OPERATOR-VALUED MARTINGALE TRANSFORMS AND R-BOUNDEDNESS

MARIA GIRARDI AND LUTZ WEIS

ABSTRACT. Banach space X-valued martingale transforms by a $\mathcal{B}(X)$-valued multiplier sequence are bounded on $L_p(X)$, where $1 < p < \infty$ and X is a UMD space, if and only if the multiplier sequence is pointwise R-bounded. This is also true for unconditionally convergent martingales in arbitrary Banach spaces.

1. INTRODUCTION

Let X be a Banach space. The *martingale transform* of an X-valued martingale $\{f_n\}_{n\in\mathbb{N}}$ by a $\mathbb{R}$-valued, predictable, uniformly bounded sequence $\{v_n\}_{n\in\mathbb{N}}$ is the martingale $\{g_n\}_{n\in\mathbb{N}}$ where

$$g_n := \sum_{k=1}^{n} v_k d_k \qquad \text{and} \qquad f_n := \sum_{k=1}^{n} d_k \ ; \tag{1.1}$$

so $\{d_n\}_{n\in\mathbb{N}}$ is the *martingale difference sequence* of $\{f_n\}_{n\in\mathbb{N}}$.

Burkholder [6] introduced UMD (unconditionality property for martingale differences) Banach spaces: for $1 < p < \infty$, the UMD constant of X is the smallest $\beta_p(X) \in [1, \infty]$ so that

$$\|\varepsilon_1 d_1 + \ldots + \varepsilon_m d_m\|_{L_p(\Omega,X)} \ \le\ \beta_p(X)\ \|d_1 + \ldots + d_m\|_{L_p(\Omega,X)} \tag{1.2}$$

for each X-valued martingale difference sequence $\{d_n\}_{n\in\mathbb{N}}$ with respect to some filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$, choice $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of signs from $\{\pm 1\}$, and $m \in \mathbb{N}$. A Banach space X is UMD provided that its UMD constant is finite for some (or equivalently, by Pisier [29], for each) $p \in (1, \infty)$.

In this setting, the underlying probability space (unless it is nonatomic) and filtration must vary. Burkholder [6] showed that (1.2) holds, with the same constant $\beta_p(X)$, if one replaces the choices $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of signs by $\{\mathcal{F}_n\}$-predictable sequences $\{v_n\}_{n\in\mathbb{N}}$ of functions valued in $[-1, 1]$.

Over the years, the interplay between probability and harmonic analysis has been very fruitful (see, e.g., [10, 11]) Indeed, the study of the martingale transform uses, for example, *Doob's maximal function* ($f^*(\omega) = \sup_n |f_n(\omega)|$) and the *square function* ($Sf = (\sum_{n\in\mathbb{N}} |d_n|^2)^{1/2}$). Also [4, 8], X

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has UMD if and only if the Hilbert transform is bounded on $L_p(\mathbb{R}, X)$ for some (or equivalently for each) $p \in (1, \infty)$.

Martínez and Torrea [27] studied *operator-valued martingale transforms* where the *multiplier sequences* $\{v_n\}_{n\in\mathbb{N}}$ are valued in $\mathcal{B}(X, Y)$ instead of $\mathbb{R}$. They derived a theory that parallels the $\mathbb{R}$-valued case. For example, they obtained a martingale version of the well-known theorem of Fefferman and Stein [17] for Hardy-Littlewood maximal operator.

However, they did not give a criteria on a fixed $\mathcal{B}(X, Y)$-valued multiplier sequence $\{v_n\}_{n\in\mathbb{N}}$ to ensure that, for some $C_p \in \mathbb{R}$,

$$\|v_1 d_1 + \ldots + v_m d_m\|_{L_p(\Omega, Y)} \ \leq \ C_p \ \|d_1 + \ldots + d_m\|_{L_p(\Omega, X)} \tag{1.3}$$

for each admissible X-valued martingale difference sequence $\{d_n\}_{n\in\mathbb{N}}$ and $m \in \mathbb{N}$ and for some (or for each) $p \in (1, \infty)$. This paper gives such a criteria, in which R-bounded plays a key role. Indeed, Theorems 3.2, 3.3, and 4.1 led to the following crystallizing corollary.

Corollary 1.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space with filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ and $p \in (1, \infty)$. Let X and Y be UMD spaces. Let $\{v_n\}_{n\in\mathbb{N}}$ be a $\mathcal{B}(X, Y)$-valued $\{\mathcal{F}_n\}$-multiplier sequence.*

(A) *For arbitrary filtrations, the following are equivalent.*
  (1) *There exists $R_p \in \mathbb{R}$ so that $R_p(\{v_n(\omega) : n \in \mathbb{N}\}) \leq R_p$ for a.e. $\omega \in \Omega$.*
  (2) *There exists $C_p \in \mathbb{R}$ so that for each (uniformly bounded) X-valued martingale difference sequence $\{d_n\}_{n=1}^m$ with respect to some subfiltration $\left\{\widehat{\mathcal{F}}_n\right\}_{n=1}^m$ of $\left\{\widehat{\mathcal{F}}_n\right\}_{n\in\mathbb{N}}$, where $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mu}\right)$ is an extension of $(\Omega, \mathcal{F}, \mu)$,*
  $$\left\|\sum_{n=1}^m \widehat{v}_n d_n\right\|_{L_p(\widehat{\Omega}, Y)} \ \leq \ C_p \left\|\sum_{n=1}^m d_n\right\|_{L_p(\widehat{\Omega}, X)} .$$

(B) *For atomic filtrations satisfying (4.1), the following are equivalent.*
  (1) *There exists $R_p \in \mathbb{R}$ so that $R_p(\{v_n(\omega) : n \in \mathbb{N}\}) \leq R_p$ for each (or equivalently, for a.e.) $\omega \in \Omega$.*
  (2) *There exists $C_p \in \mathbb{R}$ so that for each (uniformly bounded) X-valued martingale difference sequence $\{d_n\}_{n=1}^m$ with respect to some subfiltration $\{\mathcal{F}_n\}_{n=1}^m$ of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$,*
  $$\left\|\sum_{n=1}^m v_n d_n\right\|_{L_p(\Omega, Y)} \ \leq \ C_p \left\|\sum_{n=1}^m d_n\right\|_{L_p(\Omega, X)} .$$

If (1) holds, then C_p in (2) can be taken to be $\beta_p(X)\, \beta_p(Y)\, R_p$. If (2) holds, then R_p in (1) can be taken to be C_p. (Needed definitions and notation to come.)

R-boundedness was introduced by Berkson and Gillespie in [2]. The notion grew out of work of J. Bourgain on vector-valued Fourier transform [5] and has been central to recent results on operator-valued Fourier multipliers and singular integrals with operator-valued kernels on Bochner spaces (e.g. [1, 20, 19, 33]). Through these tools, R-boundedness became important for maximal regularity of parabolic differential equations (e.g. [13, 14, 25, 33]) and the holomorphic functional calculus of sectorial operators (e.g. [21, 22, 25]). Results of the present paper are especially useful for the theory of stochastic integration on Banach spaces, which recently was developed in [31] and [32]. For more information on R-boundedness and its properties, see [12, 18, 25].

This paper is organized as follows. Section 2 collects the needed definitions and notation. The main results are in Sections 3 and 4. Section 5 gives further corollaries to these main theorems. Section 6 gives a technical proof of a lemma needed in Section 4.

2. DEFINITIONS AND NOTATION

Throughout this paper, the Banach spaces that appear are over the fixed scalar field of either the real or complex numbers. X, Y, and Z are Banach spaces. $B(X)$ is the closed unit ball of X while $S(X)$ is the unit sphere of X. The space $\mathcal{B}(X,Y)$ of bounded linear operators from X into Y is endowed with the usual operator norm topology. $([0,1], \mathcal{M}, m)$ is the usual Lebesgue measure space. $(\Omega, \mathcal{F}, \mu)$ is an arbitrary (complete) probability measure space; corresponding to it is the usual Bochner-Lebesgue space $L_p(\Omega, X)$ of measurable functions from Ω into X with finite $L_p(\Omega, X)$-norm where $1 \leq p \leq \infty$. A sequence $\{d_n\}_{n\in\mathbb{N}}$ of functions from Ω into X is *uniformly bounded* (by $M \in \mathbb{R}$) provided

$$\sup_{n\in\mathbb{N}} \sup_{\omega\in\Omega} \|d_n(\omega)\|_X \leq M \ .$$

Following Burkholder [7], a sequence $\{d_n\}_{n=1}^m$ of functions in $L_p(\Omega, X)$ is called τ-*unconditional* in $L_p(\Omega, X)$ provided

$$\left\| \sum_{n=1}^m \varepsilon_n \lambda_n d_n \right\|_{L_p(\Omega,X)} \leq \tau \left\| \sum_{n=1}^m \lambda_n d_n \right\|_{L_p(\Omega,X)}$$

for each choice $\{\varepsilon_n\}_{n=1}^m$ of signs from $\{\pm 1\}$ and choice $\{\lambda_n\}_{n=1}^m$ of scalars.

$\mathbb{N}$ is the set of natural numbers while $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space with a *filtration* $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ (i.e., $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ is a nondecreasing sequence of sub-σ-fields of $\mathcal{F}$). Let $m \in \mathbb{N} \cup \{\infty\}$. A sequence $\{f_n\}_{n=1}^m$ of functions from Ω into X is a *martingale* with respect to $\{\mathcal{F}_n\}_{n=1}^m$ provided $f_n \in L_1((\Omega, \mathcal{F}_n, \mu), X)$ and $\mathbb{E}(f_{n+1} \mid \mathcal{F}_n) = f_n$ for each admissible n. A sequence $\{d_n\}_{n=1}^m$ of functions from Ω into X is a *martingale difference* *sequence* with respect to $\{\mathcal{F}_n\}_{n=1}^m$ provided $d_n \in L_1((\Omega, \mathcal{F}_n, \mu), X)$ and $\mathbb{E}(d_{n+1} \mid \mathcal{F}_n) = 0$ for each admissible n. There is a one-to-one correspondence between martingales $\{f_n\}_{n=1}^m$ and martingale difference sequence $\{d_n\}_{n=1}^m$ given by

$$f_n \;=\; \sum_{k=1}^{n} d_k \qquad \text{and} \qquad d_n = f_n - f_{n-1}$$

where $f_0 \equiv 0$. Note that for a finite (i.e. $m \in \mathbb{N}$) X-valued martingale difference sequence $\{d_n\}_{n=1}^m$ and $p \in [1, \infty)$, each d_n is in $L_p(\Omega, X)$ if and only if $\sum_{n=1}^m d_n$ is in $L_p(\Omega, X)$. A sequence $\{v_n\}_{n\in\mathbb{N}}$ of functions from Ω into Z is *predictable* with respect to $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ (in short, $\{\mathcal{F}_n\}$-predictable) provided v_n is $\mathcal{F}_{n-1}$-measurable for each $n \in \mathbb{N}$. Note that if $\{v_n\}_{n\in\mathbb{N}}$ is predictable with respect to $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$, then it is predictable with respect to each subfiltration (i.e. subsequence) $\{\mathcal{F}_{j_n}\}_{n\in\mathbb{N}_0}$ of $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$.

Definition 2.1. To ease the statements of theorems to come, for a probability space $(\Omega, \mathcal{F}, \mu)$ with filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$, let

$$\mathcal{M}(\{\mathcal{F}_n\}, X) \;:=\; \left\{ \{f_n\}_{n\in\mathbb{N}} : \{f_n\}_{n\in\mathbb{N}} \text{ is an } X\text{-valued martingale with respect to } \{\mathcal{F}_n\}_{n\in\mathbb{N}} \right\}$$

and

$$\begin{aligned}\mathcal{D}(\{\mathcal{F}_n\}, X) \;:=\; \Big\{ \{d_n\}_{n=1}^m : \{d_n\}_{n=1}^m &\text{ is an } X\text{-valued martingale difference sequence} \\ &\text{with respect to some subfiltration } \{\mathcal{F}_{j_n}\}_{n=1}^m \text{ of } \{\mathcal{F}_n\}_{n\in\mathbb{N}} \text{ and } m \in \mathbb{N} \Big\} \,.\end{aligned}$$

Definition 2.2. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space with filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$.

(1) A $\mathcal{B}(X, Y)$*-valued* $\{\mathcal{F}_n\}$*-multiplier sequence* is a sequence $\{v_n\}_{n\in\mathbb{N}}$ of functions from Ω into $\mathcal{B}(X, Y)$ that is predictable with respect to $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ and is uniformly bounded by one.

(2) For such a multiplier sequence $v := \{v_n\}_{n\in\mathbb{N}}$, the *martingale transform* of a martingale $f := \{f_n\}_{n\in\mathbb{N}} \in \mathcal{M}(\{\mathcal{F}_n\}, X)$ is the martingale $\{(T_v f)_n\}_{n\in\mathbb{N}} \in \mathcal{M}(\{\mathcal{F}_n\}, Y)$ where

$$f_n := \sum_{k=1}^{n} d_k \qquad \text{and} \qquad (T_v f)_n := \sum_{k=1}^{n} v_k d_k$$

for each $n \in \mathbb{N}$.

The dyadic sigma-fields $\{\mathcal{D}_n\}_{n\in\mathbb{N}_0}$ are given by

$$\mathcal{D}_n = \sigma\left(I_k^n : 1 \le k \le 2^n\right)$$

and the Rademacher functions $\{r_n\}_{n\in\mathbb{N}}$ are given by

$$r_n = \sum_{k=1}^{2^n} (-1)^{k+1} 1_{I_k^n}$$

where, for $n \in \mathbb{N}_0$, $I_1^n = \left[\frac{0}{2^n}, \frac{1}{2^n}\right]$ and

$$I_k^n = \left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]$$

if $k \in \mathbb{N}$ and $1 < k \le 2^n$.

A proof of the next fact can be found at [16, Contraction Principle 12.2]. In the special case of when the independent symmetric sequence is the Rademacher functions $\{r_n\}_{n\in\mathbb{N}}$, it is known as Kahane's Contraction Principle.

Fact 2.3 (Contraction Principle)**.** *Let $\{\widetilde{d_n^o}\}_{n\in\mathbb{N}}$ be a sequence of independent, symmetric, $\mathbb{R}$-valued random variables on a probability space $(\Omega, \mathcal{F}, \mu)$. If $\{z_n\}_{n=1}^m$ is a sequence in any Banach space Z and $\{\lambda_n\}_{n=1}^m$ is a sequence from $\mathbb{R}$, then*

$$\left\| \sum_{n=1}^m \lambda_n \, z_n \, \widetilde{d_n^o} \right\|_{L_p(\Omega,Z)} \le \left[\max_{1\le n\le m} |\lambda_n| \right] \left\| \sum_{n=1}^m z_n \, \widetilde{d_n^o} \right\|_{L_p(\Omega,Z)}$$

for each $p \in [1, \infty)$.

R-boundedness is the central notion of this paper.

Definition 2.4. Let τ be a subset of $\mathcal{B}(X, Y)$ and $p \in [1, \infty)$. Let $R_p(\tau)$ be the smallest constant $R \in [0, \infty]$ with the property that for each $n \in \mathbb{N}$ and subset $\{T_j\}_{j=1}^n$ of τ and subset $\{x_j\}_{j=1}^n$ of X,

$$\left\| \sum_{j=1}^n r_j(\cdot) T_j(x_j) \right\|_{L_p([0,1],Y)} \le R \left\| \sum_{j=1}^n r_j(\cdot) x_j \right\|_{L_p([0,1],X)} .$$

The set τ is *R-bounded* provided $R_p(\tau)$ is finite for some (and thus then, by Kahane's inequality [16], for each) $p \in [1, \infty)$.

Thus a set τ is R-bounded provided Kahane's Contraction Principle holds for *operator coefficients* from τ. Pisier [1] showed that X is isomorphic to a Hilbert space if and only if each (norm) bounded subset of $\mathcal{B}(X, X)$ is R-bounded. Note that if X and Y are q-concave Banach lattices for some finite q (e.g. $X = Y = L_q(\Omega, \mathbb{C})$ where $1 \le q < \infty$) then R-boundedness is equivalent to the square function estimate

$$\left\| \left(\sum_{j=1}^m |T_j x_j|^2 \right)^{1/2} \right\|_Y \le R \left\| \left(\sum_{j=1}^n |x_j|^2 \right)^{1/2} \right\|_X$$

known from harmonic analysis (cf. [26, Thm. II.1.d.6]). For basic properties of R-bounded sets and further references, see [12, 33].

All notation and terminology, not otherwise explained, are as in [9, 15, 26].

3. MAIN RESULTS FOR ARBITRARY FILTRATIONS

Part (A) of Corollary 1.1 follows easily from Theorems 3.2 and 3.3.

For arbitrary filtrations, the notion of an *extension* (cf. eg. [23]) of a probability space is used.

Definition 3.1. Let $(\Omega, \mathcal{F}, \mu)$ and $(\Omega', \mathcal{F}', \mu')$ be probability spaces with filtrations $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ and $\{\mathcal{F}'_n\}_{n\in\mathbb{N}_0}$ respectively. The *extension* of $(\Omega, \mathcal{F}, \mu)$ by $(\Omega', \mathcal{F}', \mu')$ is their product probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mu})$, along with the filtration $\{\widehat{\mathcal{F}}_n\}_{n\in\mathbb{N}_0}$ where $\widehat{\mathcal{F}}_n = \sigma(\mathcal{F}_n \times \mathcal{F}'_n)$. For $h \in L_0(\Omega, Z)$, define $\widehat{h} \in L_0\left(\widehat{\Omega}, Z\right)$ by

$$\widehat{h}(\omega, \omega') := h(\omega) \ .$$

In the special case that $(\Omega', \mathcal{F}', \mu') = ([0,1], \mathcal{M}, m)$ and $\{\mathcal{F}'_n\}_{n\in\mathbb{N}_0} = \{\mathcal{D}_n\}_{n\in\mathbb{N}_0}$, one calls $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mu})$ the *dyadic extension* of $(\Omega, \mathcal{F}, \mu)$.

Note that if $h \in L_p(\Omega, Z)$ then $\|h\|_{L_p(\Omega,Z)} = \|\widehat{h}\|_{L_p(\widehat{\Omega},Z)}$ for $1 \leq p \leq \infty$. Also, if $\{v_n\}_{n\in\mathbb{N}}$ is a $\{\mathcal{F}_n\}$-multiplier sequence then $\{\widehat{v}_n\}_{n\in\mathbb{N}}$ is a $\left\{\widehat{\mathcal{F}}_n\right\}$-multiplier sequence.

Theorem 3.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space with filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ and $p \in [1, \infty)$. Let $\{v_n\}_{n\in\mathbb{N}}$ be a $\mathcal{B}(X,Y)$-valued $\{\mathcal{F}_n\}$-multiplier sequence that satisfies, for some $C_p \in \mathbb{R}$,*

$$R_p\left(\{v_n(\omega) : n \in \mathbb{N}\}\right) \leq C_p$$

for a.e. $\omega \in \Omega$.

(a) *If $\{d_n\}_{n=1}^m \in \mathcal{D}(\{\mathcal{F}_n\}, X)$ is so that $\{d_n\}_{n=1}^m$ is α_p-unconditional in $L_p(\Omega, X)$ and $\{v_n d_n\}_{n=1}^m$ is β_p-unconditional in $L_p(\Omega, Y)$, then*

$$\left\| \sum_{n=1}^m v_n d_n \right\|_{L_p(\Omega,Y)} \leq \alpha_p \ \beta_p \ C_p \ \left\| \sum_{n=1}^m d_n \right\|_{L_p(\Omega,X)} .$$

(b) *If $\{d_n\}_{n=1}^m \in \mathcal{D}\left(\left\{\widehat{\mathcal{F}}_n\right\}, X\right)$, for some extension $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mu})$ of $(\Omega, \mathcal{F}, \mu)$, is so that $\{d_n\}_{n=1}^m$ is α_p-unconditional in $L_p\left(\widehat{\Omega}, X\right)$ and $\{\widehat{v}_n d_n\}_{n=1}^m$ is β_p-unconditional in $L_p\left(\widehat{\Omega}, Y\right)$, then*

$$\left\| \sum_{n=1}^m \widehat{v}_n d_n \right\|_{L_p(\widehat{\Omega},Y)} \leq \alpha_p \ \beta_p \ C_p \ \left\| \sum_{n=1}^m d_n \right\|_{L_p(\widehat{\Omega},X)} .$$

Proof. Part (a) follows easily from (b). Towards (b), note that for each fixed $t \in [0,1]$

$$\begin{aligned}
\left\| \sum_{n=1}^{m} r_n(t)\, d_n \right\|_{L_p(\widehat{\Omega},X)}^{p} &\leq \alpha_p^p \left\| \sum_{n=1}^{m} d_n \right\|_{L_p(\widehat{\Omega},X)}^{p} \\
\left\| \sum_{n=1}^{m} \widehat{v}_n d_n \right\|_{L_p(\widehat{\Omega},Y)}^{p} &\leq \beta_p^p \left\| \sum_{n=1}^{m} r_n(t)\, \widehat{v}_n d_n \right\|_{L_p(\widehat{\Omega},Y)}^{p} .
\end{aligned}$$

Thus

$$\begin{aligned}
\left\| \sum_{n=1}^{m} \widehat{v}_n d_n \right\|_{L_p(\widehat{\Omega},Y)}^{p} &\leq \beta_p^p \int\limits_{[0,1]} \left\| \sum_{n=1}^{m} r_n(t)\, \widehat{v}_n d_n \right\|_{L_p(\widehat{\Omega},Y)}^{p} dt \\
&= \beta_p^p \int\limits_{\Omega} \int\limits_{\Omega'} \int\limits_{[0,1]} \left\| \sum_{n=1}^{m} r_n(t)\, v_n(\omega)\, d_n\left(\omega,\omega'\right) \right\|_{Y}^{p} dt\, d\mu'(\omega')\, d\mu(\omega) \\
&\leq \beta_p^p\, C_p^p \int\limits_{\Omega} \int\limits_{\Omega'} \int\limits_{[0,1]} \left\| \sum_{n=1}^{m} r_n(t)\, d_n\left(\omega,\omega'\right) \right\|_{X}^{p} dt\, d\mu'(\omega')\, d\mu(\omega) \\
&= \beta_p^p\, C_p^p \int\limits_{[0,1]} \left\| \sum_{n=1}^{m} r_n(t)\, d_n \right\|_{L_p(\widehat{\Omega},X)}^{p} dt \\
&\leq \beta_p^p\, C_p^p\, \alpha_p^p \left\| \sum_{n=1}^{m} d_n \right\|_{L_p(\widehat{\Omega},X)}^{p} .
\end{aligned}$$

This finishes the proof of (b). $\square$

Theorem 3.3. *Let $(\Omega,\mathcal{F},\mu)$ be a probability space with filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ and $p \in [1,\infty)$. Let $\{v_n\}_{n\in\mathbb{N}}$ be a $\mathcal{B}(X,Y)$-valued $\{\mathcal{F}_n\}$-multiplier sequence that satisfies, for some $C_p \in \mathbb{R}$,*

$$\begin{gathered}
\text{for each uniformly bounded } \{d_n\}_{n=1}^{m} \in \mathcal{D}\left(\left\{\widehat{\mathcal{F}}_n\right\}, X\right) , \\
\text{where } (\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mu}) \text{ is the dyadic extension of } (\Omega,\mathcal{F},\mu) , \\
\left\| \sum_{n=1}^{m} \widehat{v}_n d_n \right\|_{L_p(\widehat{\Omega},Y)} \leq C_p \left\| \sum_{n=1}^{m} d_n \right\|_{L_p(\widehat{\Omega},X)} .
\end{gathered} \tag{3.1}$$

Then

$$R_p\left(\{v_n(\omega) : n \in \mathbb{N}\}\right) \leq C_p$$

for a.e. $\omega \in \Omega$.

Remark 3.4. Condition (3.1) can be replaced by the (apparently) weaker Condition (3.1′).

Condition (3.1′):

$$\begin{gathered}\text{for each } \mathbb{R}\text{-valued finite martingale difference sequence } \{d_n\}_{n=1}^m \\ \text{with respect to } \{\widehat{\mathcal{F}}_{m-1+n}\}_{n=1}^m, \\ \text{where } (\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mu}) \text{ is the dyadic extension of } (\Omega, \mathcal{F}, \mu), \\ \text{of the form } d_n(\omega, t) = r_{m-1+n}(t)\, 1_A(\omega) \text{ where } A \in \mathcal{F}_{m-1}, \\ \text{one has that} \\ \left\| \sum_{n=1}^m \widehat{v}_n x_n d_n \right\|_{L_p(\widehat{\Omega}, Y)} \;\le\; C_p \left\| \sum_{n=1}^m x_n d_n \right\|_{L_p(\widehat{\Omega}, X)} \\ \text{for each choice } \{x_n\}_{n=1}^m \text{ from } B(X).\end{gathered}$$

Thus Condition (3.1′) reduces, from Condition (3.1), the class of martingale difference sequences that one must test. Note that for such a martingale difference sequence $\{d_n\}_{n=1}^m$ in Condition (3.1′), if $\{z_n\}_{n=1}^m$ is from any Banach space Z, then $\{z_n d_n\}_{n=1}^m$ is 1-unconditional in $L_p\left(\widehat{\Omega}, Z\right)$ by Fact 2.3.

Proof of Theorem 3.3. Assume condition (3.1′) of Remark 3.4 holds (but not that condition (3.1) necessarily holds). Let $\{\varepsilon_j\}_{j\in\mathbb{N}}$ be a sequence of real numbers tending to zero.

For each $n \in \mathbb{N}$, since $v_n \in L_\infty((\Omega, \mathcal{F}_{n-1}, \mu), \mathcal{B}(X,Y))$, there is a sequence $\{v_n^j\}_{j\in\mathbb{N}}$ of countably-valued functions in $L_\infty((\Omega, \mathcal{F}_{n-1}, \mu), \mathcal{B}(X,Y))$ so that $\lim_{j\to\infty} \left\| v_n - v_n^j \right\|_{L_\infty} = 0$. Note that for any sub-$\sigma$-field $\mathcal{G}_n^j$ containing $\sigma(v_n^j)$

$$\left\| v_n - \mathbb{E}\left(v_n \mid \mathcal{G}_n^j\right)\right\|_{L_\infty} \;\le\; \left\| v_n - \mathbb{E}\left(v_n^j \mid \mathcal{G}_n^j\right)\right\|_{L_\infty} \;+\; \left\| \mathbb{E}\left(v_n^j - v_n \mid \mathcal{G}_n^j\right)\right\|_{L_\infty} \;\le\; 2\; \left\| v_n - v_n^j \right\|_{L_\infty} .$$

So, for each $j \in \mathbb{N}$, there is a sequence $\{\mathcal{G}_n^j\}_{n\in\mathbb{N}_0}$ of sub-σ-fields of $\mathcal{F}$ so that

(1) $\left\| v_n - w_n^j \right\|_{L_\infty(\Omega, \mathcal{B}(X,Y))} \;<\; \frac{\varepsilon_j}{2^n}$ where $w_n^j \;:=\; \mathbb{E}\left(v_n \mid \mathcal{G}_{n-1}^j\right)$

(2) $\mathcal{G}_{n-1}^j \subset \mathcal{F}_{n-1}$ and $\mathcal{G}_{n-1}^j \subset \mathcal{G}_n^j$

(3) $\mathcal{G}_{n-1}^j$ is generated by a partition of Ω into (finitely or countably many) sets of (strictly) positive measure

for each $n \in \mathbb{N}$. So there exists $G \in \mathcal{F}$ so that $\mu(G) = 1$ and

$$\left\| v_n(u) - w_n^j(u) \right\|_{\mathcal{B}(X,Y)} \;<\; \frac{\varepsilon_j}{2^n}$$

for each $u \in G$ and $j, n \in \mathbb{N}$.

Fix $u \in G$. Fix $\{x_n\}_{n=1}^m$ from $B(X)$. Fix $j \in \mathbb{N}$. It suffices to show

$$\left\| \sum_{n=1}^{m} r_n v_n (u)\, x_n \right\|_{L_p([0,1],Y)} \;\le\; C_p \left\| \sum_{n=1}^{m} r_n x_n \right\|_{L_p([0,1],X)} \;+\; 2\varepsilon_j \ . \tag{3.2}$$

Find the atom A of $\mathcal{G}_{m-1}^{j}$ so that $u \in A$. Note that

$$w_n^j (u) \;=\; w_n^j (\omega) \qquad \text{for each} \quad \omega \in A \ , \ n \in \{1, \dots, m\} \ .$$

So

$$\begin{aligned}
\left\| \sum_{n=1}^{m} r_n v_n (u) x_n \right\|_{L_p([0,1],Y)} &\le \left\| \sum_{n=1}^{m} w_n^j (u)\, x_n r_{m-1+n} \right\|_{L_p([0,1],Y)} \;+\; \varepsilon_j \\
&= \left[\int_A \int_{[0,1]} \left\| \sum_{n=1}^{m} w_n^j (\omega)\, x_n r_{m-1+n} (t) \right\|_Y^p dt \ \frac{d\mu(\omega)}{\mu(A)} \right]^{1/p} \;+\; \varepsilon_j \\
&\le \frac{1}{\mu^{1/p}(A)} \left[\int_\Omega \int_{[0,1]} \left\| \sum_{n=1}^{m} v_n (\omega)\, x_n r_{m-1+n} (t)\, 1_A (\omega) \right\|_Y^p dt\, d\mu(\omega) \right]^{1/p} \;+\; 2\varepsilon_j \\
&\le \frac{C_p}{\mu^{1/p}(A)} \left[\int_\Omega \int_{[0,1]} \left\| \sum_{n=1}^{m} x_n r_{m-1+n} (t)\, 1_A (\omega) \right\|_X^p dt\, d\mu(\omega) \right]^{1/p} \;+\; 2\varepsilon_j \\
&= \frac{C_p}{\mu^{1/p}(A)} \left[\mu(A) \int_{[0,1]} \left\| \sum_{n=1}^{m} x_n r_{m-1+n} (t) \right\|_X^p dt \right]^{1/p} \;+\; 2\varepsilon_j \\
&= C_p \left\| \sum_{n=1}^{m} r_n x_n \right\|_{L_p([0,1],X)} \;+\; 2\varepsilon_j \ .
\end{aligned}$$

Thus (3.2) holds. □

4. MAIN RESULTS FOR ATOMIC FILTRATIONS

Now consider a probability space $(\Omega, \mathcal{F}, \mu)$ with a filtration $\{\mathcal{F}_n\}_{n \in \mathbb{N}_0}$ satisfying:

$$\begin{gathered}
\text{each } \mathcal{F}_n \text{ is generated by (finitely or countably many) atoms of (strictly) positive measure} \\
\text{and } \ \lim_{n \to \infty} \sup \{\mu(B) \colon B \text{ is an atom of } \mathcal{F}_n\} = 0 \ .
\end{gathered} \tag{4.1}$$

Part (B) of Corollary 1.1 follows easily from Theorems 4.1 and 3.2.

Theorem 4.1 is the atomic version of the general filtration Theorem 3.3. Note that Theorem 4.1 *reduces* the *test class* of martingale difference sequences from the test class needed in Theorem 3.3 in that, for the atomic case, one need not have to pass to extensions.

Theorem 4.1. *Let* $(\Omega, \mathcal{F}, \mu)$ *be a probability space with a filtration* $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ *satisfying* (4.1) *and* $p \in [1, \infty)$.

Let $\{v_n\}_{n\in\mathbb{N}}$ *be a* $\mathcal{B}(X, Y)$*-valued* $\{\mathcal{F}_n\}$*-multiplier sequence that satisfies, for some* $C_p \in \mathbb{R}$,

$$\begin{gathered} \text{for each uniformly bounded } \{d_n\}_{n=1}^m \in \mathcal{D}(\{\mathcal{F}_n\}, X) \\ \left\| \sum_{n=1}^m v_n d_n \right\|_{L_p(\Omega,Y)} \le C_p \left\| \sum_{n=1}^m d_n \right\|_{L_p(\Omega,X)} . \end{gathered} \tag{4.2}$$

Then

$$R_p(\{v_n(\omega) : n \in \mathbb{N}\}) \le C_p$$

for each $\omega \in \Omega$.

Remark 4.2. Condition (4.2) can be replaced by the (apparently) weaker Condition (4.2′).
Condition (4.2′):

$$\text{there exists } \tau > 1 \text{ so that}$$

$$\text{for each uniformly bounded } \{d_n\}_{n=1}^m \in \mathcal{D}(\{\mathcal{F}_n\}, \mathbb{R}) \text{ satisfying that}$$

$$\begin{gathered} \text{if } \{z_n\}_{n=1}^m \text{ is from any Banach space } Z \\ \text{then } \{z_n d_n\}_{n=1}^m \text{ is } \tau\text{-unconditional in } L_p(\Omega, Z) \end{gathered} \tag{4.3}$$

$$\text{one has that}$$

$$\left\| \sum_{n=1}^m v_n x_n d_n \right\|_{L_p(\Omega,Y)} \le C_p \left\| \sum_{n=1}^m x_n d_n \right\|_{L_p(\Omega,X)} \tag{4.4}$$

$$\text{for each choice } \{x_n\}_{n=1}^m \text{ from } X.$$

Thus Condition (4.2′) reduces, from Condition (4.2), the class of martingale difference sequences that one must test.

The proof of Theorem 4.1 uses the following lemma, whose long technical proof is in Section 6.

Lemma 4.3. *Let* $(\Omega, \mathcal{F}, \mu)$ *be a probability space with a filtration* $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ *satisfying* (4.1).

Let A *be an atom of some* $\mathcal{F}_j$. *Let* $p \in [1, \infty)$ *and* $\tau, \tau_1 > 1$ *and* $\varepsilon > 0$ *and* $m \in \mathbb{N}$. *Then there exists a uniformly bounded* $\mathbb{R}$*-valued martingale difference sequence* $\{d_n\}_{n=1}^m$ *with respect to some subfiltration* $\{\mathcal{F}_{j_n}\}_{n=1}^m$ *such that*

(1) $j_1 > j$

(2) $\operatorname{supp} d_n \subset A$ *for each* $n \in \{1, \ldots, m\}$

(3) $1 \le |d_n(\omega)| \le \tau_1$ *for each* $n \in \{1, \ldots, m\}$ *and* $\omega \in G_m := \operatorname{supp} d_m$

and furthermore, for any choice $\{z_n\}_{n=1}^m$ *from any Banach space* Z*,*

(4) $\{z_n d_n\}_{n=1}^m$ *is* τ*-unconditional in* $L_p(\Omega, Z)$

(5) *if* $z_{n_0} \neq 0$*, then*

$$\int_A \left\| \sum_{n=1}^m d_n(\omega) z_n r_n \right\|_{L_p([0,1],Z)}^p d\mu(\omega) \ \le\ [1 + \varepsilon M^p] \int_{G_m} \left\| \sum_{n=1}^m d_n(\omega) z_n r_n \right\|_{L_p([0,1],Z)}^p d\mu(\omega)$$

where $M \, \|z_{n_0}\|_Z \ = \ \tau_1 \sum_{n=1}^m \|z_n\|_Z$.

Proof of Theorem 4.1. Assume condition (4.2′) of Remark 4.2 holds (thus giving $\tau > 1$) (but not that condition (4.2) necessarily holds).

Fix $u \in \Omega$. Fix $m \in \mathbb{N}$ and $\{x_n\}_{n=1}^m$ from X. Let $\tau_1, \tau_2 > 1$. It suffices to show

$$\left\| \sum_{n=1}^m r_n v_n(u) x_n \right\|_{L_p([0,1],Y)} \ \le\ \tau_1 \, \tau_2 \, C_p \left\| \sum_{n=1}^m r_n x_n \right\|_{L_p([0,1],X)} . \tag{4.5}$$

Without loss of generality, there exists $n_0 \in \{1, \ldots m\}$ so that $x_{n_0} \neq 0$.

Find the atom A of $\mathcal{F}_{m-1}$ so that $u \in A$. Note that

$$v_n(u) = v_n(\omega) \qquad \text{for each} \quad \omega \in A \ , \ n \in \{1, \ldots m\} \ . \tag{4.6}$$

Find $\varepsilon > 0$ so that

$$1 \ + \ \varepsilon \left[\frac{\tau_1 \sum_{n=1}^m \|x_n\|_X}{\|x_{n_0}\|_X} \right]^p \ < \ \tau_2^p \ . \tag{4.7}$$

Apply Lemma 4.3 (with $\mathcal{F}_j := \mathcal{F}_{m-1}$ and other notation consistent) to find the corresponding uniformly bounded $\{d_n\}_{n=1}^m \in \mathcal{D}(\{\mathcal{F}_n\}, \mathbb{R})$. Let $G_m := \operatorname{supp} d_m$. Note that $\{d_n\}_{n=1}^m$ satisfies condition (4.3) and so (4.4) holds for the choice $\{r_n(t) x_n\}_{n=1}^m$ from X for each fixed $t \in [0,1]$. By Kahane's Contraction Principle (Fact 2.3), for each fixed $w \in G_m$,

$$\left\| \sum_{n=1}^m d_n(\omega) x_n r_n \right\|_{L_p([0,1],X)} \ \le\ \tau_1 \left\| \sum_{n=1}^m x_n r_n \right\|_{L_p([0,1],X)} \tag{4.8}$$

$$\left\| \sum_{n=1}^m v_n(u) x_n r_n \right\|_{L_p([0,1],Y)} \ \le\ \left\| \sum_{n=1}^m d_n(\omega) v_n(u) x_n r_n \right\|_{L_p([0,1],Y)} \tag{4.9}$$

by (3) of Lemma 4.3. Thus

$$\left\| \sum_{n=1}^m r_n v_n(u) x_n \right\|_{L_p([0,1],Y)}^p \ \le\ \int_{G_m} \left\| \sum_{n=1}^m d_n(\omega) r_n v_n(u) x_n \right\|_{L_p([0,1],Y)}^p \frac{d\mu(\omega)}{\mu(G_m)}$$

$$
\begin{aligned}
&\le \int\limits_{[0,1]}\int\limits_{\Omega}\left\|\sum_{n=1}^{m} v_n(\omega)\left[r_n(t)\,x_n\right] d_n(\omega)\right\|_Y^p \frac{d\mu(\omega)}{\mu(G_m)}\,dt \\
&\le C_p^p \int\limits_{[0,1]}\int\limits_{\Omega}\left\|\sum_{n=1}^{m} r_n(t)\,x_n d_n(\omega)\right\|_X^p \frac{d\mu(\omega)}{\mu(G_m)}\,dt \\
&= C_p^p \int\limits_{A}\left\|\sum_{n=1}^{m} d_n(\omega)\left[x_n r_n\right]\right\|_{L_p([0,1],X)}^p \frac{d\mu(\omega)}{\mu(G_m)} \\
&\le C_p^p\ \tau_2^p \int\limits_{G_m}\left\|\sum_{n=1}^{m} d_n(\omega)\left[x_n r_n\right]\right\|_{L_p([0,1],X)}^p \frac{d\mu(\omega)}{\mu(G_m)} \\
&\le C_p^p\ \tau_2^p\ \tau_1^p \left\|\sum_{n=1}^{m} x_n r_n\right\|_{L_p([0,1],X)}^p
\end{aligned}
$$

where the *inequalities* (in order) follow from: (4.9), the monotonicity of the integral for nonnegative functions and (4.6), (4.4), Lemma 4.3 and (4.7), and (4.8). So (4.5) holds. □

The next example shows that the condition

$$\lim_{n\to\infty} \sup\{\mu(B)\colon B \text{ is an atom of } \mathcal{F}_n\} = 0$$

of (4.1) in Theorem 4.1 is necessary.

Example 4.4. Consider any filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ on $([0,1],\mathcal{M},m)$ satisfying that $\left(\frac{1}{2},1\right]$ is an atom of $\mathcal{F}_n$ for each $n\in\mathbb{N}_0$. Let $v_n\colon \Omega\to\mathcal{B}(X,X)$ have the form

$$v_n(\omega) = \begin{cases} T_n & \text{if } \omega\in\left(\frac{1}{2},1\right] \\ 0 & \text{if } \omega\in\left[0,\frac{1}{2}\right] \end{cases}.$$

Any $\{d_n\}_{n=1}^m \in \mathcal{D}(\{\mathcal{F}_n\},X)$ satisfies $d_n(\omega)=0$ if $\omega\in\left(\frac{1}{2},1\right]$ for $n>1$. So (4.2) holds. But if X is not Hilbertian, then there is a non-R-bounded set $\{T_n\}_{n\in\mathbb{N}}$ in $\mathcal{B}(X,X)$.

5. COROLLARIES TO THE MAIN RESULTS

As in the scalar case, boundedness of operator-valued martingale transforms in one sense is equivalent to other notions of boundedness. To be precise, for a Z-valued martingale $f:=\{f_n\}_{n\in\mathbb{N}}$, define

$$
\begin{aligned}
\|f\|_{L_p(\Omega,Z)} &:= \sup_{n\in\mathbb{N}} \|f_n\|_{L_p(\Omega,Z)} && \text{for } 1\le p<\infty \\
f_n^*(\omega) &:= \sup_{1\le k\le n} \|f_k(\omega)\|_Z && \text{for } n\in\mathbb{N}
\end{aligned}
$$

$$f^*(\omega) := \sup_{n\in\mathbb{N}} \|f_n(\omega)\|_Z \qquad \text{Doob's maximal function .}$$

Let's keep with the notation in Definitions 2.1 and 2.2.

Fact 5.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space with filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$. Let $\{v_n\}_{n\in\mathbb{N}}$ be a $\mathcal{B}(X,Y)$-valued $\{\mathcal{F}_n\}$-multiplier sequence. Then conditions (1) through (5) are equivalent.*

(1) *For each (or equivalently, for some) $p \in (1,\infty)$ there exists $C_p \in \mathbb{R}$ so that*

$$\|(T_v f)_m\|_{L_p(\Omega,Y)} \;\le\; C_p \;\|f_m\|_{L_p(\Omega,X)}$$

for each $f := \{f_n\} \in \mathcal{M}(\{\mathcal{F}_n\}, X)$ and $m \in \mathbb{N}$.

(2) *For each (or equivalently, for some) $p \in (1,\infty)$ there exists $C_p \in \mathbb{R}$ so that*

$$\|(T_v f)^*\|_{L_p(\Omega,\mathbb{R})} \;\le\; C_p \;\|f\|_{L_p(\Omega,X)}$$

for each $f := \{f_n\} \in \mathcal{M}(\{\mathcal{F}_n\}, X)$.

(3) *For each $p \in [1,\infty)$ there exists $C_p \in \mathbb{R}$ so that*

$$\|(T_v f)^*\|_{L_p(\Omega,\mathbb{R})} \;\le\; C_p \;\|f^*\|_{L_p(\Omega,\mathbb{R})}$$

for each $f := \{f_n\} \in \mathcal{M}(\{\mathcal{F}_n\}, X)$.

(4) *There exists $C \in \mathbb{R}$ so that*

$$\lambda\,\mu\left[(T_v f)^* > \lambda\right] \;\le\; C \;\|f^*\|_{L_1(\Omega,\mathbb{R})}$$

for each $f := \{f_n\} \in \mathcal{M}(\{\mathcal{F}_n\}, X)$ and $\lambda > 0$.

(5) *There exists $C \in \mathbb{R}$ so that*

$$\lambda\,\mu\left[(T_v f)^* > \lambda\right] \;\le\; C \;\|f\|_{L_1(\Omega,X)}$$

for each $f := \{f_n\} \in \mathcal{M}(\{\mathcal{F}_n\}, X)$ and $\lambda > 0$.

If, furthermore, Y has the Radon-Nikodym property, then (3) implies (6).

(6) *For each $f := \{f_n\} \in \mathcal{M}(\{\mathcal{F}_n\}, X)$, if $\|f\|_{L_1(\Omega,X)}$ is finite then $(T_v f)$ converges a.e..*

Martínez and Torrea [27] showed the equivalence of (2) through (5) and the implication to (6) indicated above. Of course, that (1) implies (2) follows from standard techniques (such as those found in [27, Remark 1]) while that (2) implies (1) follows easily from the definitions.

Corollary 5.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space with filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$. Let X and Y be UMD spaces. Let $\{v_n\}_{n\in\mathbb{N}}$ be a $\mathcal{B}(X,Y)$-valued $\{\mathcal{F}_n\}$-multiplier sequence that satisfies, for some $C \in \mathbb{R}$,*

$$R_2(\{v_n(\omega) : n \in \mathbb{N}\}) \;\le\; C \qquad \text{for a.e. } \omega \in \Omega\ .$$

Then (1) *through* (6) *of Fact 5.1 hold* (*with the constants appearing depending also on the UMD constants of X and Y*).

Proof. Let $f := \{f_n\} \in \mathcal{M}(\{\mathcal{F}_n\}, X)$ and $m \in \mathbb{N}$. It follows from Theorem 3.2 that

$$\|(T_v f)_m\|_{L_2(\Omega,Y)} \leq \beta_2(X)\,\beta_2(Y)\,C\ \|f_m\|_{L_2(\Omega,X)} .$$

Now apply Fact 5.1. □

Remark 5.3. Martínez and Torrea [28] showed the equivalence of (1) in Fact 5.1 to the boundedness of the martingale transform on various Banach space valued BMO and Hardy spaces. Thus, similar to Corollary 5.2, a pointwise R-bounded $\mathcal{B}(X,Y)$-valued multiplier sequence $\{v_n\}_{n\in\mathbb{N}}$, where X and Y are UMD spaces, yields bounded martingale transform operators between BMO and Hardy spaces.

Burkholder [6] showed that if X is a UMD space then (1.2) holds, with the same constant $\beta_p(X)$, if one replaces the choices $\{\varepsilon_n\}_{n\in\mathbb{N}}$ of signs by $[-1,1]$-valued $\{\mathcal{F}_n\}$-multiplier sequences $\{v_n\}_{n\in\mathbb{N}}$. A similar result is true for operator-valued multiplier sequences.

Corollary 5.4. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space with filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ and $p \in [1,\infty)$. Assume that there is $\tau_p(\{\mathcal{F}_n\}, X) \in \mathbb{R}$ so that for each $\{d_n\}_{n=1}^m \in \mathcal{D}(\{\mathcal{F}_n\}, X)$*

$$\left\|\sum_{n=1}^m \varepsilon_n d_n\right\|_{L_p(\Omega,X)} \leq \tau_p(\{\mathcal{F}_n\}, X)\ \left\|\sum_{n=1}^m d_n\right\|_{L_p(\Omega,X)}$$

for each choice $\{\varepsilon_n\}_{n=1}^m$ of signs from $\{\pm 1\}$.

If a $\mathcal{B}(X,X)$-valued $\{\mathcal{F}_n\}$-multiplier sequence $\{v_n\}_{n\in\mathbb{N}}$ satisfies, for some $C_p \in \mathbb{R}$,

$$R_p(\{v_n(\omega) : n \in \mathbb{N}\}) \leq C_p \qquad \text{for a.e. } \omega \in \Omega$$

then

$$\left\|\sum_{n=1}^m v_n d_n\right\|_{L_p(\Omega,X)} \leq [\tau_p(\{\mathcal{F}_n\}, X)]^2\ C_p\ \left\|\sum_{n=1}^m d_n\right\|_{L_p(\Omega,X)}$$

for each $\{d_n\}_{n=1}^m \in \mathcal{D}(\{\mathcal{F}_n\}, X)$.

Note that if $\{v_n\}_{n\in\mathbb{N}}$ is a $[-1,1]$-valued $\{\mathcal{F}_n\}$-multiplier sequence then $\{v_n 1_X\}_{n\in\mathbb{N}}$ is a $\mathcal{B}(X,X)$-valued $\{\mathcal{F}_n\}$-multiplier sequence and $R_p(\{v_n(\omega)\,1_X : n \in \mathbb{N}\}) = \sup\{|v_n(\omega)| : n \in \mathbb{N}\}$.

Proof. The result follows directly from Theorem 3.2. □

This section closes with a special case of Theorem 4.1: note that here one must only test condition (4.2) for translated filtration rather than for all subfiltration.

Proposition 5.5. *Consider the Lebesgue measure space* $([0,1],\mathcal{M},m)$ *along with its dyadic filtration* $\{\mathcal{D}_n\}_{n\in\mathbb{N}_0}$. *Let* $p\in[1,\infty)$.

Let $\{v_n\}_{n\in\mathbb{N}}$ *be a* $\mathcal{B}(X,Y)$*-valued* $\{\mathcal{D}_n\}$*-multiplier sequence that satisfies, for some* $C_p\in\mathbb{R}$,

$$\begin{gathered}\text{for each } X\text{-valued finite martingale difference sequence } \{d_n\}_{n=1}^m \\ \text{with respect to } \{\mathcal{D}_{m-1+n}\}_{n=1}^m \\ \text{of the form } d_n = x_n r_{m-1+n} 1_{I_k^{m-1}} \\ \text{for some } \{x_n\}_{n=1}^m \subset B(X) \text{ and } k\in\left\{1,\ldots,2^{m-1}\right\} \\ \left\|\sum_{n=1}^m v_n d_n\right\|_{L_p([0,1],Y)} \ \le\ C_p \left\|\sum_{n=1}^m d_n\right\|_{L_p([0,1],X)} .\end{gathered} \tag{5.1}$$

Then

$$R_p\left(\{v_n(u) : n\in\mathbb{N}\}\right) \le C_p$$

for each $u\in[0,1]$.

Note that any martingale difference sequence of the above form is 1-unconditional in $L_p([0,1],X)$.

Proof. Fix $u\in[0,1]$.

Fix $m\in\mathbb{N}$ and $\{x_n\}_{n=1}^m$ from $B(X)$. Find $k\in\left\{1,\ldots,2^{m-1}\right\}$ so that $u\in I_k^{m-1}$. Note that, for $n\in\{1,\ldots,m\}$, each v_n is constant on I_k^{m-1}. Thus, by changes of variables and (5.1),

$$\begin{aligned}\int_{[0,1]} \left\|\sum_{n=1}^m r_n(t)\, v_n(u)\, x_n\right\|_Y^p dt &= 2^{m-1} \int_{I_k^{m-1}} \left\|\sum_{n=1}^m v_n(u)\left[x_n r_n\left(2^{m-1}t-k+1\right)\right]\right\|_Y^p dt \\ &= 2^{m-1} \int_{[0,1]} \left\|\sum_{n=1}^m v_n(t)\left[x_n r_{m-1+n}(t)\, 1_{I_k^{m-1}}(t)\right]\right\|_Y^p dt \\ &\le 2^{m-1}\, C_p^p \int_{[0,1]} \left\|\sum_{n=1}^m x_n r_{m-1+n}(t)\, 1_{I_k^{m-1}}(t)\right\|_X^p dt \\ &= 2^{m-1}\, C_p^p \int_{I_k^{m-1}} \left\|\sum_{n=1}^m x_n r_n\left(2^{m-1}t-k+1\right)\right\|_X^p dt \\ &= C_p^p \int_{[0,1]} \left\|\sum_{n=1}^m x_n r_n(t)\right\|_X^p dt .\end{aligned}$$

Thus $R_p\left(\{v_n(u) : n\in\mathbb{N}\}\right) \le C_p$. □

6. Proof of Lemma 4.3

A tree-structured sequence $\{\Gamma_n^*\}_{n\in\mathbb{N}_0}$ of indexing sets is needed. Let $\Gamma_0^* := \{\emptyset\}$ and, for $n \in \mathbb{N}$,

$$\Gamma_n^* = ((0, \pm 1) \times \mathbb{N}_0)^n \ .$$

There is a natural identification of Γ_n^* with $\Gamma_{n-1}^* \times ((0, \pm 1) \times \mathbb{N}_0)$ and so one can express Γ_n^* as

$$\begin{aligned} \Gamma_n^* &= \{((\delta_1, k_1), \dots (\delta_n, k_n)) : \delta_j \in \{0 \pm 1\} \text{ and } k_j \in \mathbb{N}_0 \text{ for each } j \in \{1, \dots, n\}\} \\ &= \left\{(\gamma, (\delta, k)) : \gamma \in \Gamma_{n-1}^* \ , \ \delta \in \{0, \pm 1\} \ , \ k \in \mathbb{N}_0\right\} \end{aligned}$$

for $n \in \mathbb{N}$. The notation

$$A \uplus B \ = \ C$$

indicates that C is the *disjoint* union of A and B.

Lemma 6.1. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space with a filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ satisfying (4.1). Let A be an atom of some $\mathcal{F}_j$ and $0 < \delta < \frac{1}{2}$. Let $n \in \mathbb{N}$ satisfy*

$$\sup\{\mu(B) : B \subset A \ , \ B \text{ is an atom of } \mathcal{F}_n\} \ < \ \delta\mu(A) \ . \tag{6.1}$$

(Note $n > j$). Then there exists A_1 and A_{-1} in $\mathcal{F}_n$ so that, for $\varepsilon = \pm 1$,

1. $A_1 \uplus A_{-1} \subset A$
2. A_ε *is a* FINITE *union of atoms of* $\mathcal{F}_n$
3. $\frac{1}{2} - \delta \ < \ \frac{\mu(A_\varepsilon)}{\mu(A)} \ < \ \frac{1}{2}$

and so

4. $(1 - 2\delta)\,\mu(A) \ < \ \mu(A_1 \cup A_{-1})$
5. $1 \ < \ \frac{\mu(A)}{2\mu(A_\varepsilon)} \ < \ \frac{1}{1-2\delta}$
6. $\left|\frac{\varepsilon\mu(A)}{2\mu(A_\varepsilon)} - \varepsilon\right| \ < \ \frac{2\delta}{1-2\delta}$.

Proof. One can express A as

$$A = \biguplus_{k=1}^{m} A_{(0,k)}$$

where the $A_{(0,k)}$'s are (disjoint) atoms of $\mathcal{F}_n$ and $m \in \mathbb{N} \cup \{\infty\}$. Note that

$$\frac{\mu\left(A_{(0,k)}\right)}{\mu(A)} \ < \ \delta \ < \ \frac{1}{2}$$

and so $m \geq 3$.

So there exists $l_1 \in \mathbb{N}$ (with $1 + l_1 < m$) so that

$$\sum_{k=1}^{l_1} \frac{\mu\left(A_{(0,k)}\right)}{\mu\left(A\right)} \;<\; \frac{1}{2} \;\leq\; \sum_{k=1}^{1+l_1} \frac{\mu\left(A_{(0,k)}\right)}{\mu\left(A\right)} \;.$$

Let

$$A_1 \;:=\; \bigcup_{k=1}^{l_1} A_{(0,k)} \;.$$

Note

$$\frac{1}{2} - \delta \;<\; \frac{1}{2} \;-\; \left[\sum_{k=1}^{1+l_1} \frac{\mu\left(A_{(0,k)}\right)}{\mu\left(A\right)} \;-\; \sum_{k=1}^{l_1} \frac{\mu\left(A_{(0,k)}\right)}{\mu\left(A\right)}\right] \;\leq\; \sum_{k=1}^{l_1} \frac{\mu\left(A_{(0,k)}\right)}{\mu\left(A\right)} \;=\; \frac{\mu\left(A_1\right)}{\mu\left(A\right)} \;<\; \frac{1}{2}$$

and

$$\frac{1}{2} \;<\; \sum_{k=1+l_1}^{m} \frac{\mu\left(A_{(0,k)}\right)}{\mu\left(A\right)} \;=\; 1 \;-\; \sum_{k=1}^{l_1} \frac{\mu\left(A_{(0,k)}\right)}{\mu\left(A\right)} \;<\; \frac{1}{2} \;+\; \delta \;.$$

So there exists $l_{-1} \in \mathbb{N}$ (with $1 + l_{-1} + l_1 \leq m$) so that

$$\sum_{k=1+l_1}^{l_{-1}+l_1} \frac{\mu\left(A_{(0,k)}\right)}{\mu\left(A\right)} \;<\; \frac{1}{2} \;\leq\; \sum_{k=1+l_1}^{1+l_{-1}+l_1} \frac{\mu\left(A_{(0,k)}\right)}{\mu\left(A\right)} \;.$$

Let

$$A_{-1} \;:=\; \bigcup_{k=1}^{l_{-1}} A_{(0,k+l_1)} \;.$$

Note

$$\frac{1}{2} - \delta \;<\; \frac{1}{2} \;-\; \left[\sum_{k=1+l_1}^{1+l_{-1}+l_1} \frac{\mu\left(A_{(0,k)}\right)}{\mu\left(A\right)} \;-\; \sum_{k=1+l_1}^{l_{-1}+l_1} \frac{\mu\left(A_{(0,k)}\right)}{\mu\left(A\right)}\right] \;\leq\; \sum_{k=1+l_1}^{l_{-1}+l_1} \frac{\mu\left(A_{(0,k)}\right)}{\mu\left(A\right)} \;=\; \frac{\mu\left(A_{-1}\right)}{\mu\left(A\right)} \;<\; \frac{1}{2} \;.$$

Thus (1), (2), and (3) hold, from which (4), (5), and (6) follow easily. □

The ultimate goal of (the long) Lemma 6.2 is to find the functions mentioned in Remark 6.3 along with some sets $\{G_n\}_{n\in\mathbb{N}}$, all of which satisfy conditions (F7) through (F11) of Lemma 6.2.

Lemma 6.2. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space with a filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}_0}$ satisfying* (4.1). *Let A be an atom of $\mathcal{F}_j$ and $A^o \in \mathcal{M}$ be so that $\mu(A) = m(A^o)$. Let $\{\delta_n\}_{n\in\mathbb{N}}$ be a sequence from $\left(0, \frac{1}{2}\right)$.*

Then there exists, for $n \in \mathbb{N}_0$,

- (E1) *good sets $\Gamma_n^G \subset \Gamma_n^*$*
- (E2) *bad sets $\Gamma_n^B \subset \Gamma_n^*$*
- (E3) *subsets $\{l_\gamma\}_{\gamma\in\Gamma_n^G}$ of $\mathbb{N}$*
- (E4) *expansions of the good sets $\vec{\Gamma}_n \subset \Gamma_{n+1}^*$*
- (E5) *$j_n \in \mathbb{N}$*

(E6) *subsets* $\{A_\gamma\}_{\gamma\in\Gamma_n^G}$ *of* $\mathcal{F}_{j_n}$

(E7) *atoms* $\{A_\gamma\}_{\gamma\in\vec{\Gamma}_n}$ *of* $\mathcal{F}_{j_n}$

(E8) *subsets* $\left\{A^o_\gamma\right\}_{\gamma\in\Gamma_n^G\cup\vec{\Gamma}_n}$ *and* $\left\{B^o_\gamma\right\}_{\gamma\in\Gamma_n^G\cup\Gamma_n^B}$ *of* $\mathcal{M}$

where the items in (E1) through (E8) corresponding to $n=0$ *are*

(Z1) $\Gamma_0^G=\{\emptyset\}$

(Z2) $\Gamma_0^B=\emptyset$

(Z3) $l_\gamma=1$ *for* $\gamma\in\Gamma_0^G$

(Z4) $\vec{\Gamma}_0=\{(\emptyset,(0,1))\}\subset\Gamma_1^*$

(Z5) $j_0=j$

(Z6) *if* $\gamma\in\Gamma_0^G$, *then* $A_\gamma=\emptyset$ *and* $A^o_\gamma=\emptyset=B^o_\gamma$

(Z7) *if* $\gamma\in\vec{\Gamma}_0$, *then* $A_\gamma=A$ *and* $A^o_\gamma=A^o$

and the indexing sets take the form, for $n\in\mathbb{N}$,

(I1) $\Gamma_n^G=\left\{(\gamma,(\varepsilon,k))\in\Gamma_n^*\colon \gamma\in\Gamma_{n-1}^G\ ,\ \varepsilon=\pm1\ ,\ 1\le k\le l_\gamma\right\}$

(I2) $\Gamma_n^B=\left\{(\gamma,(\varepsilon,0))\in\Gamma_n^*\colon \gamma\in\Gamma_{n-1}^G\cup\Gamma_{n-1}^B\ ,\ \varepsilon=\pm1\right\}$

(the zero is a notationally convenient way to ensure $\Gamma_n^G\cap\Gamma_n^B=\emptyset$*)*

(I3) $\vec{\Gamma}_n=\left\{(\gamma,(0,l))\in\Gamma_{n+1}^*\colon \gamma\in\Gamma_n^G\ ,\ 1\le l\le l_\gamma\right\}$ *(which also holds for* $n=0$*)*

and so one can write

(I3′) $\vec{\Gamma}_n=\left\{(\gamma,(\varepsilon,k),(0,l))\in\Gamma_{n+1}^*\colon \gamma\in\Gamma_{n-1}^G\ ,\ \varepsilon=\pm1\ ,\ 1\le k\le l_\gamma\ ,\ 1\le l\le l_{(\gamma,(\varepsilon,k))}\right\}$

and it easily follows that

(I1′) $\Gamma_n^G=\left\{(\gamma,(\varepsilon,k))\in\Gamma_n^*\colon \varepsilon=\pm1\ ,\ (\gamma\ ,\ (0,k))\in\vec{\Gamma}_{n-1}\right\}$

so that, for $n\in\mathbb{N}$,

(C0) Γ_n^G, Γ_n^B, *and* $\vec{\Gamma}_n$ *each have finitely many elements*

(C1) $j_n>j_{n-1}$

(C2) *if* $\gamma_1,\gamma_2\in\vec{\Gamma}_n$ *and* $\gamma_1\ne\gamma_2$, *then* $A_{\gamma_1}\cap A_{\gamma_2}=\emptyset$ *and* $A^o_{\gamma_1}\cap A^o_{\gamma_2}=\emptyset$

(C3) *if* $\gamma\in\Gamma_n^G$ *then*

$$A_\gamma=\bigcup_{(\gamma,(0,l))\in\vec{\Gamma}_n}A_{(\gamma,(0,l))}\qquad\textit{and}\qquad A^o_\gamma=\bigcup_{(\gamma,(0,l))\in\vec{\Gamma}_n}A^o_{(\gamma,(0,l))}$$

(C4) *if* $(\gamma,(\pm1,k))\in\Gamma_n^G$ *then for* $\varepsilon=\pm1$

$$A_{(\gamma,(1,k))}\uplus A_{(\gamma,(-1,k))}\subset A_{(\gamma,(0,k))}\qquad\textit{and}\qquad \frac{1}{2}-\delta_n\ <\ \frac{\mu\left(A_{(\gamma,(\varepsilon,k))}\right)}{\mu\left(A_{(\gamma,(0,k))}\right)}\ <\ \frac{1}{2}$$

(C5) *if* $\gamma\in\vec{\Gamma}_n$ *then* $m(A^o_\gamma)=\mu(A_\gamma)$

(C6) *if* $(\gamma,(\pm 1,k)) \in \Gamma_n^G$ *then for* $\varepsilon = \pm 1$

$$\left[A^o_{(\gamma,(1,k))} \uplus A^o_{(\gamma,(-1,k))}\right] \uplus \left[B^o_{(\gamma,(1,k))} \uplus B^o_{(\gamma,(-1,k))}\right] \;=\; A^o_{(\gamma,(0,k))}$$

$$m\left(A^o_{(\gamma,(\varepsilon,k))} \cup B^o_{(\gamma,(\varepsilon,k))}\right) \;=\; \frac{1}{2} m\left(A^o_{(\gamma,(0,k))}\right)$$

(C7) *if* $(\gamma,(\pm 1,0)) \in \Gamma_n^B$ *then*

$$B^o_{(\gamma,(1,0))} \uplus B^o_{(\gamma,(-1,0))} = B^o_\gamma \qquad \text{and} \qquad m\left(B^o_{(\gamma,(1,0))}\right) = m\left(B^o_{(\gamma,(-1,0))}\right)$$

(C8) *the family* $\mathcal{M}_n := \left\{A^o_\gamma\right\}_{\gamma\in\Gamma_n^G} \cup \left\{B^o_\gamma\right\}_{\gamma\in\Gamma_n^G\cup\Gamma_n^B}$ *is pairwise disjoint*

(C9) $\left[\cup_{\gamma\in\Gamma_n^G} A^o_\gamma\right] \cup \left[\cup_{\gamma\in\Gamma_n^G\cup\Gamma_n^B} B^o_\gamma\right] = A^o$.

Furthermore, if for $n \in \mathbb{N}$ *one defines*

(D1) $d_n \;:=\; \sum_{(\gamma,(\varepsilon,k))\in\Gamma_n^G} \frac{\varepsilon\ \mu\left(A_{(\gamma,(0,k))}\right)}{2\ \mu\left(A_{(\gamma,(\varepsilon,k))}\right)}\ 1_{A_{(\gamma,(\varepsilon,k))}} : \Omega \to \mathbb{R}$

(D2) $d_n^o \;:=\; \sum_{(\gamma,(\varepsilon,k))\in\Gamma_n^G} \frac{\varepsilon\ m\left(A^o_{(\gamma,(0,k))}\right)}{2\ m\left(A^o_{(\gamma,(\varepsilon,k))}\right)}\ 1_{A^o_{(\gamma,(\varepsilon,k))}} : [0,1] \to \mathbb{R}$

(D3) $\widetilde{d}_n^o \;:=\; \sum_{(\gamma,(\varepsilon,k))\in\Gamma_n^G} \varepsilon\ 1_{A^o_{(\gamma,(\varepsilon,k))}\cup B^o_{(\gamma,(\varepsilon,k))}} \;+\; \sum_{(\gamma,(\varepsilon,0))\in\Gamma_n^B} \varepsilon\ 1_{B^o_{(\gamma,(\varepsilon,0))}} : [0,1] \to \mathbb{R}$

(D4) $\mathcal{F}^o_{j_n} := \sigma\left(\left\{A^o_\gamma \colon \gamma \in \vec{\Gamma}_n\right\}\right)$ *and* $\mathcal{F}^o_{j_0} := \sigma\left(\left\{A^o_\gamma \colon \gamma \in \vec{\Gamma}_0\right\}\right)$

(D5) $G_n \;:=\; \cup_{\gamma\in\Gamma_n^G} A_\gamma$

then for each $n \in \mathbb{N}$

(F1) d_n *is* $\mathcal{F}_{j_n}$*-measurable*

(F2) *if* B *is an atom of* $\mathcal{F}_{j_{n-1}}$, *then* $\int_B d_n\, d\mu = 0$

(F3) d_n^o *is* $\mathcal{F}^o_{j_n}$*-measurable*

(F4) *if* B^o *is an atom of* $\mathcal{F}^o_{j_{n-1}}$, *then* $\int_{B^o} d_n^o\, dm = 0$

(F5) $m\left[\widetilde{d}_n^o = 1\right] = \frac{m(A^o)}{2} = m\left[\widetilde{d}_n^o = -1\right]$

(F6) *for each choice* $\{\varepsilon_l\}_{l=1}^n$ *of signs there exists* $\widetilde{\Gamma}_n^G \subset \Gamma_n^G$ *and* $\widetilde{\Gamma}_n^B \subset \Gamma_n^B$ *such that*
- $\cap_{l=1}^n \left[\widetilde{d}_l^o = \varepsilon_l\right] = \left[\cup_{\gamma\in\widetilde{\Gamma}_n^G} A^o_\gamma\right] \cup \left[\cup_{\gamma\in\widetilde{\Gamma}_n^G\cup\widetilde{\Gamma}_n^B} B^o_\gamma\right]$
- $m\left(\cap_{l=1}^n \left[\widetilde{d}_l^o = \varepsilon_l\right]\right) \;=\; \left(\frac{1}{2}\right)^n\ m\left(A^o\right)$

(F7) $G_n \subset G_{n-1}$ *where* $G_0 := A$

(F8) $1 < |d_n(\omega)| < \frac{1}{1-2\delta_n}$ *if* $\omega \in G_n$ *and* $d_n(\omega) = 0$ *if* $\omega \in \Omega \setminus G_n$

(F9) $\mu(G_n) > \left[\ \prod_{k=1}^n (1-2\delta_k)\ \right]\ \mu(A)$

(F10) *if* $1 \le p < \infty$ *and* $\{z_n\}_{k=1}^n$ *are from any Banach space* Z *then*
$\left\|\sum_{k=1}^n z_k d_k\right\|_{L_p(\Omega,Z)} \;=\; \left\|\sum_{k=1}^n z_k d_k^o\right\|_{L_p([0,1],Z)}$

(F11) *if* $1 \le p < \infty$ *then*
$\left\|d_n^o - \widetilde{d}_n^o\right\|^p_{L_p([0,1],\mathbb{R})} < \left[\left(\frac{2\delta_n}{1-2\delta_n}\right)^p \;+\; \left(1 - \prod_{k=1}^n (1-2\delta_k)\right)\right]\ \mu(A)$.

Remark 6.3. Note that, in Lemma 6.2,

(1) $\{d_n\}_{n\in\mathbb{N}}$ is a martingale difference sequence with respect to the filtration $\{\mathcal{F}_{j_n}\}_{n\in\mathbb{N}}$
(2) $\{d_n^o\}_{n\in\mathbb{N}}$ is a martingale difference sequence with respect to the filtration $\left\{\mathcal{F}_{j_n}^o\right\}_{n\in\mathbb{N}}$
(3) $\left\{\widetilde{d}_n^o\ 1_{A^o}\right\}_{n\in\mathbb{N}}$ is an independent sequence of $\{\pm 1\}$-valued symmetric random variables on the probability space $\left(A^o, \{B\in\mathcal{M}\colon B\subset A^o\}, \frac{m(\cdot)}{m(A^o)}\right)$.

Proof of Lemma 6.2. Let the desired items in (E1) through (E8) corresponding to $n=0$ be as in (Z1) through (Z7). The proof now continues by induction on n.

Let $n=1$. Let

$$\begin{aligned}\Gamma_1^G &:= \{(\emptyset,(\varepsilon,1))\in\Gamma_1^*\colon \varepsilon=\pm1\}\equiv\{(\varepsilon,1)\in\Gamma_1^*\colon\varepsilon=\pm1\}\\ \Gamma_1^B &:= \{(\emptyset,(\varepsilon,0))\in\Gamma_1^*\colon \varepsilon=\pm1\}\equiv\{(\varepsilon,0)\in\Gamma_1^*\colon\varepsilon=\pm1\}\ .\end{aligned}$$

Recall $\vec{\Gamma}_0=\{(\emptyset,(0,1))\}\subset\Gamma_1^*$.

Since $A=A_{(\emptyset,(0,1))}$ is an atom of $\mathcal{F}_{j_0}$, there exists $j_1>j_0$ so that

$$\sup\left\{\mu(B) : B\subset A_{(\emptyset,(0,1))}\ ,\ B \text{ is an atom of } \mathcal{F}_{j_1}\right\}\ <\ \delta_1\ \mu\left(A_{(\emptyset,(0,1))}\right)\ .$$

By Lemma 6.1, there are sets $A_{(\emptyset,(\pm1,1))}\in\mathcal{F}_{j_1}$ so that

$$A_{(\emptyset,(1,1))}\uplus A_{(\emptyset,(-1,1))}\subset A_{(\emptyset,(0,1))} \tag{6.2}$$

and, for $\varepsilon=\pm1$, each $A_{(\emptyset,(\varepsilon,1))}$ is a finite union of atoms of $\mathcal{F}_{j_1}$, say

$$A_{(\emptyset,(\varepsilon,1))}\ =\ \biguplus_{l=1}^{l_{(\emptyset,(\varepsilon,1))}} A_{(\emptyset,(\varepsilon,1),(0,l))}\ , \tag{6.3}$$

and

$$\frac{1}{2}-\delta_1\ <\ \frac{\mu\left(A_{(\emptyset,(\varepsilon,1))}\right)}{\mu\left(A_{(\emptyset,(0,1))}\right)}\ <\ \frac{1}{2}\ . \tag{6.4}$$

Let

$$\vec{\Gamma}_1\ :=\ \left\{(\emptyset,(\varepsilon,1),(0,l))\in\Gamma_2^*\colon \varepsilon=\pm1, 1\le l\le l_{(\emptyset,(\varepsilon,1))}\right\}\ .$$

This completes the construction of the desired items in (E1) through (E7) that satisfy their conditions in (C0) through (C4).

Since $A_{(\emptyset,(0,1))}=A$ and $A^o_{(\emptyset,(0,1))}=A^o$ and $\mu(A)=m(A^o)$, by (6.2), for $\varepsilon=\pm1$, there exists $A^o_{(\emptyset,(\varepsilon,1))}\in\mathcal{M}$ so that

$$\begin{aligned}&A^o_{(\emptyset,(1,1))}\ \uplus\ A^o_{(\emptyset,(-1,1))}\subset A^o_{(\emptyset,(0,1))}\\ &m\left(A^o_{(\emptyset,(\varepsilon,1))}\right)\ =\ \mu\left(A_{(\emptyset,(\varepsilon,1))}\right)\ .\end{aligned} \tag{6.5}$$

So by (6.4), for $\varepsilon = \pm 1$,

$$\frac{1}{2} - \delta_1 \ < \ \frac{m\left(A^o_{(\emptyset,(\varepsilon,1))}\right)}{m\left(A^o_{(\emptyset,(0,1))}\right)} \ < \ \frac{1}{2} \ .$$

So, for $\varepsilon = \pm 1$, there exists $B^o_{(\emptyset,(\varepsilon,1))} \in \mathcal{M}$ so that (C6) holds. It follows from (6.3) and (6.5) that, for $\varepsilon = \pm 1$, there exists $A^o_{(\emptyset,(\varepsilon,1),(0,l))} \in \mathcal{M}$ so that

$$\begin{aligned} A^o_{(\emptyset,(\varepsilon,1))} \ &= \ \biguplus_{l=1}^{l_{(\emptyset,(\varepsilon,1))}} \ A^o_{(\emptyset,(\varepsilon,1),(0,l))} \\ m\left(A^o_{(\emptyset,(\varepsilon,1),(0,l))}\right) \ &= \ \mu\left(A_{(\emptyset,(\varepsilon,1),(0,l))}\right) \ . \end{aligned}$$

If $\gamma \in \Gamma_1^B$, let $B^o_\gamma = \emptyset$. This completes the construction of the desired items in (E8) that satisfy their conditions in (C2), (C3), and (C5) through (C9).

Note that

$$\begin{aligned} d_1 \ &= \ \frac{\mu\left(A_{(\emptyset,(0,1))}\right)}{2\,\mu\left(A_{(\emptyset,(1,1))}\right)} 1_{A_{(\emptyset,(1,1))}} \ - \ \frac{\mu\left(A_{(\emptyset,(0,1))}\right)}{2\,\mu\left(A_{(\emptyset,(-1,1))}\right)} 1_{A_{(\emptyset,(-1,1))}} \\ d_1^o \ &= \ \frac{m\left(A^o_{(\emptyset,(0,1))}\right)}{2\,m\left(A^o_{(\emptyset,(1,1))}\right)} 1_{A^o_{(\emptyset,(1,1))}} \ - \ \frac{m\left(A^o_{(\emptyset,(0,1))}\right)}{2\,m\left(A^o_{(\emptyset,(-1,1))}\right)} 1_{A^o_{(\emptyset,(-1,1))}} \\ \widetilde{d}_1^o \ &= \ 1_{A^o_{(\emptyset,(1,1))} \cup B^o_{(\emptyset,(1,1))}} \ - \ 1_{A^o_{(\emptyset,(-1,1))} \cup B^o_{(\emptyset,(-1,1))}} \ . \end{aligned}$$

So, clearly, (F1) through (F7) along with (F10) hold. A quick look at Lemma 6.1 gives (F8) and (F9) and also that

$$\begin{aligned} \left\| d_1^o - \widetilde{d}_1^o \right\|^p_{L_p([0,1],\mathbb{R})} \ &< \ \sum_{\gamma \in \Gamma_n^G} \left(\frac{2\delta_1}{1 - 2\delta_1} \right)^p \ m\left(A^o_\gamma\right) \ + \ \sum_{\gamma \in \Gamma_n^G} m\left(B^o_\gamma\right) \\ &= \ \left(\frac{2\delta_1}{1 - 2\delta_1} \right)^p \ \mu\left(G_1\right) \ + \ \mu\left(A \setminus G_1\right) \ . \end{aligned}$$

So (F11) now follows from (F9).

This completes the $n = 1$ base step.

Fix $n \in \mathbb{N}$ with $n \geq 2$ and assume that the desired items in (E1) through (E8) have been found for $k \in \{0, 1, \ldots, n-1\}$. Let

$$\begin{aligned} \Gamma_n^G \ &:= \ \left\{ (\gamma, (\varepsilon, k)) \in \Gamma_n^* \colon \gamma \in \Gamma_{n-1}^G \ , \ \varepsilon = \pm 1 \ , \ 1 \leq k \leq l_\gamma \right\} \\ \Gamma_n^B \ &:= \ \left\{ (\gamma, (\varepsilon, 0)) \in \Gamma_n^* \colon \gamma \in \Gamma_{n-1}^G \cup \Gamma_{n-1}^B \ , \ \varepsilon = \pm 1 \right\} \ . \end{aligned}$$

If $(\gamma, (0, k)) \in \vec{\Gamma}_{n-1}$, then $A_{(\gamma,(0,k))}$ is an atom of $\mathcal{F}_{j_{n-1}}$; find $j_n > j_{n-1}$ so that

$$\sup\left\{ \mu\left(B\right) : B \subset A_{(\gamma,(0,k))} \ , \ B \text{ is an atom of } \mathcal{F}_{j_n} \right\} \ < \ \delta_n \ \mu\left(A_{(\gamma,(0,k))}\right)$$

for each $(\gamma, (0,k)) \in \vec{\Gamma}_{n-1}$.

Fix $(\gamma, (0,k)) \in \vec{\Gamma}_{n-1}$ (and so $\gamma \in \Gamma^G_{n-1}$ and $1 \le k \le l_\gamma$). By Lemma 6.1, there are sets $A_{(\gamma,(\pm 1,k))} \in \mathcal{F}_{j_n}$ so that

$$A_{(\gamma,(1,k))} \uplus A_{(\gamma,(-1,k))} \subset A_{(\gamma,(0,k))} \tag{6.6}$$

and, for $\varepsilon = \pm 1$, each $A_{(\gamma,(\varepsilon,k))}$ is a finite union of atoms of $\mathcal{F}_{j_n}$, say

$$A_{(\gamma,(\varepsilon,k))} \;=\; \biguplus_{l=1}^{l_{(\gamma,(\varepsilon,k))}} A_{(\gamma,(\varepsilon,k),(0,l))} \;, \tag{6.7}$$

and

$$\frac{1}{2} - \delta_n \;<\; \frac{\mu\left(A_{(\gamma,(\varepsilon,k))}\right)}{\mu\left(A_{(\gamma,(0,k))}\right)} \;<\; \frac{1}{2}\;. \tag{6.8}$$

Let

$$\vec{\Gamma}_n \;:=\; \left\{ (\gamma, (\varepsilon,k)\,, (0,l)) \in \Gamma^*_{n+1} \colon \gamma \in \Gamma^G_{n-1}\;,\; \varepsilon = \pm 1\;,\; 1 \le k \le l_\gamma\;,\; 1 \le l \le l_{(\gamma,(\varepsilon,k))} \right\}\;.$$

Towards (C2), note that for distinct elements $(\gamma_1, (\varepsilon_1, k_1)\,, (0, l_1))$ and $(\gamma_2, (\varepsilon_2, k_2)\,, (0, l_2))$ from $\vec{\Gamma}_n$

$$A_{(\gamma_1,(\varepsilon_1,k_1),(0,l_1))} \cap A_{(\gamma_2,(\varepsilon_2,k_2),(0,l_2))} \;=\; \emptyset\;; \tag{6.9}$$

indeed, it follows from (6.6) and (6.7) that, for $i \in \{1,2\}$,

$$A_{(\gamma_i,(\varepsilon_i,k_i),(0,l_i))} \;\subset\; A_{(\gamma_i,(\varepsilon_i,k_i))} \;\subset\; A_{(\gamma_i,(0,k_i))}$$

and so if $\gamma_1 \ne \gamma_2$ or $k_1 \ne k_2$ then (6.9) follows from the inductive hypothesis (specifically (C2)) while if $\gamma_1 = \gamma_2$ and $k_1 = k_2$ then (6.9) follows from (6.6) if $\varepsilon_1 \ne \varepsilon_2$ and from (6.7) if $\varepsilon_1 = \varepsilon_2$. This completes the construction of the desired items in (E1) through (E7) that satisfy their conditions in (C0) through (C4).

Towards (E8), fix $(\gamma, (0,k)) \in \vec{\Gamma}_{n-1}$. Thus $m\left(A^o_{(\gamma,(0,k))}\right) = \mu\left(A_{(\gamma,(0,k))}\right)$. By (6.6), for $\varepsilon = \pm 1$, there exists $A^o_{(\gamma,(\varepsilon,k))} \in \mathcal{M}$ so that

$$A^o_{(\gamma,(1,k))} \;\uplus\; A^o_{(\gamma,(-1,k))} \subset A^o_{(\gamma,(0,k))} \tag{6.6$'$}$$

$$m\left(A^o_{(\gamma,(\varepsilon,k))}\right) \;=\; \mu\left(A_{(\gamma,(\varepsilon,k))}\right)\;. \tag{6.10}$$

So by (6.8), for $\varepsilon = \pm 1$,

$$\frac{1}{2} - \delta_n \;<\; \frac{m\left(A^o_{(\gamma,(\varepsilon,k))}\right)}{m\left(A^o_{(\gamma,(0,k))}\right)} \;<\; \frac{1}{2}\;. \tag{6.8$'$}$$

So, for $\varepsilon = \pm 1$, there exists $B^o_{(\gamma,(\varepsilon,k))} \in \mathcal{M}$ so that (C6) holds. It follows from (6.7) and (6.10) that, for $\varepsilon = \pm 1$, there exists $A^o_{(\gamma,(\varepsilon,k),(0,l))} \in \mathcal{M}$ so that

$$A^o_{(\gamma,(\varepsilon,k))} \;=\; \biguplus_{l=1}^{l_{(\gamma,(\varepsilon,k))}} A^o_{(\gamma,(\varepsilon,k),(0,l))} \tag{6.7$'$}$$
$$m\left(A^o_{(\gamma,(\varepsilon,k),(0,l))}\right) \;=\; \mu\left(A_{(\gamma,(\varepsilon,k),(0,l))}\right) \;.$$

Fix $(\gamma,(\pm 1,0)) \in \Gamma^B_n$. Thus $\gamma \in \Gamma^G_{n-1} \cup \Gamma^B_{n-1}$. Find $B^o_{(\gamma,(\pm 1,0))} \in \mathcal{M}$ so that (C7) holds. This completes the construction of the items in (E8). Clearly, their conditions in (C3), (C5), (C6) and (C7) hold. As (C2) holds for the A_γ's follows from the inductive hypothesis, (6.6), and (6.7), that (C2) holds for the A^o_γ's follows from the inductive hypothesis, (6.6$'$), and (6.7$'$).

Towards (C9), note that by (C6) and (C3)

$$\begin{aligned}\bigcup_{\gamma\in\Gamma^G_n} \left(A^o_\gamma \cup B^o_\gamma\right) \;&=\; \bigcup_{(\gamma,(0,k))\in\vec{\Gamma}_{n-1}} \;\bigcup_{\varepsilon=\pm 1} \left[A^o_{(\gamma,(\varepsilon,k))} \cup B^o_{(\gamma,(\varepsilon,k))}\right] \\ &=\; \bigcup_{(\gamma,(0,k))\in\vec{\Gamma}_{n-1}} A^o_{(\gamma,(0,k))} \;=\; \bigcup_{\gamma\in\Gamma^G_{n-1}} \;\bigcup_{(\gamma,(0,k))\in\vec{\Gamma}_{n-1}} A^o_{(\gamma,(0,k))} \;=\; \bigcup_{\gamma\in\Gamma^G_{n-1}} A^o_\gamma\end{aligned}$$

and by (C7)

$$\begin{aligned}\bigcup_{\gamma\in\Gamma^B_n} B^o_\gamma \;&=\; \bigcup_{(\gamma,(1,0))\in\Gamma^B_n} \left[B^o_{(\gamma,(1,0))} \cup B^o_{(\gamma,(-1,0))}\right] \\ &=\; \bigcup_{(\gamma,(1,0))\in\Gamma^B_n} B^o_\gamma \;=\; \left[\bigcup_{\gamma\in\Gamma^G_{n-1}} B^o_\gamma\right] \cup \left[\bigcup_{\gamma\in\Gamma^B_{n-1}} B^o_\gamma\right] \;.\end{aligned}$$

So (C9) holds by the inductive hypothesis.

Now to show (C8). Note that the family

$$\mathcal{M}^1_n \;:=\; \left\{A^o_\gamma\right\}_{\gamma\in\Gamma^G_n} \cup \left\{B^o_\gamma\right\}_{\gamma\in\Gamma^G_n} \qquad \text{is pairwise disjoint.} \tag{6.11}$$

Indeed, if $\gamma \in \Gamma^G_n$ then $A^o_\gamma \cap B^o_\gamma = \emptyset$. So fix $\widetilde{\gamma}_i = (\gamma_i,(\varepsilon_i,k_i)) \in \Gamma^G_n$ with $\widetilde{\gamma}_1 \neq \widetilde{\gamma}_2$ and consider $C_{\gamma_i} \in \mathcal{M}^1_n$. If $\gamma_1 = \gamma_2$ and $k_1 = k_2$, then (6.11) follows from (C6). If $\gamma_1 \neq \gamma_2$ or $k_1 \neq k_2$, then (6.11) follows from (C2) since

$$C_{(\gamma_i,(\varepsilon_i,k_i))} \;\subset\; A^o_{(\gamma_i,(0,k_i))} \;.$$

and $(\gamma_i,(0,k_i)) \in \vec{\Gamma}_{n-1}$. Next note that the family

$$\mathcal{M}^2_n \;:=\; \left\{B^o_\gamma\right\}_{\gamma\in\Gamma^B_n} \qquad \text{is pairwise disjoint.} \tag{6.12}$$

Indeed, if $\widetilde{\gamma}_i = (\gamma_i, (\varepsilon_i, 0)) \in \Gamma_n^B$ then

$$B^o_{(\gamma_i,(\varepsilon_i,0))} \ \subset \ B^o_{\gamma_i} \ .$$

If $\gamma_1 = \gamma_2$, then (6.12) follows from (C7). If $\gamma_1 \neq \gamma_2$, then (6.12) follows by the inductive hypothesis (specifically, (C8)) since $\gamma_i \in \Gamma^G_{n-1} \cup \Gamma^B_{n-1}$. Now if

$$C_{(\gamma_1,(\varepsilon_1,k_1))} \in \mathcal{M}^1_n \qquad \text{and} \qquad B^o_{(\gamma_2,(\varepsilon_2,0))} \in \mathcal{M}^2_n$$

then $C_{(\gamma_1,(\varepsilon_1,k_1))} \subset A^o_{\gamma_1}$ with $\gamma_1 \in \Gamma^G_{n-1}$ and $B^o_{(\gamma_2,(\varepsilon_2,0))} \subset B^o_{\gamma_2}$ with $\gamma_2 \in \Gamma^G_{n-1} \cup \Gamma^B_{n-1}$. So by the inductive hypothesis on (C8), $C_{(\gamma_1,(\varepsilon_1,k_1))} \cap B^o_{(\gamma_2,(\varepsilon_2,0))} = \emptyset$. So (C8) holds.

Now to show that (F1) through (F11) hold. (F1) follows from (E6). Towards (F2), rewrite d_n as

$$d_n \ = \sum_{(\gamma,(0,k))\in\vec{\Gamma}_{n-1}} \left[\frac{\mu\left(A_{(\gamma,(0,k))}\right)}{2\,\mu\left(A_{(\gamma,(1,k))}\right)} 1_{A_{(\gamma,(1,k))}} \ - \ \frac{\mu\left(A_{(\gamma,(0,k))}\right)}{2\,\mu\left(A_{(\gamma,(-1,k))}\right)} 1_{A_{(\gamma,(-1,k))}} \right]$$

and note that, by (C4) and (E7)

$$A_{(\gamma,(1,k))} \ \uplus \ A_{(\gamma,(-1,k))} \ \subset A_{(\gamma,(0,k))} \ \in \ \left\{ B \colon B \text{ is an atom of } \mathcal{F}_{j_{n-1}} \right\} \ .$$

So (F2) holds. (F3) follows from (C3). Note that

$$d^o_n \ = \sum_{(\gamma,(0,k))\in\vec{\Gamma}_{n-1}} \left[\frac{m\left(A^o_{(\gamma,(0,k))}\right)}{2\,m\left(A^o_{(\gamma,(1,k))}\right)} 1_{A^o_{(\gamma,(1,k))}} \ - \ \frac{m\left(A^o_{(\gamma,(0,k))}\right)}{2\,m\left(A^o_{(\gamma,(-1,k))}\right)} 1_{A^o_{(\gamma,(-1,k))}} \right]$$

and by (C6) and the definition of $\mathcal{F}^o_{j_{n-1}}$

$$A^o_{(\gamma,(1,k))} \ \uplus \ A^o_{(\gamma,(-1,k))} \ \subset A^o_{(\gamma,(0,k))} \ \in \ \left\{ B \colon B \text{ is an atom of } \mathcal{F}^o_{j_{n-1}} \right\} \ .$$

So (F4) holds. Towards (F5), note that by (C8), (C9), (C6), and (C7)

$$\begin{aligned} m\left[\widetilde{d}^o_n = 1\right] \ &= \sum_{(\gamma,(1,k))\in\Gamma^G_n} m\left(A^o_{(\gamma,(1,k))} \cup B^o_{(\gamma,(1,k))}\right) \ + \sum_{(\gamma,(1,0))\in\Gamma^B_n} m\left(B^o_{(\gamma,(1,0))}\right) \\ &= \sum_{(\gamma,(-1,k))\in\Gamma^G_n} m\left(A^o_{(\gamma,(-1,k))} \cup B^o_{(\gamma,(-1,k))}\right) \ + \sum_{(\gamma,(-1,0))\in\Gamma^B_n} m\left(B^o_{(\gamma,(-1,0))}\right) \\ &= m\left[\widetilde{d}^o_n = -1\right] \ . \end{aligned}$$

So (F5) holds (again by using (C8) and (C9)).

Towards (F6), fix a choice $\{\varepsilon_l\}_{l=1}^n$ of signs. Find $\widetilde{\Gamma}^G_{n-1} \subset \Gamma^G_{n-1}$ and $\widetilde{\Gamma}^B_{n-1} \subset \Gamma^B_{n-1}$ such that

$$\bigcap_{l=1}^{n-1} \left[\widetilde{d}^o_l = \varepsilon_l\right] \ = \ \left[\bigcup_{\gamma\in\widetilde{\Gamma}^G_{n-1}} A^o_\gamma \right] \cup \left[\bigcup_{\gamma\in\widetilde{\Gamma}^G_{n-1}\cup\widetilde{\Gamma}^B_{n-1}} B^o_\gamma \right] \ .$$

Let

$$\begin{aligned}
\widetilde{\Gamma}_n^G &= \left\{(\gamma,(\varepsilon_n,k)) \in \Gamma_n^G \colon \gamma \in \widetilde{\Gamma}_{n-1}^G\right\} \\
\widetilde{\Gamma}_n^B &= \left\{(\gamma,(\varepsilon_n,0)) \in \Gamma_n^B \colon \gamma \in \widetilde{\Gamma}_{n-1}^G \cup \widetilde{\Gamma}_{n-1}^B\right\} .
\end{aligned}$$

It follows from (C3), (C6), and (C7) that

$$\begin{aligned}
\left[\widetilde{d}_n^o = \varepsilon_n\right] \cap \left(\bigcup_{\gamma\in\widetilde{\Gamma}_{n-1}^G} A_\gamma^o\right) &= \bigcup_{\gamma\in\widetilde{\Gamma}_n^G} \left(A_\gamma^o \cup B_\gamma^o\right) \\
\left[\widetilde{d}_n^o = \varepsilon_n\right] \cap \left(\bigcup_{\gamma\in\widetilde{\Gamma}_{n-1}^G\cup\widetilde{\Gamma}_{n-1}^B} B_\gamma^o\right) &= \bigcup_{\gamma\in\widetilde{\Gamma}_n^B} B_\gamma^o .
\end{aligned}$$

Thus

$$\bigcap_{l=1}^{n}\left[\widetilde{d}_l^o = \varepsilon_l\right] = \left(\bigcup_{\substack{(\gamma,(\varepsilon_n,k))\in\Gamma_n^G \\ \gamma\in\widetilde{\Gamma}_{n-1}^G}} \left(A^o_{(\gamma,(\varepsilon_n,k))} \cup B^o_{(\gamma,(\varepsilon_n,k))}\right)\right) \cup \left(\bigcup_{\substack{(\gamma,(\varepsilon_n,0))\in\Gamma_n^B \\ \gamma\in\widetilde{\Gamma}_{n-1}^G\cup\widetilde{\Gamma}_{n-1}^B}} B^o_{(\gamma,(\varepsilon_n,0))}\right) . \tag{6.13}$$

By (C6) and (C7), for the set on the right-hand side of (6.13), replacing ε_n by $-\varepsilon_n$ does not change its measure. So (F6) holds.

(F7) follows from (C3) and (C4) while (F8) follows from (C4).

Fix $\gamma \in \Gamma_{n-1}^G$ and $1 \le k \le l_\gamma$. So $(\gamma,(0,k)) \in \vec{\Gamma}_{n-1}$ and $(\gamma,(\pm1,k)) \in \Gamma_n^G$. It follows from (C4) that

$$(1-2\delta_n)\ \mu\left(A_{(\gamma,(0,k))}\right) < \mu\left(A_{(\gamma,(1,k))} \uplus A_{(\gamma,(-1,k))}\right) .$$

Taking the double sum $\sum_{\gamma\in\Gamma_{n-1}^G}\sum_{1\le k\le l_\gamma}$ of both sides gives (via (C3))

$$(1-2\delta_n)\ \mu\left(G_{n-1}\right) < \mu\left(G_n\right) .$$

So (F9) holds. (F10) is clear. Since

$$\begin{aligned}
\left\| d_n^o - \widetilde{d}_n^o \right\|^p_{L_p([0,1],\mathbb{R})} &< \sum_{\gamma\in\Gamma_n^G} \left(\frac{2\delta_n}{1-2\delta_n}\right)^p m\left(A_\gamma^o\right) + \sum_{\gamma\in\Gamma_n^G\cup\Gamma_n^B} m\left(B_\gamma^o\right) \\
&= \left(\frac{2\delta_n}{1-2\delta_n}\right)^p \mu\left(G_n\right) + \mu\left(A\setminus G_n\right) .
\end{aligned}$$

(F11) now follows from (F9). □

The next lemma follows easily from the Contraction Principle (Fact 2.3) and a standard [3, 24] perbutation argument. A proof is included for completeness sake.

Lemma 6.4. *Let $p \in [1, \infty)$ and $0 < \delta < 1$.*

Let $\{\widetilde{d}_n^o\}_{n=1}^m$ be a sequence of independent, symmetric, $\{\pm 1\}$-valued random variables on a probability space $(\Omega, \mathcal{F}, \mu)$ and $\{d_n^o\}_{n=1}^m$ be a sequence in $L_p(\Omega, \mathbb{R})$. If

$$\sum_{n=1}^m \left\| \widetilde{d}_n^o - d_n^o \right\|_{L_p(\Omega,\mathbb{R})} \leq \frac{\delta}{2}$$

then for any choice $\{x_n\}_{n=1}^m$ from a Banach space X, $\{x_n d_n^o\}_{n=1}^m$ is a $\left(\frac{1+\delta}{1-\delta}\right)$-unconditional sequence in $L_p(\Omega, X)$.

Proof. Fix $\{x_n\}_{n=1}^m$ from some Banach space X, a choice $\{\varepsilon_n\}_{n=1}^m$ of signs from $\{\pm 1\}$, and scalars $\{\lambda_n\}_{n=1}^m$. It needs to be shown that

$$\left\| \sum_{n=1}^m \varepsilon_n \, \lambda_n \, x_n \, d_n^o \right\|_{L_p(\Omega,X)} \leq \left(\frac{1+\delta}{1-\delta} \right) \left\| \sum_{n=1}^m \lambda_n \, x_n \, d_n^o \right\|_{L_p(\Omega,X)} . \tag{6.14}$$

Find $\{\widetilde{x}_n\}_{n=1}^m$ from $S(X)$ and $\{\widetilde{\lambda}_n\}_{n=1}^m$ from $\mathbb{R}$ so that $\lambda_n \, x_n = \widetilde{\lambda}_n \widetilde{x}_n$ for each $n \in \{1, \ldots, m\}$.

It follows from Fact 2.3 that $\{\widetilde{x}_n \widetilde{d}_n^o\}_{n=1}^m$ is a (normalized) 1-unconditional basic sequence in $L_p(\Omega, X)$. Since $\sum_{n=1}^m \left\| \widetilde{x}_n \widetilde{d}_n^o - \widetilde{x}_n d_n^o \right\|_{L_p(\Omega,X)} \leq \frac{\delta}{2}$, for any choice $\{\alpha_n\}_{n=1}^m$ of scalars

$$(1-\delta) \left\| \sum_{n=1}^m \alpha_n \widetilde{x}_n \widetilde{d}_n^o \right\|_{L_p(\Omega,X)} \leq \left\| \sum_{n=1}^m \alpha_n \widetilde{x}_n d_n^o \right\|_{L_p(\Omega,X)} \leq (1+\delta) \left\| \sum_{n=1}^m \alpha_n \widetilde{x}_n \widetilde{d}_n^o \right\|_{L_p(\Omega,X)}$$

(cf., eg., [26, Prop. I.1.a.9]) Thus $\{\widetilde{x}_n d_n^o\}_{n=1}^m$ is a $(\frac{1+\delta}{1-\delta})$-unconditional basic sequence. So (6.14) holds. □

Proof of Lemma 4.3. Let's keep with the notation in Lemma 6.2.

Pick $\{\delta_n\}_{n=1}^m$ so that

(a) $0 < \delta_n < \frac{1}{2}$ for each $n \in \{1, \ldots, m\}$

(b) $\frac{1}{1-2\delta_n} \leq \tau_1$ for each $n \in \{1, \ldots, m\}$

(c) $\sum_{n=1}^m \left[\left(\frac{2\delta_n}{1-2\delta_n} \right)^p + \left(1 - \prod_{k=1}^m (1-2\delta_k)\right) \right]^{\frac{1}{p}} \leq \frac{1}{2} \frac{\tau - 1}{\tau + 1}$

(d) $\prod_{n=1}^m (1-2\delta_n)^{-1} \leq 1 + \varepsilon$.

Apply Lemma 6.2 to find $\{d_n\}_{n=1}^m \in L_p(\Omega, \mathbb{R})$, along with everything else. Thus (1) of Lemma 4.3 holds.

Condition (b) above along with (F7) and (F8) of Lemma 6.2 imply (2) and (3) of Lemma 4.3. By (c) above and (F11) of Lemma 6.2,

$$\sum_{n=1}^m \left[\int_{A^o} \left| d_n^o - \widetilde{d}_n^o \right|^p \frac{dm}{m(A^o)} \right]^{\frac{1}{p}} = \sum_{n=1}^m \left\| d_n^o - \widetilde{d}_n^o \right\|_{L_p([0,1],\mathbb{R})} \left[\mu(A) \right]^{\frac{-1}{p}} \leq \frac{1}{2} \left(\frac{\tau - 1}{\tau + 1} \right) .$$

So (4) of Lemma 4.3 holds by Lemma 6.4, Remark 6.3, and (F10) of Lemma 6.2.

Towards (5) of Lemma 4.3, let $z_{n_0} \neq 0$. Then for each $\omega \in G_m$, by Kahane's Contraction Principle (Fact 2.3) and (3) of Lemma 4.3

$$\left\| \sum_{n=1}^{m} d_n(\omega)\, z_n r_n \right\|_{L_p([0,1],Z)} \geq \|d_{n_0}(\omega)\, z_{n_0} r_{n_0}\|_{L_p([0,1],Z)} \geq \|z_{n_0}\|_Z .$$

Thus, by (b) and (F8), along with (d) and (F9)

$$\begin{aligned}
\int_{A\setminus G_m} \|\sum_{n=1}^{m} d_n(\omega) z_n r_n\|^p_{L_p([0,1],Z)}\, d\mu(\omega) &\leq \int_{A\setminus G_m} \left[\sum_{n=1}^{m} \|d_n(\omega)\, z_n r_n\|_{L_\infty([0,1],Z)}\right]^p d\mu(\omega) \\
&\leq \int_{A\setminus G_m} \left[\sum_{n=1}^{m} \tau_1 \|z_n\|_Z\right]^p d\mu(\omega) \\
&= \frac{\mu(A\setminus G_m)}{\mu(G_m)} \left[\frac{\tau_1 \sum_{n=1}^{m} \|z_n\|_Z}{\|z_{n_0}\|_Z}\right]^p \mu(G_m)\ \|z_{n_0}\|_Z^p \\
&\leq \left[\frac{\mu(A)}{\mu(G_m)} - 1\right] M^p \int_{G_m} \left\|\sum_{n=1}^{m} d_n(\omega)\, z_n r_n\right\|^p_{L_p([0,1],Z)} d\mu(\omega) \\
&\leq \varepsilon\, M^p \int_{G_m} \left\|\sum_{n=1}^{m} d_n(\omega)\, z_n r_n\right\|^p_{L_p([0,1],Z)} d\mu(\omega) .
\end{aligned}$$

So (5) of Lemma 4.3 holds. □

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Department of Mathematics, University of South Carolina, Columbia, SC 29208, U.S.A.
E-mail address: girardi@math.sc.edu

Mathematisches Institut I, Universität Karlsruhe, Englerstrasse 2, 76128 Karlsruhe, Germany
E-mail address: Lutz.Weis@math.uni-karlsruhe.de