INTEGRAL OPERATORS WITH OPERATOR-VALUED KERNELS

MARIA GIRARDI AND LUTZ WEIS

ABSTRACT. Under fairly mild measurability and integrability conditions on operator-valued kernels, boundedness results for integral operators on Bochner spaces $L_p(X)$ are given. In particular, these results are applied to convolutions operators.

1. INTRODUCTION

One of the most commonly used boundedness criterion for integral operators states that, for $1 \le p \le \infty$ and σ-finite measure spaces (T, Σ_T, μ) and (S, Σ_S, ν), a measurable kernel $k \colon T \times S \to \mathbb{C}$ defines a bounded linear operator

$$K \colon L_p(S, \mathbb{C}) \to L_p(T, \mathbb{C}) \qquad \text{via} \qquad (Kf)(\cdot) := \int_S k(\cdot, s)\ f(s)\ d\nu(s)$$

provided

$$\sup_{s \in S} \int_T |k(t,s)|\ d\mu(t) \ \le\ C \qquad \text{and} \qquad \sup_{t \in T} \int_S |k(t,s)|\ d\nu(s) \ \le\ C \tag{1.1}$$

(see, e.g. [5, Theorem 6.18]). In the theory of evolution equations one frequently uses operator-valued analogs of this situation, where the kernel k maps $T \times S$ into the space $\mathcal{B}(X, Y)$ of bounded linear operators from a Banach space X into a Banach space Y and then one desires the boundedness of the corresponding integral operator

$$K \colon L_p(S, X) \to L_p(T, Y)\ .$$

Such integral operators appear, for example, in solution formulas for inhomogeneous Cauchy problems (see, e.g. [10]) and for Volterra integral equations (see, e.g. [11]) as well as in control theory (see, e.g. [2]); furthermore, the stability of such solutions is often expressed in terms of the boundedness of these operators.

However, difficulties can easily arise since in many situations the kernel k is not measurable with respect to the operator norm because the range of k is not (essentially) valued in a separable subspace of $\mathcal{B}(X, Y)$. This paper presents boundedness results for integral operators with operator-valued kernels under relatively mild measurability and integrability conditions on the kernels.

Date: 4 August 2003.

2000 *Mathematics Subject Classification.* Primary 45P05, 47G10, 46E40. Secondary 46B20.

Key words and phrases. integral operators on Bochner spaces, operator-valued kernels, convolution with operator-valued kernels.

The first step is to place a mild measurability condition on a kernel $k\colon T \times S \to \mathcal{B}(X,Y)$ to guarantee that if f is in the space $\mathcal{E}(S,X)$ of finitely-valued finitely-supported measurable functions then the Bochner integrals

$$(Kf)(\cdot) \;:=\; \int_S k(\cdot,s)\;[f(s)]\;d\nu(s) \tag{1.2}$$

define a measurable function from T into Y, thus defining a mapping

$$K\colon \mathcal{E}(S,X) \to L_0(T,Y)\;\;.$$

Then, to ensure that K linearly extends to a desired superspace, one adds integability conditions, which replace (1.1) in the scalar case and, roughly speaking, take the form

$$\sup_{s\in S} \int_T \|k(t,s)\,x\|_Y\;d\mu(t) \;\le\; C\,\|x\|_X \qquad \text{for each } x \in X \tag{1.3}$$

$$\sup_{t\in T} \int_S \|k^*(t,s)\,y^*\|_{X^*}\;d\nu(s) \le\; C\,\|y^*\|_{Y^*} \qquad \text{for each } y^* \in Y^* \tag{1.4}$$

along with appropriate measurability conditions (see Section 3 for the precise formulations). Assume k has the appropriate measurability conditions. Theorem 3.4 shows that if k satisfies (1.3) then K extends to a bounded linear operator from $L_1(S,X)$ into $L_1(T,Y)$; Theorem 3.6 shows that if k satisfies (1.4) then K extends to a bounded linear operator from the closure of $\mathcal{E}(S,X)$ in the L_∞-norm into $L_\infty(T,Y)$. Then Theorem 3.8 uses an interpolation argument to show that if k satisfies (1.3) and (1.4) then K extends to a bounded linear operator from $L_p(S,X)$ into $L_p(T,Y)$ for $1 < p < \infty$. The case $p = \infty$ is more delicate since $\mathcal{E}(S,X)$ is not necessarily dense in $L_\infty(S,X)$. Theorem 3.11 shows that if k satisfies (1.4) then K can be extended to a bounded linear operator from $L_\infty(S,X)$ into the space of w^*-measurable μ-essentially bounded functions from T into Y^{**} where the integrals in (1.2) exists (a.e) as Dunford integrals for each $f \in L_\infty(S,X)$; also, sufficient conditions are given to guarantee that K maps $L_\infty(S,X)$ into $L_\infty(T,Y)$. Using ideas from the Geometry of Banach Spaces, Example 3.13 shows that, without further assumptions, it is necessary to pass to Y^{**} in Theorem 3.11.

As an immediate consequence of these results, Corollary 5.1 gives boundedness results for convolution operators with operator-valued kernels. A similar result, which inspired this paper, was used to obtain operator-valued Fourier multiplier results [8, 7].

2. Notation and Basics

Throughout this paper, X, Y, and Z are Banach spaces over the field $\mathbb{K}$ of $\mathbb{R}$ or $\mathbb{C}$. Also, X^* is the (topological) dual of X and $B(X)$ is the (closed) unit ball of X. The space $\mathcal{B}(X,Y)$ of bounded linear operators from X into Y is endowed with the usual uniform operator topology.

A subspace Z of Y^* τ-*norms* Y, where $\tau \ge 1$, provided

If Z τ-norms Y, then the natural mapping

$$j\colon Y \to Z^* \quad \text{given by} \quad \langle\, z, jy \,\rangle \ := \ \langle\, y, z \,\rangle \ \text{ for } \ z \in Z$$

is an isomorphic embedding with

$$\frac{1}{\tau}\left\| y \right\|_Y \ \leq \ \left\| j(y) \right\|_{Z^*} \ \leq \ \left\| y \right\|_Y \ ,$$

in which case Y is identified as a subspace of Z^*.

(T, Σ_T, μ) and (S, Σ_S, ν) are σ-finite (positive) measure spaces;

$$\begin{aligned} \Sigma_S^{\text{finite}} \ &:= \ \{A \in \Sigma_S \colon \nu(A) < \infty\} \\ \Sigma_S^{\text{full}} \ &:= \ \{A \in \Sigma_S \colon \nu\left(S \setminus A\right) = 0\} \ , \end{aligned}$$

with similar notation for the corresponding subsets of Σ_T.

$\mathcal{E}\left(S, X\right)$ is the space of finitely-valued finitely-supported measurable functions from S into X, i.e.

$$\mathcal{E}\left(S, X\right) \ = \ \left\{ \sum_{i=1}^{n} x_i 1_{A_i} \colon x_i \in X, \ A_i \in \Sigma_S^{\text{finite}}, \ n \in \mathbb{N} \right\} \ .$$

Let Γ be a subspace of X^*. A function $f\colon S \to X$ is

- *measurable* provided there is a sequence $(f_n)_{n=1}^{\infty}$ from $\mathcal{E}\left(S, X\right)$ so that $\lim_{n\to\infty} \left\| f(s) - f_n(s) \right\|_X = 0$ for ν-a.e. s
- $\sigma\left(X, \Gamma\right)$-*measurable* provided $\langle\, f\left(\cdot\right), x^* \,\rangle \colon S \to \mathbb{K}$ is measurable for each $x^* \in \Gamma$.

The following fact will be used (c.f., e.g., [3, Corollary II.1.4]).

Fact 2.1 (Pettis's Measurability Theorem). A function $f\colon S \to X$ is measurable if and only if

(i) f is essentially separably valued
(ii) f is $\sigma\left(X, \Gamma\right)$-measurable for some subspace Γ of X^* that 1-norms X. □

$L_0\left(S, X\right)$ is the space of (equivalence classes of) measurable functions from S into X. The Bochner-Lebesgue space $L_p\left(S, X\right)$, where $1 \leq p \leq \infty$, is endowed with its usual norm topology. The space $L_\infty^{w^*}\left(T, Z^*\right)$ of μ-essentially bounded $\sigma\left(Z^*, Z\right)$-measurable functions from T into Z^* is endowed with the μ-essential supremum norm, under which it becomes a Banach space.

$\mathcal{E}\left(S, X\right)$ is norm dense in $L_p\left(S, X\right)$ for $1 \leq p < \infty$. Let $L_\infty^0\left(S, X\right)$ be the closure of $\mathcal{E}\left(S, X\right)$ in the $L_\infty\left(S, X\right)$-norm. If X is infinite-dimensional, then $L_\infty^0\left(S, X\right) \neq L_\infty\left(S, X\right)$ (provided Σ_S contains a countable number of pairwise disjoint sets of strictly positive measure). $L_\infty^0\left(S, X\right)$ can be described as follows.

Proposition 2.2. *Let $f \in L_0\left(S, X\right)$. Then $f \in L_\infty^0\left(S, X\right)$ if and only if*

(1) $\inf\left\{ \left\| f 1_{S\setminus A} \right\|_{L_\infty(S,X)} \colon A \in \Sigma_S^{\text{finite}} \right\} \ = \ 0$
(2) *there is $B \in \Sigma_S^{\text{full}}$ so that the set $\left\{f\left(s\right) \colon s \in B\right\}$ is relatively compact in X.*

Conversely, for $\varepsilon > 0$, conditions (1) and (2) give a set $G\ (:= A \cap B) \in \Sigma_S^{\text{finite}}$ so that

$$\left\| f 1_{S \setminus G} \right\|_{L_\infty(S,X)} < \varepsilon \quad \text{and} \quad \{f(s) : s \in G\} \text{ is relatively compact ;}$$

thus allowing one to find, via a finite covering of the set $f(G)$ by ε-balls, a function $f_\varepsilon \in \mathcal{E}(S,X)$, with support in G, so that $\|f - f_\varepsilon\|_{L_\infty(S,X)} < \varepsilon$. □

Lemma 2.3 will help to deal with the fact that $\mathcal{E}(S,X)$ is (usually) not norm dense in $L_\infty(S,X)$.

Lemma 2.3. *Let $f \in L_\infty(S,X)$ and $\varepsilon > 0$. There is a sequence $\{g_n\}_{n=1}^\infty$ from $\mathcal{E}(S,X)$ so that*

$$\begin{aligned} f(s) &= \sum_{n=1}^\infty g_n(s) \\ \sum_{n=1}^\infty \|g_n(s)\|_X &\le (1+\varepsilon)\ \|f\|_{L_\infty(S,X)} \end{aligned}$$

for a.e. $s \in S$.

Proof. Fix a sequence $\{\varepsilon_j\}_{j=1}^\infty$ of positive numbers so that $\varepsilon_1 = 1$ and $\sum_{j=1}^\infty \varepsilon_j < 1 + \varepsilon$.

Choose a sequence $\{f_j\}_{j=1}^\infty$ from $\mathcal{E}(S,X)$ so that, for a.e. $s \in S$,

$$\begin{aligned} f_j(s) &\to f(s) && \text{as } j \to \infty \\ \|f_j(s)\|_X &\le \|f(s)\|_X && \text{for each } j \in \mathbb{N}\ . \end{aligned}$$

Find a sequence $\{S_k\}_{k=1}^\infty$ of pairwise disjoint sets from Σ_S^{finite} so that $\nu(S \setminus \cup_{k=1}^\infty S_k) = 0$ and, for each S_k,

$$\begin{aligned} f_j &\to f && \text{uniformly on } S_k \\ \|f_j(s)\|_X &\le \|f(s)\|_X && \text{for each } s \in S_k \text{ and } j \in \mathbb{N}\ . \end{aligned}$$

Hence, on each S_k, there is a sequence $\{g_j^k\}_{j=1}^\infty$ from $\mathcal{E}(S_k,X)$ so that

$$\begin{aligned} f(s) &= \sum_{j=1}^\infty g_j^k(s) && \text{for each } s \in S_k \\ \left\| g_j^k \right\|_{L_\infty(S_k,X)} &\le \varepsilon_j \|f\|_{L_\infty(S,X)} && \text{for each } j \in \mathbb{N}\ . \end{aligned}$$

For $n \in \mathbb{N}$, let

$$g_n := \sum_{k<n} \left(g_n^k 1_{S_k} \right) + \left(\sum_{j=1}^n g_j^n \right) 1_{S_n}\ .$$

Thus

$$\begin{aligned} g_1 &= g_1^1\ 1_{S_1} \\ g_2 &= g_2^1\ 1_{S_1} + (g_1^2 + g_2^2)\ 1_{S_2} \\ g_3 &= g_3^1\ 1_{S_1} + \quad g_3^2 \quad 1_{S_2} + (g_1^3 + g_2^3 + g_3^3)\ 1_{S_3} \end{aligned}$$

Note that if $s \in S_k$ then

$$\sum_{n=1}^{\infty} g_n(s) = \sum_{j=1}^{\infty} g_j^k(s)$$

and, by the triangle inequality,

$$\sum_{n=1}^{\infty} \|g_n(s)\|_X \le \sum_{j=1}^{\infty} \left\| g_j^k(s) \right\|_X .$$

So clearly the g_n's do as they should. $\square$

Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. There is a natural isometric embedding of $L_{p'}(T, Z^*)$ into $[L_p(T, Z)]^*$ given by

$$\langle f, g \rangle := \int_T \langle f(t), g(t) \rangle \, d\mu(t) \quad \text{for } g \in L_{p'}(T, Z^*) \ , \ f \in L_p(T, Z) \ .$$

There also is a natural isometric embedding of $L_1(T, Y^*)$ into $\left[L_\infty^{w^*}(T, Y^{**})\right]^*$; indeed, for $g = \sum_{i=1}^{n} y_i^* 1_{B_i} \in L_1(T, Y^*)$ and $f \in L_\infty^{w^*}(T, Y^{**})$ let

$$\langle f, g \rangle := \int_T \langle g(t), f(t) \rangle \, d\mu(t) = \sum_{i=1}^{n} \int_{B_i} \langle y_i^*, f(t) \rangle \, d\mu(t)$$

and observe that $\|g\|_{\left[L_\infty^{w^*}(T, Y^{**})\right]^*} = \|g\|_{L_1(T, Y^*)}$.

For a mapping

$$k \colon T \times S \to \mathcal{B}(X, Y)$$

the mapping

$$k^* \colon T \times S \to \mathcal{B}(Y^*, X^*)$$

is defined by $k^*(t, s) := [k(t, s)]^*$.

Non-numerical subscripts on constants indicate dependency. All other notation and terminology, not otherwise explained, are as in [3, 9].

3. MAIN RESULTS

Several conditions on a kernel $k \colon T \times S \to \mathcal{B}(X, Y)$ will be considered. The first one is a mild measurability condition.

Definition 3.1. $k \colon T \times S \to \mathcal{B}(X, Y)$ satisfies condition (C_0) provided that for each $A \in \Sigma_S^{\text{finite}}$ and each $x \in X$

- there is $T_{A,x} \in \Sigma_T^{\text{full}}$ so that if $t \in T_{A,x}$ then the Bochner integral

$$\int_A k(t, s) \, x \, d\nu(s)$$

exists

- the mapping

$$T_{A,x} \ni t \mapsto \int_A k(t, s) \, x \, d\nu(s) \in Y$$

Remark 3.2. Let $k\colon T\times S\to\mathcal{B}(X,Y)$ satisfy condition (C_0). Then

$$\begin{aligned}&\text{for each}\quad f\in\mathcal{E}(S,X)\quad\text{there is}\quad T_f\in\Sigma_T^{\text{full}}\quad\text{so that}\\&\text{if}\quad t\in T_f\quad\text{then the Bochner integral}\quad (Kf)(t)\ :=\ \int_S k(t,s)\left[f(s)\right]\,d\nu(s)\quad\text{exists}\end{aligned}\tag{3.1}$$

and (3.1) defines a linear mapping

$$K\colon\mathcal{E}(S,X)\to L_0(T,Y)\ .\tag{3.2}$$

Next integrability conditions on k are added to ensure that the mapping K in (3.2) extends to the desired superspaces. □

Condition (C_1) will be used for the L_1-case in Theorem 3.4.

Definition 3.3. $k\colon T\times S\to\mathcal{B}(X,Y)$ satisfies condition (C_1) provided there is a constant C_1 so that for each $x\in X$

- the mapping $T\times S\ni(t,s)\to\left\|k(t,s)\,x\right\|_Y\in\mathbb{R}$ is product measurable
- there is $S_x\in\Sigma_S^{\text{full}}$ so that

$$\int_T\left\|k(t,s)\,x\right\|_Y\,d\mu(t)\leq C_1\left\|x\right\|_X$$

  for each $s\in S_x$. □

Note that the first condition guarantees that the mapping $T\ni t\to\left\|k(t,s)\,x\right\|_Y\in\mathbb{R}$ is measurable for ν-a.e. s. Also, the first condition is often satisfied even though the mapping $T\times S\ni(t,s)\to k(t,s)\,x\in Y$ may not be product measurable.

The L_1-case is a straightforward extension of the scalar-valued situation.

Theorem 3.4. *Let $k\colon T\times S\to\mathcal{B}(X,Y)$ satisfy conditions (C_0) and (C_1). Then the integral operator*

$$(Kf)(\cdot)\ :=\ \int_S k(\cdot,s)\,f(s)\,d\nu(s)\qquad\text{for } f\in\mathcal{E}(S,X)$$

extends to a bounded linear operator

$$K\colon L_1(S,X)\to L_1(T,Y)$$

of norm at most the constant C_1 from Definition 3.3 .

Proof. Fix $f=\sum_{i=1}^n x_i 1_{A_i}\in\mathcal{E}(S,X)$ with the A_i's disjoint. By condition (C_1), for each i,

$$\begin{aligned}\int_T\int_S\left\|k(t,s)\,x_i1_{A_i}(s)\right\|_Y\,d\nu(s)\,d\mu(t)\ &=\ \int_S\left[\int_T\left\|k(t,s)\,x_i\right\|_Y\,d\mu(t)\right]1_{A_i}(s)\,d\nu(s)\\&\leq\ \int C_1\left\|x_i\right\|_X 1_{A_i}(s)\,d\nu(s)\end{aligned}\tag{3.3}$$

Condition (C_0) gives that $Kf \in L_0(T,Y)$ and also, combined with (3.3), that

$$\begin{aligned}
\|Kf\|_{L_1(T,Y)} &= \int_T \left\| \sum_{i=1}^{n} \int_S k(t,s)\, x_i 1_{A_i}(s)\ d\nu(s) \right\|_Y d\mu(t) \\
&\leq \sum_{i=1}^{n} \int_T \int_S \|k(t,s)\, x_i 1_{A_i}(s)\|_Y\ d\nu(s)\ d\mu(t) \\
&\leq \sum_{i=1}^{n} C_1 \|x_i\|_X\, \nu(A_i) \ = \ C_1 \|f\|_{L_1(S,X)} \ .
\end{aligned}$$

This completes the proof. □

Condition (C^0_∞) will be used for the L^0_∞-case in Theorem 3.6.

Definition 3.5. Let Z be a subspace of Y^*. Then $k\colon T \times S \to \mathcal{B}(X,Y)$ satisfies condition (C^0_∞), with respect to Z, provided there is a constant C^0_∞ so that for each $y^* \in Z$ there is $T_{y^*} \in \Sigma_T^{\mathrm{full}}$ so that for each $t \in T_{y^*}$

- the mapping $S \ni s \to \|k^*(t,s)\, y^*\|_{X^*} \in \mathbb{R}$ is measurable
- $\int_S \|k^*(t,s)\, y^*\|_{X^*}\ d\nu(s) \leq C^0_\infty \|y^*\|_{Y^*}$. □

Theorem 3.6. *Let Z be a subspace of Y^* that τ-norms Y. Let $k\colon T \times S \to \mathcal{B}(X,Y)$ satisfy conditions (C_0) and (C^0_∞) with respect to Z. Then the integral operator*

$$(Kf)(\cdot) \ := \ \int_S k(\cdot,s)\, f(s)\ d\nu(s) \qquad \text{for } f \in \mathcal{E}(S,X)$$

extends to a bounded linear operator

$$K\colon L^0_\infty(S,X) \to L_\infty(T,Y)$$

of norm at most $\tau \cdot C^0_\infty$ where the constant C^0_∞ is from Definition 3.5.

Proof. Fix $f \in \mathcal{E}(S,X)$. Fix $y^* \in Z$. Find the corresponding sets $T_f, T_{y^*} \in \Sigma_T^{\mathrm{full}}$ from the definitions of conditions (C_0) and (C^0_∞). If $t \in T_f \cap T_{y^*}$ then

$$\begin{aligned}
|\langle (Kf)(t), y^* \rangle| &= \left| \left\langle \int_S k(t,s)\, f(s)\ d\nu(s), y^* \right\rangle \right| \\
&\leq \int_S |\langle f(s), k^*(t,s)\, y^* \rangle|\ d\nu(s) \\
&\leq \int_S \|k^*(t,s)\, y^*\|_{X^*} \|f(s)\|_X\ d\nu(s) \\
&\leq \|f\|_{L_\infty(S,X)}\ C^0_\infty\ \|y^*\|_{Y^*} \ .
\end{aligned}$$

Since $T_f \cap T_{y^*} \in \Sigma_T^{\mathrm{full}}$ and Z τ-norms Y,

$$\|Kf\|_{L_\infty(T,Y)} \ \leq \ \tau\, C^0_\infty\ \|f\|_{L_\infty(S,X)} \ .$$

Remark 3.7 (on Theorem 3.6). Note that K maps $L_{\infty}^{0}(S,X)$ into $L_{\infty}^{0}(T,Y)$ provided that for each $x \in X$ and $A \in \Sigma_S^{\text{finite}}$

$$T_f \ni t \to \int_A k(t,s)\, x\, d\nu(s) \in Y \tag{3.4}$$

defines a function in $L_{\infty}^{0}(T,Y)$. This will be the case, for example, if μ is a Radon measure on a locally compact Hausdorff space T (e.g., T is a Borel subset of $\mathbb{R}^N$, endowed with the Lebesgue measure) and (3.4) defines a function in

$$C_0(T,Y) \; := \; \left\{ g \colon T \to Y \mid g \text{ is continuous and } \inf \left\{ \left\| g 1_{T\setminus B} \right\|_{L_\infty} : B \text{ is compact} \right\} = 0 \right\} \; ;$$

indeed, conditions (1) and (2) of Proposition 2.2 are then fulfilled. □

Interpolating between Theorems 3.4 and 3.6 gives the L_p-case for $1 < p < \infty$.

Theorem 3.8. *Let Z be a subspace of Y^* that τ-norms Y and $1 < p < \infty$. Let $k \colon T \times S \to \mathcal{B}(X,Y)$ satisfy conditions* (C_0)*,* (C_1)*, and* (C_{∞}^{0}) *with respect to Z. Then the integral operator*

$$(Kf)(\cdot) \; := \; \int_S k(\cdot,s)\, f(s)\; d\nu(s) \qquad \text{for } f \in \mathcal{E}(S,X)$$

extends to a bounded linear operator

$$K \colon L_p(S,X) \to L_p(T,Y)$$

of norm at most $(C_1)^{1/p}\left(\tau \cdot C_{\infty}^{0}\right)^{1/p'}$ where the constants C_1 and C_{∞}^{0} are from Definitions 3.3 and 3.5.

Proof. The proof follows directly from Theorems 3.4 and 3.6 and Lemma 3.9. □

The below interpolation lemma is a slight improvement on [1, Thm. 5.1.2].

Lemma 3.9. *Let the linear mapping*

$$K \colon \mathcal{E}(S,X) \to L_1(T,Y) + L_{\infty}(T,Y)$$

satisfy, for each $f \in \mathcal{E}(S,X)$,

$$\begin{aligned} \|Kf\|_{L_1(T,Y)} \; &\le \; c_1 \|f\|_{L_1(S,X)} \; < \; \infty \\ \|Kf\|_{L_\infty(T,Y)} \; &\le \; c_\infty \|f\|_{L_\infty(S,X)} \; < \; \infty \, . \end{aligned}$$

Then, for each $1 < p < \infty$, the mapping K extends to a bounded linear operator

$$K \colon L_p(S,X) \to L_p(T,Y)$$

of norm at most $(c_1)^{1/p}(c_\infty)^{1/p'}$.

Proof. Fix $B \in \Sigma_T^{\text{finite}}$ and a finite measurable partition π of B. Let $\Sigma_0 := \sigma_B(\pi)$ be the σ-algebra of subsets of B that is generated by π. Then the linear mapping given by $K_0 f := \mathbb{E}\left(\left(Kf\right)1_B \mid \Sigma_0\right)$ where $\mathbb{E}\left(\cdot \mid \Sigma_0\right)$ is the conditional expectation operator relative to Σ_0, satisfies

$$\begin{aligned} \left\|K_0 f\right\|_{L_1(T,Y)} &\leq c_1 \left\|f\right\|_{L_1(S,X)} < \infty \\ \left\|K_0 f\right\|_{L_\infty(T,Y)} &\leq c_\infty \left\|f\right\|_{L_\infty(S,X)} < \infty \end{aligned}$$

for each $f \in \mathcal{E}\left(S,X\right)$. Furthermore, $K_0 \colon L^0_\infty\left(S,X\right) \to L^0_\infty\left(T,Y\right)$. Thus, by [1, Thm. 5.1.2], for each $p \in (1,\infty)$, the linear mapping K_0 extends to a bounded linear operator from $L_p\left(S,X\right)$ to $L_p\left(T,Y\right)$ of norm at most $\left(c_1\right)^{1/p}\left(c_\infty\right)^{1/p'}$.

Next, fix $f \in \mathcal{E}\left(S,X\right)$ and $p \in (1,\infty)$. By assumption, $Kf \in L_1\left(T,Y\right) \cap L_\infty\left(T,Y\right)$; thus, $Kf \in L_p\left(T,Y\right)$. Fix $B \in \Sigma_T^{\text{finite}}$. Since T is σ-finite, it suffices to show

$$\left\|\left(Kf\right)1_B\right\|_{L_p(T,Y)} \leq \left(c_1\right)^{1/p}\left(c_\infty\right)^{1/p'} \left\|f\right\|_{L_p(S,X)} . \tag{3.5}$$

Find a sequence $\{g_n\}_{n=1}^\infty$ of functions from $\mathcal{E}\left(T,Y\right)$ that are supported on B and, for μ-a.e. t,

$$\begin{aligned} g_n &\to \left(Kf\right)(t)\ 1_B(t) \\ \left\|g_n(t)\right\|_Y &\leq \left\|\left(Kf\right)(t)\ 1_B(t)\right\|_Y . \end{aligned} \tag{3.6}$$

Let $\Sigma_n := \sigma_B\left(g_1, \ldots g_n\right)$ be the σ-algebra of subsets of B that is generated by $\{g_1, \ldots, g_n\}$. Note that $\left(Kf\right)1_B$ is the limit in $L_p\left(T,Y\right)$ of $\{g_n\}_{n=1}^\infty$ (by (3.6)) and thus also of $\{\mathbb{E}\left(\left(Kf\right)1_B \mid \Sigma_n\right)\}_{n=1}^\infty$ since

$$\begin{aligned} &\left\|\left(Kf\right)1_B - \mathbb{E}\left(\left(Kf\right)1_B \mid \Sigma_n\right)\right\|_{L_p(T,Y)} \\ &\qquad \leq \left\|\left(Kf\right)1_B - g_n\right\|_{L_p(T,Y)} + \left\|\mathbb{E}\left(g_n - \left(Kf\right)1_B \mid \Sigma_n\right)\right\|_{L_p(T,Y)} . \end{aligned}$$

But by the previous paragraph, for each $n \in \mathbb{N}$,

$$\left\|\mathbb{E}\left(\left(Kf\right)1_B \mid \Sigma_n\right)\right\|_{L_p(T,Y)} \leq \left(c_1\right)^{1/p}\left(c_\infty\right)^{1/p'} \left\|f\right\|_{L_p(S,X)} .$$

Thus (3.5) holds. □

Condition (C_∞), a strengthening of condition (C^0_∞), will be used for the L_∞-case in Theorem 3.11.

Definition 3.10. Let Z be a subspace of Y^*. Then $k \colon T \times S \to \mathcal{B}\left(X,Y\right)$ satisfies condition (C_∞), with respect to Z, provided there is a constant C_∞ and $T_0 \in \Sigma_T^{\text{full}}$ so that for each $t \in T_0$ and $y^* \in Z$

- the mapping $S \ni s \to k^*\left(t,s\right) y^* \in X^*$ is measurable
- $\int_S \left\|k^*\left(t,s\right) y^*\right\|_{X^*} \, d\nu\left(s\right) \leq C_\infty \left\|y^*\right\|_{Y^*}$. □

The L_∞-case is more delicate since, in general, $\mathcal{E}\left(S,X\right)$ is not norm dense in $L_\infty\left(S,X\right)$.

Theorem 3.11. *Let Z be a subspace of Y^* that τ-norms Y. Let $k \colon T \times S \to \mathcal{B}\left(X,Y\right)$ satisfy conditions* (C_0) *and* (C_∞) *with respect to Z. In particular,* (C_∞) *gives that for each $t \in T_0 \in \Sigma_T^{\text{full}}$ and each $y^* \in Z$*

$$\int_S \left\|k^*\left(t,s\right) y^*\right\|_{X^*} \, d\nu\left(s\right) \leq C_\infty \left\|y^*\right\|_{Y^*} .$$

defined by

$$(Kf)(\cdot) \;:=\; \int_S k(\cdot, s)\, f(s)\, d\nu(s) \qquad \textit{for } f \in \mathcal{E}(S, X) \tag{3.7}$$

extends (identifying Y *as a subspace of* Z^**) to a bounded (of norm at most* C_∞*) linear operator*

$$K\colon L_\infty(S, X) \to L_\infty^{w^*}(T, Z^*) \tag{3.8}$$

that is given by

$$\begin{aligned} &\langle\, y^*, (Kf)(t)\,\rangle \;:=\; \int_S \langle\, k(t,s)\, f(s)\,, y^*\,\rangle\; d\nu(s) \\ &\textit{for } f \in L_\infty(S, X) \textit{ and } t \in T_0 \textit{ and } y^* \in Z\ . \end{aligned} \tag{3.9}$$

Furthermore, the K *of* (3.8) *maps* $L_\infty(S, X)$ *into* $L_\infty(T, Y)$ *provided either*

(i) Y *does not (isomorphically) contain* c_0

or

(ii) *for each* $t \in T_0$ *the subset* $\{\|k^*(t,\cdot)\, y^*\|_{X^*} : y^* \in B(Z)\}$ *of* $L_0(S, \mathbb{R})$ *is equi-integrable.*

Recall that the subset in (ii) is *equi-integrable* provided if $\{A_n\}_{n=1}^\infty$ is a sequence from Σ_S with $A_n \supseteq A_{n+1}$ and $\nu(\cap_{n=1}^\infty A_n) = 0$ then

$$\lim_{n\to\infty} \sup_{y^* \in B(Z)} \int_{A_n} \|k^*(t,s)\, y^*\|_{X^*}\; d\nu(s) \;=\; 0\ . \tag{3.10}$$

Remark 3.12 (on Theorem 3.11).

(a) There can be advantages in taking a proper norming subspace $Z \subsetneq Y^*$ over taking $Z = Y^*$. First, it eases the assumptions of k. Second, Z^* may be much *smaller* than Y^{**} and so the conclusion $K(L_\infty(S,X)) \subset L_\infty^{w^*}(T, Z^*)$ may be more useful than $K(L_\infty(S,X)) \subset L_\infty^{w^*}(T, Y^{**})$.

For example, if $Y = C[0,1]$ then $Z := L_1[0,1] \subsetneq Y^*$ 1-norms Y; furthermore, $Z^* \simeq L_\infty[0,1]$ is *nicer* than $Y^{**} \simeq (M[0,1])^*$, which is very *large*.

(b) If $Y = (Y_*)^*$ is a separable dual space, then $Z := Y_* \subset Y^*$ 1-norms Y and, by Pettis's measurability theorem (Fact 2.1), one has that $L_\infty^{w^*}(T, Z^*) = L_\infty(T, Y)$.

(c) If $\nu(S) < \infty$ and for each $t \in T_0$ there exists $q_t \in (1, \infty]$ and $C_t \in (0, \infty)$ such that

$$\sup_{y^* \in B(Z)} \|k^*(t,\cdot)\, y^*\|_{L_{q_t}(S,X)} \;\le\; C_t\ ,$$

then the equi-integrability condition in (ii) holds; indeed, just apply Hölder's inequality.

(d) Remark 3.7 is valid in this setting also. □

Proof of Theorem 3.11. Fix $f \in L_\infty(S, X)$. Fix $t \in T_0$. For each $y^* \in Z$ the function

$$\langle\, k(t,\cdot)\, f(\cdot)\,, y^*\,\rangle : S \to \mathbb{K}$$

(b) in $L_1(S,\mathbb{K})$ with

$$\begin{aligned}\int_S |\langle\, k(t,s)\, f(s)\,, y^*\,\rangle|\; d\nu(s) \;&\leq\; \int_S \|k^*(t,s)\, y^*\|_{X^*}\, \|f(s)\|_X\; d\nu(s) \\ &\leq\; C_\infty\, \|y^*\|_Z\, \|f\|_{L_\infty(S,X)}\ .\end{aligned} \tag{3.11}$$

Thus by the Closed Graph Theorem, applied to the mapping

$$Z \ni y^* \to \langle\, k(t,\cdot)\, f(\cdot)\,, y^*\,\rangle \in L_1(S,\mathbb{K})\ ,$$

the mapping

$$Z \ni y^* \to \int_S \langle\, k(t,s)\, f(s)\,, y^*\,\rangle\; d\nu(s) \in \mathbb{K}$$

defines an element $(Kf)(t)$ of Z^* that satisfies (3.9).

Let $f \in \mathcal{E}(S,X)$. By condition (C_0), there is $T_f \in \Sigma_T^{\mathrm{full}}$ such that if $t \in T_f$ then $k(t,\cdot)\, f(\cdot) \in L_1(S,Y)$. For each $t \in T_f \cap T_0 \in \Sigma_T^{\mathrm{full}}$ and each $y^* \in Z$

$$\left\langle \int_S k(t,s)\, f(s)\; d\nu(s)\,, y^* \right\rangle \;=\; \int_S \langle\, k(t,s)\, f(s)\,, y^*\,\rangle\; d\nu(s) \;=\; \langle\, y^*, (Kf)(t)\,\rangle\ .$$

Hence (3.7) holds. Thus, by Theorem 3.6, K maps $\mathcal{E}(S,X)$ into $L_\infty(T,Y)$.

Fix $f \in L_\infty(S,X)$. To see that Kf is $\sigma(Z^*,Z)$-measurable, fix a sequence $\{f_n\}_{n\in\mathbb{N}}$ from $\mathcal{E}(S,X)$ that converges a.e. to f and $\|f_n\|_{L_\infty(S,X)} \;\leq\; \|f\|_{L_\infty(S,X)}$ for each $n \in \mathbb{N}$. Then, by the Lebesgue Dominated Convergence Theorem, for each $y^* \in Z$ and for a.e. $t \in T$,

$$\begin{aligned}\langle\, y^*, (Kf)(t)\,\rangle \;&=\; \int_S \langle\, f(s)\,, k^*(t,s)\, y^*\,\rangle\; d\nu(s) \\ &=\; \lim_{n\to\infty} \int_S \langle\, f_n(s)\,, k^*(t,s)\, y^*\,\rangle\; d\nu(s) \;=\; \lim_{n\to\infty}\ \langle\, y^*, (Kf_n)(t)\,\rangle\end{aligned}$$

and the latter functions $\langle\, y^*, (Kf_n)(\cdot)\,\rangle$ are μ-measurable functions by condition (C_0). Furthermore, by (3.11)

$$\sup_{t\in T_0}\ \|(Kf)(t)\|_{Z^*} \;=\; \sup_{t\in T_0}\ \sup_{y^*\in B(Z)}\ \left|\int_S \langle\, k(t,s)\, f(s)\,, y^*\,\rangle\; d\nu(s)\right| \;\leq\; C_\infty\, \|f\|_{L_\infty(S,X)}\ ;$$

thus, $Kf \in L_\infty^{w^*}(T,Z^*)$ and the K of (3.8) is of norm at most C_∞

<u>PROOF OF (i)</u> Assume that c_0 does not isomorphically embed into Y.

Fix $f \in L_\infty(S,X)$. By Lemma 2.3, there is a sequence $\{f_n\}_{n=1}^\infty$ from $\mathcal{E}(S,X)$ so that

$$f(s) \;=\; \sum_{n=1}^{\infty} f_n(s)$$

$$\sum_{n=1}^{\infty} \|f_n(s)\|_X \;\leq\; 2\, \|f\|_{L_\infty(S,X)}$$

for a.e. $s \in S$. Since each f_n is in $\mathcal{E}(S,X)$, by (3.7) and condition (C_0), there is $T_1 \in \Sigma_T^{\mathrm{full}}$, with $T_1 \subseteq T_0$, so that for each $t \in T_1$ and each $n \in \mathbb{N}$

$$(Kf_n)(t) \;=\; \int k(t,s)\, f_n(s)\; d\nu(s) \;\in\; Y$$

Fix $t \in T_1$. By the Lebesgue Dominated Convergence Theorem,

$$(Kf)(t) \;=\; \lim_{m\to\infty} \sum_{n=1}^{m} (Kf_n)(t) \quad \text{in the } \sigma(Z^*, Z)\text{-topology} \tag{3.13}$$

since, for each $y^* \in Z$,

$$\begin{aligned}
\langle\, y^*, (Kf)(t)\,\rangle \;&=\; \int_S \langle\, f(s), k^*(t,s)\, y^*\,\rangle \; d\nu(s) \\
&=\; \lim_{m\to\infty} \int_S \left\langle \sum_{n=1}^{m} f_n(s), k^*(t,s)\, y^* \right\rangle d\nu(s) \;=\; \lim_{m\to\infty} \left\langle \sum_{n=1}^{m} (Kf_n)(t), y^* \right\rangle .
\end{aligned}$$

It suffices to show that the series in (3.13) converges also in the Z^*-norm topology; thus, since Y does not contain c_0, it suffices to show that

$$\sum_{n=1}^{\infty} \left|\langle\, (Kf_n)(t), y^*\,\rangle\right| \;<\; \infty \quad \text{for each } y^* \in Y^* \tag{3.14}$$

by a theorem of Bessaga and Pełczyński (cf., e.g., [4, Thm. V.8]).

For each $y^* \in Z$,

$$\begin{aligned}
\sum_{n=1}^{\infty} \left|\langle\, (Kf_n)(t), y^*\,\rangle\right| \;&=\; \sum_{n=1}^{\infty} \left| \int_S \langle\, k(t,s)\, f_n(s), y^*\,\rangle \; d\nu(s) \right| \\
&\le\; \sum_{n=1}^{\infty} \int_S \left|\langle\, f_n(s), k^*(t,s)\, y^*\,\rangle\right| \; d\nu(s) \\
&\le\; \sum_{n=1}^{\infty} \int_S \left\|k^*(t,s)\, y^*\right\|_{X^*} \left\|f_n(s)\right\|_X \; d\nu(s) \\
&\le\; \int_S \left\|k^*(t,s)\, y^*\right\|_{X^*} \left(\sum_{n=1}^{\infty} \left\|f_n(s)\right\|_X \right) d\nu(s) \\
&\le\; 2 \left\|f\right\|_{L_\infty(S,X)} \; C_\infty \;\; \left\|y^*\right\|_{Y^*} \; .
\end{aligned}$$

Thus the mapping

$$Z \ni y^* \xrightarrow{\;U\;} \left\{\langle\, (Kf_n)(t), y^*\,\rangle\right\}_{n=1}^{\infty} \in \ell_1$$

is a bounded linear operator. Fix $\{\alpha_n\}_{n=1}^{\infty} \in B(\ell_\infty)$ and $m \in \mathbb{N}$. If $y^* \in B(Z)$ then

$$\left| \left\langle \sum_{n=1}^{m} \alpha_n (Kf_n)(t), y^* \right\rangle \right| \;=\; \left| \sum_{n=1}^{m} \alpha_n \langle\, (Kf_n)(t), y^*\,\rangle \right| \;\le\; \|U\|_{\mathcal{B}(Z,\ell_1)}$$

and so

$$\left\| \sum_{n=1}^{m} \alpha_n (Kf_n)(t) \right\|_Y \;\le\; \tau \, \|U\|_{\mathcal{B}(Z,\ell_1)} \; .$$

Fix $f \in L_\infty(S,X)$. Choose a sequence $\{f_n\}_{n=1}^\infty$ from $\mathcal{E}(S,X)$ such that

$$\begin{aligned} \lim_{n\to\infty} f_n(s) &= f(s) \quad \text{for a.e. } s \in S \\ \|f_n\|_{L_\infty(S,X)} &\le \|f\|_{L_\infty(S,X)} \quad \text{for each } n \in \mathbb{N} \ . \end{aligned}$$

As in the proof of (i), since the f_n's are in $\mathcal{E}(S,X)$, there is $T_1 \in \Sigma_T^{\text{full}}$, with $T_1 \subseteq T_0$, so that (3.12) holds for each $t \in T_1$ and each $n \in \mathbb{N}$. It suffices to show that $\{(Kf_n)(t)\}_{n=1}^\infty$ converges in Z^*-norm to $(Kf)(t)$ for each $t \in T_1$.

Fix $t \in T_1$. Fix $\delta > 0$ and let $B_n := \{s \in S \colon \|f(s) - f_n(s)\|_X > \delta\}$ and $A_n := \cup_{k=n}^\infty B_k$. Then

$$\begin{aligned} \|(Kf)(t) - (Kf_n)(t)\|_{Z^*} &= \sup_{y^* \in B(Z)} \left| \int_S \langle\, k(t,s)(f(s) - f_n(s)), y^* \,\rangle \, d\nu(s) \right| \\ &\le \sup_{y^* \in B(Z)} \int_S \|k^*(t,s)\, y^*\|_{X^*} \, \|f(s) - f_n(s)\|_X \, d\nu(s) \\ &\le \sup_{y^* \in B(Z)} \left[\delta \int_{B_n^C} \|k^*(t,s)\, y^*\|_{X^*} \, d\nu(s) \;+\; 2\ \|f\|_{L_\infty(S,X)} \int_{B_n} \|k^*(t,s)\, y^*\|_{X^*} \, d\nu(s) \right] \\ &\le \delta\, C_\infty \;+\; 2\ \|f\|_{L_\infty(S,X)} \left[\sup_{y^* \in B(Z)} \int_{A_n} \|k^*(t,s)\, y^*\|_{X^*} \, d\nu(s) \right] . \end{aligned}$$

Note that $\cap_{n=1}^\infty A_n \subseteq \{s \in S \colon f_n(s) \text{ does not converge to } f(s)\}$. Thus, by the equi-integrability assumption, $\{(Kf_n)(t)\}_{n=1}^\infty$ converges in Z^*-norm to $(Kf)(t)$. $\square$

The following example illustrates the limitations on the conclusions in Theorems 3.6 and 3.11.

Example 3.13. Let $X = \mathbb{C}$ and $Y = c_0$. Thus

$$\mathcal{B}(X,Y) \;\simeq\; c_0 \qquad \text{and} \qquad \mathcal{B}(Y^*,X^*) \;\simeq\; \ell_\infty \ .$$

Let $S = T = \mathbb{R}$. Define

$$\begin{aligned} &k_0 \colon \mathbb{R} \to c_0 & k_0(\cdot) &:= \sum_{n=1}^\infty e_n 1_{I_n}(\cdot) \\ &k \colon T \times S \to \mathcal{B}(X,Y) & k(t,s) &:= k_0(t-s) \end{aligned}$$

where $\{e_n\}_{n=1}^\infty$ is the standard unit vector basis of c_0 and $I_n = [n-1, n)$.

Since $k_0 \in L_\infty(\mathbb{R}, c_0)$, for each $f \in L_1(S,X)$ the Bochner integral

$$(Kf)(t) \;:=\; \int_S k(t,s)\, f(s)\; ds \;=\; \int_{\mathbb{R}} k_0(t-s)\, f(s)\; ds \;=\; (k_0 * f)(t)$$

exists for each $t \in T$; furthermore, $Kf \in L_\infty(T,Y)$ and Kf is uniformly continuous. From this it follows that k satisfies condition (C_0). The kernel k also satisfies condition (C_∞) with $Z = Y^*$ and $T_0 = T$ since for each $y^* \in Y^* \simeq \ell_1$ and $t \in T$

$$k^*(t,s)\, y^* \;=\; \sum_{n=1}^\infty y^*(e_n)\, 1_{I_n}(t-s) \quad \text{for each } s \in S$$

Theorem 3.6 gives that $K\left(L_{\infty}^{0}(S,X)\right)\subseteq L_{\infty}(T,Y)$. However, $K(\mathcal{E}(S,X))\nsubseteq L_{\infty}^{0}(T,Y)$. Indeed, $(K1_{I_1})(n)=e_n$ for each $n\in\mathbb{N}$ and so, since $K1_{I_1}$ is uniformly continuous, $K1_{I_1}$ does not satisfy (2) of Proposition 2.2 and so $K1_{I_1}\notin L_{\infty}^{0}(T,Y)$.

Theorem 3.11 gives that $K(L_{\infty}(S,X))\subseteq L_{\infty}^{w^*}(T,Y^{**})$. However, $K(L_{\infty}(S,X))\nsubseteq L_{\infty}(T,Y)$. Indeed, consider $f=1_{(-\infty,0)}\in L_{\infty}(S,X)$. If $n\in\mathbb{N}$ and $0<\delta\leq 1$ then

$$(Kf)(n-\delta)=\delta e_n+\sum_{j\in\mathbb{N}}e_{n+j}\in Y^{**}\setminus Y\ .$$

Thus $Kf\notin L_{\infty}(T,Y)$. □

4. REMARKS ON DUALITY AND WEAK CONTINUITY

The remarks in this section explore the duality and weak continuity of K. For this, dual versions of the four conditions in Section 3 are needed.

Definition 4.1. Let (C) (possibly with respect to a subspace Z of Y^*) be one of the four conditions in Section 3 on a kernel

$$k\colon T\times S\to\mathcal{B}(X,Y)\ .$$

Then k satisfies condition (C*) (possibly with respect to a subspace Z of X^{**}) provided the mapping

$$\begin{aligned}\widetilde{k}\colon S\times T&\to\mathcal{B}(Y^*,X^*)\\ \widetilde{k}(s,t)\ &:=\ [k(t,s)]^*\end{aligned}$$

satisfies condition (C) (possibly with respect to a subspace Z of X^{**}).

For example, $k\colon T\times S\to\mathcal{B}(X,Y)$ satisfies condition (C$_0^*$) provided that for each $B\in\Sigma_T^{\text{finite}}$ and each $y^*\in Y^*$

- there is $S_{B,y^*}\in\Sigma_S^{\text{full}}$ so that if $s\in S_{B,y^*}$ then the Bochner integral

$$\int_B k^*(t,s)\,y^*\,d\mu(t)$$

  exists
- the mapping

$$S_{B,y^*}\ni s\to\int_B k^*(t,s)\,y^*\,d\mu(t)\in X^*$$

  defines a measurable function from S into X^*. □

Remark 4.2. Let $1<p<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. Let $k\colon T\times S\to\mathcal{B}(X,Y)$ be so that

(i) k satisfies conditions (C$_0$), (C$_1$), (C$_{\infty}^0$) with respect to $Z=Y^*$, and (C$_0^*$)

(ii) for each $y^*\in Y^*$ and each $x\in X$, the mapping $T\times S\ni(t,s)\to\langle k(t,s)\,x,y^*\rangle\in\mathbb{K}$ is

By Theorem 3.8 (with $Z = Y^*$), there is a bounded linear operator

$$K\colon L_p(S,X) \to L_p(T,Y) \tag{4.1}$$

defined by

$$(Kf)(\cdot) \;=\; \int_S k(\cdot,s)\, f(s)\, d\nu(s) \in L_p(T,Y) \quad \text{for } f \in \mathcal{E}(S,X) \;.$$

Note that k satisfies

(iv) (C_1^*) by (iii) and the fact that k satisfies (C_∞^0) with respect to $Z = Y^*$

(v) (C_∞^{0*}) with respect to $Z = X$ since k satisfies (C_1).

So by Theorem 3.8 (with $Z = X$), there is a bounded linear operator

$$\widetilde{K}\colon L_{p'}(T,Y^*) \to L_{p'}(S,X^*)$$

defined by

$$\left(\widetilde{K}g\right)(\cdot) \;=\; \int_T k^*(t,\cdot)\, g(t)\, d\mu(t) \in L_{p'}(S,X^*) \quad \text{for } g \in \mathcal{E}(T,Y^*) \;.$$

Note that

$$K^*g \;=\; \widetilde{K}g \quad \text{for each } g \in L_{p'}(T,Y^*)$$

since if $g = y^*1_B \in \mathcal{E}(T,Y^*) \subset L_{p'}(T,Y^*)$ and $f = x1_A \in \mathcal{E}(S,X) \subset L_p(S,X)$

$$\begin{aligned}
\langle f \,,\, K^*g \rangle &= \langle Kf \,,\, g \rangle \\
&= \int_T \left\langle \int_S k(t,s)\, x1_A(s)\, d\nu(s) \;,\; y^*1_B(t) \right\rangle d\mu(t) \\
&= \int_T \int_S \langle k(t,s)\, x1_A(s) \;,\; y^*1_B(t) \rangle\, d\nu(s)\, d\mu(t) \\
&= \int_S \int_T \langle x1_A(s) \;,\; k^*(t,s)\, y^*1_B(t) \rangle\, d\mu(t)\, d\nu(s) \\
&= \int_S \left\langle x1_A(s) \;,\; \int_T k^*(t,s)\, y^*1_B(t)\, d\mu(t) \right\rangle d\nu(s) \\
&= \left\langle f \,,\, \widetilde{K}g \right\rangle
\end{aligned} \tag{4.2}$$

where assumption (ii) helps justify the use of Fubini's theorem. Thus

$$(K^*g)(\cdot) \;=\; \int_T k^*(t,\cdot)\, g(t)\, d\mu(t) \in L_{p'}(S,X^*) \quad \text{for } g \in \mathcal{E}(T,Y^*)$$

and K^* maps $L_{p'}(T,Y^*)$ into $L_{p'}(S,X^*)$. Thus the K in (4.1) is

$$\sigma\left(L_p(S,X) \;,\; L_{p'}(S,X^*)\right)\text{–to–}\sigma\left(L_p(T,Y) \;,\; L_{p'}(T,Y^*)\right)$$

continuous. □

Remark 4.3. Let $k\colon T \times S \to \mathcal{B}(X,Y)$ be so that

By Theorem 3.4, there is a bounded linear operator

$$K\colon L_1(S,X) \to L_1(T,Y) \tag{4.3}$$

defined by

$$(Kf)(\cdot) \;=\; \int_S k(\cdot,s)\, f(s)\; d\nu(s) \in L_1(T,Y) \quad \text{for } f \in \mathcal{E}(S,X)\ .$$

Since k satisfies condition (C_1), it satisfies condition (C_∞^{0*}) with respect to $Z = X$; thus, by Theorem 3.6, there is a bounded linear operator

$$\widetilde{K}\colon L_\infty^0(T,Y^*) \to L_\infty(S,X^*)$$

defined by

$$\left(\widetilde{K}g\right)(\cdot) \;=\; \int_T k^*(t,\cdot)\, g(t)\; d\mu(t) \in L_\infty(S,X^*) \quad \text{for } g \in \mathcal{E}(T,Y^*)\ .$$

Note that

$$K^*g \;=\; \widetilde{K}g \quad \text{for each } g \in L_\infty^0(T,Y^*)$$

since if $g = y^*1_B \in \mathcal{E}(T,Y^*) \subset L_\infty^0(T,Y^*)$ and $f = x1_A \in \mathcal{E}(S,X) \subset L_1(S,X)$ then the calculation in (4.2) shows that $\langle f, K^*g \rangle = \left\langle f, \widetilde{K}g \right\rangle$. Thus

$$(K^*g)(\cdot) \;=\; \int_T k^*(t,\cdot)\, g(t)\; d\mu(t) \in L_\infty(S,X^*) \quad \text{for } g \in \mathcal{E}(T,Y^*)$$

and K^* maps $L_\infty^0(T,Y^*)$ into $L_\infty(S,X^*)$. Thus the K in (4.3) is

$$\sigma\left(L_1(S,X)\ ,\ L_\infty(S,X^*)\right)\text{–to–}\sigma\left(L_1(T,Y)\ ,\ L_\infty^0(T,Y^*)\right)$$

continuous. □

Remark 4.4. Let $k\colon T \times S \to \mathcal{B}(X,Y)$ be so that

- k satisfies conditions (C_0), (C_1), (C_0^*), and (C_∞^*) with respect to $Z = X$
- condition (ii) of Remark 4.2 holds
- X^* does not contain c_0.

By Theorem 3.4, there is a bounded linear operator

$$K\colon L_1(S,X) \to L_1(T,Y) \tag{4.4}$$

defined by

$$(Kf)(\cdot) \;=\; \int_S k(\cdot,s)\, f(s)\; d\nu(s) \in L_1(T,Y) \quad \text{for } f \in \mathcal{E}(S,X)\ .$$

By Theorem 3.11, there is a bounded linear operator

$$\widetilde{K}\colon L_\infty(T,Y^*) \to L_\infty(S,X^*)$$

defined by, for some $S_0 \in \Sigma_S^{\mathrm{full}}$,

$$\left\langle x, \left(\widetilde{K}g\right)(s) \right\rangle \;=\; \int \langle x, k^*(t,s)\, g(t) \rangle\; d\mu(t)$$

Note that

$$K^*g \;=\; \widetilde{K}g \quad \text{for each } g \in L_\infty(T, Y^*)$$

since if $g \in L_\infty(T, Y^*)$ and $f = x1_A \in \mathcal{E}(S, X) \subset L_1(S, X)$ then

$$\begin{aligned}
\langle f \,,\, K^*g \rangle &= \langle Kf \,,\, g \rangle \\
&= \int_T \left\langle \int_S k(t,s)\, x1_A(s)\, d\nu(s) \,,\, g(t) \right\rangle d\mu(t) \\
&= \int_T \int_S \langle k(t,s)\, x \,,\, g(t) \rangle\, 1_A(s)\, d\nu(s)\, d\mu(t) \\
&= \int_S \int_T \langle x \,,\, k^*(t,s)\, g(t) \rangle\, 1_A(s)\, d\mu(t)\, d\nu(s) \\
&= \int_S \left\langle x \,,\, \left(\widetilde{K}g\right)(s) \right\rangle 1_A(s)\, d\nu(s) \\
&= \left\langle f \,,\, \widetilde{K}g \right\rangle
\end{aligned}$$

where assumption (ii) helps justify the use of Fubini's theorem. Thus, by (3.7),

$$(K^*g)(\cdot) \;=\; \int_T k^*(t,\cdot)\, g(t)\, d\mu(t) \in L_\infty(S, X^*) \quad \text{for } g \in \mathcal{E}(T, Y^*)$$

and K^* maps $L_\infty(T, Y^*)$ into $L_\infty(S, X^*)$. Thus the K in (4.4) is

$$\sigma\left(L_1(S,X) \,,\, L_\infty(S,X^*)\right)\text{–to–}\sigma\left(L_1(T,Y) \,,\, L_\infty(T,Y^*)\right)$$

continuous. □

Remark 4.5. Let $k\colon T \times S \to \mathcal{B}(X, Y)$ be so that

- k satisfies conditions (C_0), (C_∞) with respect to $Z = Y^*$, and (C_0^*)
- conditions (ii) and (iii) of Remark 4.2 hold.

Then (3.9) of Theorem 3.11, with $Z = Y^*$, defines a bounded linear operator

$$K\colon L_\infty(S, X) \to L_\infty^{w^*}(T, Y^{**}) \;. \tag{4.5}$$

Note that k satisfies condition (C_1^*) by (iii) and the fact that k satisfies condition (C_∞) with respect to $Z = Y^*$. So by Theorem 3.4

$$\left(\widetilde{K}g\right)(\cdot) \;:=\; \int_T k^*(t,\cdot)\, g(t)\, d\mu(t) \in L_1(S, X^*) \quad \text{for } g \in \mathcal{E}(T, Y^*)$$

defines a bounded linear operator

$$\widetilde{K}\colon L_1(T, Y^*) \to L_1(S, X^*) \;.$$

Note that

$$\widetilde{K}g \;=\; K^*g \quad \text{for each } g \in L_1(T, Y^*)$$

since for $g = y^*1_B \in \mathcal{E}(T, Y^*) \subset L_1(T, Y^*)$ and $f \in L_\infty(S, X)$

$$\langle f \,,\, K^*g \rangle \;=\; \langle Kf \,,\, g \rangle$$

$$
\begin{aligned}
&= \int_T \left[\int_S \langle\, k(t,s)\, f(s) \;,\; y^* \,\rangle \; d\nu(s) \right] 1_B(t)\; d\mu(t) \\
&= \int_S \int_T \langle\, f(s) \;,\; k^*(t,s)\, y^* 1_B(t) \,\rangle \; d\mu(t)\; d\nu(s) \\
&= \int_S \left\langle\, f(s) \;,\; \int_T k^*(t,s)\, y^* 1_B(t)\; d\mu(t) \right\rangle d\nu(s) \\
&= \int_S \left\langle\, f(s) \;,\; \left(\widetilde{K} g\right)(s) \right\rangle d\nu(s) \\
&= \left\langle\, f \;,\; \widetilde{K} g \right\rangle
\end{aligned}
$$

where assumption (ii) helps justify the use of Fubini's theorem. Thus

$$(K^* g)(\cdot) \;=\; \int_T k^*(t,\cdot)\, g(t)\; d\mu(t) \in L_1(S, X^*) \quad \text{for each } g \in \mathcal{E}(T, Y^*)$$

and K^* maps $L_1(T, Y^*)$ into $L_1(S, X^*)$. So the K of (4.5) is

$$\sigma\left(L_\infty(S,X) \;,\; L_1(S,X^*)\right)\text{–to–}\sigma\left(L_\infty^{w^*}(T, Y^{**}) \;,\; L_1(T, Y^*)\right)$$

continuous. Thus, since $\mathcal{E}(S,X)$ (resp. the Schwartz class $\mathcal{S}\left(\mathbb{R}^N, X\right)$ in the case $S = \mathbb{R}^N$) is $\sigma\left(L_\infty(S,X), L_1(S,X^*)\right)$-dense in $L_\infty(S,X)$, many of the properties of K are determined by its restriction to $\mathcal{E}(S,X)$ (resp. $\mathcal{S}\left(\mathbb{R}^N, X\right)$).

If furthermore Y does not contain c_0, then (3.9) of Theorem 3.11, with $Z = Y^*$, defines a bounded linear operator

$$K \colon L_\infty(S,X) \to L_\infty(T,Y)$$

that is $\sigma\left(L_\infty(S,X), L_1(S,X^*)\right)$–to–$\sigma\left(L_\infty(T,Y), L_1(T,Y^*)\right)$ continuous. □

5. CONVOLUTION OPERATORS

The results thus far are now applied to convolution operators on $T = S = R^N$ (endowed with the Lebesgue measure).

Corollary 5.1. *Let Z be a subspace of Y^* that τ-norms Y. Let $k \colon \mathbb{R}^N \to \mathcal{B}(X,Y)$ be strongly measurable on X and $k^* \colon \mathbb{R}^N \to \mathcal{B}(Y^*, X^*)$ be strongly measurable on Z and*

$$\int_{\mathbb{R}^N} \|k(s)\, x\|_Y \; ds \;\le\; C_1 \|x\|_X \;<\; \infty \qquad \textit{for each } x \in X \tag{5.1}$$

$$\int_{\mathbb{R}^N} \|k^*(s)\, y^*\|_{X^*} \; ds \;\le\; C_\infty \|y^*\|_{Y^*} \;<\; \infty \qquad \textit{for each } y^* \in Z\;. \tag{5.2}$$

Then the convolution operator

$$K \colon \mathcal{E}\left(\mathbb{R}^N, X\right) \to L_0\left(\mathbb{R}^N, Y\right)$$

defined by

$$(Kf)(t) \;:=\; \int_{R^N} k(t-s)\, f(s)\; ds \qquad \textit{for } f \in \mathcal{E}\left(\mathbb{R}^N, X\right) \tag{5.3}$$

- $K_\infty^0 \colon L_\infty^0 \left(\mathbb{R}^N, X\right) \to L_\infty^0 \left(\mathbb{R}^N, Y\right)$,
- *and, if* Y *does not contain* c_0*, then to* $K_\infty \colon L_\infty \left(\mathbb{R}^N, X\right) \to L_\infty \left(\mathbb{R}^N, Y\right)$ *satisfying*

$$\langle\, y^*, (K_\infty f)(t)\,\rangle \;=\; \int_{R^N} \langle\, k(t-s)\, f(s)\,, y^*\,\rangle \; d\nu(s)$$

$$\textit{for } f \in L_\infty\left(\mathbb{R}^N, X\right) \textit{ and } t \in \mathbb{R}^N \textit{ and } y^* \in Z \;.$$

Furthermore,

$$\|K_p\|_{L_p \to L_p} \;\leq\; (C_1)^{\frac{1}{p}}\; (\tau C_\infty)^{\frac{1}{p'}}$$

for $1 \leq p \leq \infty$ *and* $\left\|K_\infty^0\right\|_{L_\infty^0 \to L_\infty^0} \leq \tau C_\infty$.

Remark 5.2. In Corollary 5.1, if $Z = Y^*$ and either

- $1 < p < \infty$
- $p = 1$ and X^* does not contain c_0
- $p = \infty$ and Y does not contain c_0,

then the dual operator

$$K_p^* \colon \left[L_p\left(\mathbb{R}^N, Y\right)\right]^* \to \left[L_p\left(\mathbb{R}^N, X\right)\right]^*$$

has the form

$$\left(K_p^* g\right)(s) \;=\; \int_{\mathbb{R}^N} k^*(t-s)\, g(t)\; dt \in L_{p'}\left(\mathbb{R}^N, X^*\right) \quad \text{for } g \in \mathcal{E}\left(\mathbb{R}^N, Y^*\right)$$

and K_p^* maps $L_{p'}\left(\mathbb{R}^N, Y^*\right)$ into $L_{p'}\left(\mathbb{R}^N, X^*\right)$ and thus K_p is

$$\sigma\left(L_p\left(\mathbb{R}^N, X\right), L_{p'}\left(\mathbb{R}^N, X^*\right)\right)\text{–to–}\sigma\left(L_p\left(\mathbb{R}^N, Y\right), L_{p'}\left(\mathbb{R}^N, Y^*\right)\right)$$

continuous.

Proof of Corollary 5.1 and Remark 5.2. If $f = x 1_A \in \mathcal{E}\left(\mathbb{R}^N, X\right)$, then $Kf = [k(\cdot)\,x] * 1_A$ with $k(\cdot)\,x \in L_1\left(\mathbb{R}^N, Y\right)$ and $1_A \in L_\infty\left(\mathbb{R}^N, \mathbb{R}\right)$. Thus for each $f \in \mathcal{E}\left(\mathbb{R}^N, X\right)$: the Bochner integral in (5.3) exists for each t in $\mathbb{R}^N$, Kf is a uniformly continuous function from $\mathbb{R}^N$ to Y, and Kf vanishes at infinity. Thus $K\left(\mathcal{E}\left(\mathbb{R}^N, X\right)\right) \subset L_\infty^0\left(\mathbb{R}^N, Y\right)$.

It is straightforward to verify that the kernel

$$k_0 \colon \mathbb{R}^N \times \mathbb{R}^N \to \mathcal{B}(X, Y)$$

$$k_0(t, s) \;:=\; k(t-s)$$

satisfies conditions: (C_0), (C_1), (C_∞) with respect to Z with $T_0 = \mathbb{R}^N$, (C_0^*), (C_∞^*) with respect to $Z = X$, and (ii) and (iii) of Remark 4.2.

The corollary now follows from: Theorems 3.4, 3.6, 3.8, 3.11 and Remarks 4.2, 4.4, 4.5. $\square$

Remark 5.3 (on Corollary 5.1).

(a) The proof shows that if k is strongly measurable on X and (5.1) holds, then one has the

(b) Under the stronger assumption that $k \in L_1\left(\mathbb{R}^N, \mathcal{B}(X,Y)\right)$, for $1 \leq p \leq \infty$ the Bochner integrals

$$(K_p f)(t) \;:=\; \int_{\mathbb{R}^N} k(t-s)\, f(s)\; ds$$
$$f \in L_p\left(\mathbb{R}^N, X\right) \cap L_\infty\left(\mathbb{R}^N, X\right) \quad \text{and} \quad t \in \mathbb{R}^N$$

exist and define a bounded linear operator

$$K_p \colon L_p\left(\mathbb{R}^N, X\right) \to L_p\left(\mathbb{R}^N, Y\right) \;.$$

This fact is well-known and easy to show; indeed, for $f \in L_p\left(\mathbb{R}^N, X\right) \cap L_\infty\left(\mathbb{R}^N, X\right)$ and $f_s(t) := f(t-s)$,

$$\int_{\mathbb{R}^N} \|k(t-s)\, f(s)\|_Y \; ds \;=\; \int_{\mathbb{R}^N} \|k(s)\, f_s(t)\|_Y \; ds \;\leq\; \|k\|_{L_1(\mathbb{R}^N, \mathcal{B}(X,Y))} \; \|f\|_{L_\infty(\mathbb{R}^N, X)}$$

for each $t \in \mathbb{R}^N$ and

$$\begin{aligned} \|(Kf)(\cdot)\|_{L_p(\mathbb{R}^N, Y)} \;&\leq\; \int_{\mathbb{R}^N} \|k(s)\, f_s(\cdot)\|_{L_p(\mathbb{R}^N, Y)}\; ds \\ &\leq\; \int_{\mathbb{R}^N} \|k(s)\|_{\mathcal{B}(X,Y)}\, \|f_s(\cdot)\|_{L_p(\mathbb{R}^N, X)}\; ds \\ &=\; \|k\|_{L_1(\mathbb{R}^N, \mathcal{B}(X,Y))}\, \|f\|_{L_p(\mathbb{R}^N, X)} \;; \end{aligned}$$

thus, $\|K_p\|_{L_p \to L_p} \leq \|k\|_{L_1(\mathbb{R}^N, \mathcal{B}(X,Y))}$.

If, in addition, k satisfies (5.1) and k^* satisfies (5.2) with $Z = Y^*$, then it was shown in [8, Lemma 4.5] that $\|K_p\|_{L_p \to L_p} \leq (C_1)^{\frac{1}{p}}\, (C_\infty)^{\frac{1}{p'}}$. □

References

[1] Jöran Bergh and Jörgen Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976, Grundlehren der Mathematischen Wissenschaften, No. 223.

[2] Ruth F. Curtain and Hans Zwart, *An introduction to infinite-dimensional linear systems theory*, Springer-Verlag, New York, 1995.

[3] J. Diestel and J. J. Uhl, Jr., *Vector measures*, American Mathematical Society, Providence, R.I., 1977, With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[4] Joseph Diestel, *Sequences and series in Banach spaces*, Graduate Texts in Mathematics, vol. 92, Springer-Verlag, New York, 1984.

[5] Gerald B. Folland, *Real analysis*, Pure and Applied Mathematics, John Wiley & Sons Inc., New York, 1984, Modern techniques and their applications, A Wiley-Interscience Publication.

[6] José García-Cuerva and José L. Rubio de Francia, *Weighted norm inequalities and related topics*, North-Holland Publishing Co., Amsterdam, 1985, Notas de Matemática [Mathematical Notes], 104.

[7] Maria Girardi and Lutz Weis, *Operator-valued Fourier multiplier theorems on $L_p(X)$ and geometry of Banach spaces*, Journal of Functional Analysis **204** no. 2 (2003), 320–354.

[8] Maria Girardi and Lutz Weis, *Operator-valued Fourier multiplier theorems on Besov spaces*, Math. Nachr. **251** (2003), 34–51.

[9] Joram Lindenstrauss and Lior Tzafriri, *Classical Banach spaces. I*, Springer-Verlag, Berlin, 1977, Sequence spaces, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92.

[10] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

[11] Jan Prüss, *Evolutionary integral equations and applications*, Birkhäuser Verlag, Basel, 1993.

Mathematisches Institut I, Universität Karlsruhe, Englerstrasse 2, 76128 Karlsruhe, Germany
E-mail address: Lutz.Weis@math.uni-karlsruhe.de