X)$ to $L_q(\Omega,Y)$.

References

[1] Wolfgang Arendt and Shangquan Bu, *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity*, Math. Z. **240** (2002), no. 2, 311–343.

[2] Earl Berkson and T. A. Gillespie, *Spectral decompositions and harmonic analysis on UMD spaces*, Studia Math. **112** (1994), no. 1, 13–49.

[3] Jean Bourgain, *Vector-valued singular integrals and the H^1-BMO duality*, Probability theory and harmonic analysis (Cleveland, Ohio, 1983), Monogr. Textbooks Pure Appl. Math., vol. 98, Dekker, New York, 1986, pp. 1–19.

[4] D. L. Burkholder, *A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional*, Ann. Probab. **9** (1981), no. 6, 997–1011.

[5] ———, *Explorations in martingale theory and its applications*, École d'Été de Probabilités de Saint-Flour XIX—1989, Lecture Notes in Math., vol. 1464, Springer, Berlin, 1991, pp. 1–66.

[6] ———, *Martingales and singular integrals in Banach spaces*, Handbook of the geometry of Banach spaces, Vol. I, North-Holland, Amsterdam, 2001, pp. 233–269.

[7] P. Clément, B. de Pagter, F. A. Sukochev, and H. Witvliet, *Schauder decomposition and multiplier theorems*, Studia Math. **138** (2000), no. 2, 135–163.

[8] Robert Denk, Matthias Hieber, and Jan Prüss, *R-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Amer. Math. Soc. **166** (2003), no. 788, viii+114.

[9] J. Diestel and J. J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977, With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.
[10] Maria Girardi and Lutz Weis, *Criteria for R-boundedness of operator families*, Evolution equations, Lecture Notes in Pure and Appl. Math., vol. 234, Dekker, New York, 2003, pp. 203–221.
[11] ———, *Operator-valued Fourier multiplier theorems on $L_p(X)$ and geometry of Banach spaces*, J. Funct. Anal. **204** (2003), no. 2, 320–354.
[12] ———, *Operator-valued Martingale transforms and R-boundedness*, Illinois J. Math. **49** (2005), no. 2, 487–516 (electronic).
[13] Tuomas Hytönen and Lutz Weis, *Singular convolution integrals with operator-valued kernels*, Math. Zeitschrift, to appear.
[14] N. J. Kalton, P. Kunstmann, and L. Weis, *Perturbations and interpolation theorems for the H^∞-calculus with applications to differential operators*, Math. Ann., to appear.
[15] N. J. Kalton and L. Weis, *The H^∞-calculus and sums of closed operators*, Math. Ann. **321** (2001), no. 2, 319–345.
[16] Peer C. Kunstmann and Lutz Weis, *Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus*, Functional analytic methods for evolution equations, Lecture Notes in Math., vol. 1855, Springer, Berlin, 2004, pp. 65–311.
[17] Joram Lindenstrauss and Lior Tzafriri, *Classical Banach spaces. I (sequence spaces) and II (function spaces)*, Classics in Mathematics, Springer-Verlag, Berlin, 1996.
[18] B. Maurey, *Système de Haar*, Séminaire Maurey-Schwartz 1974–1975: Espaces Lsup p, applications radonifiantes et géométrie des espaces de Banach, Exp. Nos. I et II, Centre Math., École Polytech., Paris, 1975, pp. 26 pp. (erratum, p. 1).
[19] Igor Novikov and Evgenij Semenov, *Haar series and linear operators*, Mathematics and its Applications, vol. 367, Kluwer Academic Publishers Group, Dordrecht, 1997.
[20] Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.
[21] J. M. A. M. van Neerven, M. Veraar, and L. Weis, *Stochastic integration of processes with values in a UMD Banach space*, submitted.
[22] J. M. A. M. van Neerven and L. Weis, *Stochastic integration of functions with values in a Banach space*, Studia Math. **166** (2005), no. 2, 131–170.
[23] Lutz Weis, *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann. **319** (2001), no. 4, 735–758.

Department of Mathematics, University of South Carolina, Columbia, SC 29208, U.S.A.
E-mail address: `girardi@math.sc.edu`