INTEGRAL OPERATORS WITH OPERATOR-VALUED KERNELS

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Abstract. Under fairly mild measurability and integrability conditions on operator-valued kernels, boundedness results for integral operators on Bochner spaces $L^p(X)$ are given. In particular, these results are applied to convolutions operators.

1. INTRODUCTION

One of the most commonly used boundedness criterion for integral operators states that, for $1 \leq p \leq \infty$ and $\sigma$-finite measure spaces $(T, \Sigma_T, \mu)$ and $(S, \Sigma_S, \nu)$, a measurable kernel $k : T \times S \to \mathbb{C}$ defines a bounded linear operator

$$K : L^p(S, \mathbb{C}) \to L^p(T, \mathbb{C}) \quad \text{via} \quad (Kf)(\cdot) := \int_S k(\cdot, s) f(s) \, d\nu(s)$$

provided

$$\sup_{s \in S} \int_T |k(t, s)| \, d\mu(t) \leq C \quad \text{and} \quad \sup_{t \in T} \int_S |k(t, s)| \, d\nu(s) \leq C \quad (1.1)$$

(see, e.g. [5, Theorem 6.18]). In the theory of evolution equations one frequently uses operator-valued analogs of this situation, where the kernel $k$ maps $T \times S$ into the space $B(X, Y)$ of bounded linear operators from a Banach space $X$ into a Banach space $Y$ and then one desires the boundedness of the corresponding integral operator

$$K : L_p(S, X) \to L_p(T, Y) \ .$$

Such integral operators appear, for example, in solution formulas for inhomogeneous Cauchy problems (see, e.g. [10]) and for Volterra integral equations (see, e.g. [11]) as well as in control theory (see, e.g. [2]); furthermore, the stability of such solutions is often expressed in terms of the boundedness of these operators.

However, difficulties can easily arise since in many situations the kernel $k$ is not measurable with respect to the operator norm because the range of $k$ is not (essentially) valued in a separable subspace of $B(X, Y)$. This paper presents boundedness results for integral operators with operator-valued kernels under relatively mild measurability and integrability conditions on the kernels.
The first step is to place a mild measurability condition on a kernel $k: T \times S \to B(X, Y)$ to guarantee that if $f$ is in the space $\mathcal{E}(S, X)$ of finitely-valued finitely-supported measurable functions then the Bochner integrals

$$(Kf)(\cdot) := \int_S k(\cdot, s) [f(s)] \, d\nu(s)$$

(1.2)

define a measurable function from $T$ into $Y$, thus defining a mapping

$K: \mathcal{E}(S, X) \to L_0(T, Y).$

Then, to ensure that $K$ linearly extends to a desired superspace, one adds integrability conditions, which replace (1.1) in the scalar case and, roughly speaking, take the form

$$\sup_{s \in S} \int_T \|k(t, s)x\|_Y \, d\mu(t) \leq C \|x\|_X \quad \text{for each } x \in X$$

(1.3)

$$\sup_{t \in T} \int_S \|k^*(t, s)y^*\|_{X^*} \, d\nu(s) \leq C \|y^*\|_{Y^*} \quad \text{for each } y^* \in Y^*$$

(1.4)

along with appropriate measurability conditions (see Section 3 for the precise formulations). Assume $k$ has the appropriate measurability conditions. Theorem 3.4 shows that if $k$ satisfies (1.3) then $K$ extends to a bounded linear operator from $L_1(S, X)$ into $L_1(T, Y)$; Theorem 3.6 shows that if $k$ satisfies (1.4) then $K$ extends to a bounded linear operator from the closure of $\mathcal{E}(S, X)$ in the $L_\infty$-norm into $L_\infty(T, Y)$. Then Theorem 3.8 uses an interpolation argument to show that if $k$ satisfies (1.3) and (1.4) then $K$ extends to a bounded linear operator from $L_p(S, X)$ into $L_p(T, Y)$ for $1 < p < \infty$. The case $p = \infty$ is more delicate since $\mathcal{E}(S, X)$ is not necessarily dense in $L_\infty(S, X)$.

Theorem 3.11 shows that if $k$ satisfies (1.4) then $K$ can be extended to a bounded linear operator from $L_\infty(S, X)$ into the space of $w^*$-measurable $\mu$-essentially bounded functions from $T$ into $Y^{**}$ where the integrals in (1.2) exists (a.e) as Dunford integrals for each $f \in L_\infty(S, X)$; also, sufficient conditions are given to guarantee that $K$ maps $L_\infty(S, X)$ into $L_\infty(T, Y)$. Using ideas from the Geometry of Banach Spaces, Example 3.13 shows that, without further assumptions, it is necessary to pass to $Y^{**}$ in Theorem 3.11.

As an immediate consequence of these results, Corollary 5.1 gives boundedness results for convolution operators with operator-valued kernels. A similar result, which inspired this paper, was used to obtain operator-valued Fourier multiplier results [8, 7].

2. NOTATION AND BASICS

Throughout this paper, $X$, $Y$, and $Z$ are Banach spaces over the field $\mathbb{K}$ of $\mathbb{R}$ or $\mathbb{C}$. Also, $X^*$ is the (topological) dual of $X$ and $B(X)$ is the (closed) unit ball of $X$. The space $B(X, Y)$ of bounded linear operators from $X$ into $Y$ is endowed with the usual uniform operator topology.

A subspace $Z$ of $Y^* \tau$-norms $Y$, where $\tau \geq 1$, provided
If \( Z \) \( \tau \)-norms \( Y \), then the natural mapping
\[ j : Y \to Z^* \]
given by \( \langle z, jy \rangle := \langle y, z \rangle \) for \( z \in Z \)
is an isomorphic embedding with
\[ \frac{1}{\tau} \|y\|_Y \leq \|j(y)\|_{Z^*} \leq \|y\|_Y, \]
in which case \( Y \) is identified as a subspace of \( Z^* \).

\((T, \Sigma_T, \mu)\) and \((S, \Sigma_S, \nu)\) are \( \sigma \)-finite (positive) measure spaces;
\[ \Sigma_S^\text{finite} := \{ A \in \Sigma_S : \nu(A) < \infty \} \]
\[ \Sigma_S^\text{full} := \{ A \in \Sigma_S : \nu(S \setminus A) = 0 \}, \]
with similar notation for the corresponding subsets of \( \Sigma_T \).

\( E(S, X) \) is the space of finitely-valued finitely-supported measurable functions from \( S \) into \( X \), i.e.
\[ E(S, X) = \left\{ \sum_{i=1}^n x_i 1_{A_i} : x_i \in X, A_i \in \Sigma_S^\text{finite}, n \in \mathbb{N} \right\}. \]

Let \( \Gamma \) be a subspace of \( X^* \). A function \( f : S \to X \) is
- \emph{measurable} provided there is a sequence \( (f_n)_{n=1}^\infty \) from \( E(S, X) \) so that
  \[ \lim_{n \to \infty} \|f(s) - f_n(s)\|_X = 0 \text{ for } \nu\text{-a.e. } s \]
- \emph{\( \sigma(S, \Gamma) \)-measurable} provided \( (f(\cdot), x^*) : S \to K \) is measurable for each \( x^* \in \Gamma \).

The following fact will be used (c.f., e.g., [3, Corollary II.1.4]).

\textbf{Fact 2.1 (Pettis’s Measurability Theorem).} A function \( f : S \to X \) is measurable if and only if

(i) \( f \) is essentially separably valued

(ii) \( f \) is \( \sigma(S, \Gamma) \)-measurable for some subspace \( \Gamma \) of \( X^* \) that 1-norms \( X \).

\( L_0(S, X) \) is the space of (equivalence classes of) measurable functions from \( S \) into \( X \). The Bochner-Lebesgue space \( L_p(S, X) \), where \( 1 \leq p \leq \infty \), is endowed with its usual norm topology. The space \( L_{p\infty}^w(T, Z^*) \) of \( \mu \)-essentially bounded \( \sigma(Z^*, Z) \)-measurable functions from \( T \) into \( Z^* \) is endowed with the \( \mu \)-essential supremum norm, under which it becomes a Banach space.

\( E(S, X) \) is norm dense in \( L_p(S, X) \) for \( 1 \leq p < \infty \). Let \( L_0^\infty(S, X) \) be the closure of \( E(S, X) \) in the \( L_\infty(S, X) \)-norm. If \( X \) is infinite-dimensional, then \( L_0^\infty(S, X) \neq L_\infty(S, X) \) (provided \( \Sigma_S \) contains a countable number of pairwise disjoint sets of strictly positive measure). \( L_0^\infty(S, X) \) can be described as follows.

\textbf{Proposition 2.2.} Let \( f \in L_0(S, X) \). Then \( f \in L_0^\infty(S, X) \) if and only if

(1) \( \inf \left\{ \|f 1_{S \setminus A}\|_{L_\infty(S, X)} : A \in \Sigma_S^\text{finite} \right\} = 0 \)

(2) there is \( B \in \Sigma_S^\text{full} \) so that the set \( \{ f(s) : s \in B \} \) is relatively compact in \( X \).
Conversely, for \( \varepsilon > 0 \), conditions (1) and (2) give a set \( G := A \cap B \in \Sigma_S^{\text{finite}} \) so that

\[
\|f 1_{S \setminus G}\|_{L_\infty(S,X)} < \varepsilon \quad \text{and} \quad \{f(s) : s \in G\} \text{ is relatively compact ;}
\]

thus allowing one to find, via a finite covering of the set \( f(G) \) by \( \varepsilon \)-balls, a function \( f_\varepsilon \in \mathcal{E}(S,X) \), with support in \( G \), so that \( \|f - f_\varepsilon\|_{L_\infty(S,X)} < \varepsilon \).

Lemma 2.3 will help to deal with the fact that \( \mathcal{E}(S,X) \) is (usually) not norm dense in \( L_\infty(S,X) \).

**Lemma 2.3.** Let \( f \in L_\infty(S,X) \) and \( \varepsilon > 0 \). There is a sequence \( \{g_n\}_{n=1}^\infty \) from \( \mathcal{E}(S,X) \) so that

\[
f(s) = \sum_{n=1}^\infty g_n(s)
\]

\[
\sum_{n=1}^\infty \|g_n(s)\|_X \leq (1 + \varepsilon) \|f\|_{L_\infty(S,X)}
\]

for a.e. \( s \in S \).

**Proof.** Fix a sequence \( \{\varepsilon_j\}_{j=1}^\infty \) of positive numbers so that \( \varepsilon_1 = 1 \) and \( \sum_{j=1}^\infty \varepsilon_j < 1 + \varepsilon \).

Choose a sequence \( \{f_j\}_{j=1}^\infty \) from \( \mathcal{E}(S,X) \) so that, for a.e. \( s \in S \),

\[
f_j(s) \to f(s) \quad \text{as } j \to \infty
\]

\[
\|f_j(s)\|_X \leq \|f(s)\|_X
\]

for each \( j \in \mathbb{N} \).

Find a sequence \( \{S_k\}_{k=1}^\infty \) of pairwise disjoint sets from \( \Sigma_S^{\text{finite}} \) so that \( \nu(S \setminus \bigcup_{k=1}^\infty S_k) = 0 \) and, for each \( S_k \),

\[
f_j \to f \quad \text{uniformly on } S_k
\]

\[
\|f_j(s)\|_X \leq \|f(s)\|_X
\]

for each \( s \in S_k \) and \( j \in \mathbb{N} \).

Hence, on each \( S_k \), there is a sequence \( \{g_{kj}\}_{j=1}^\infty \) from \( \mathcal{E}(S_k,X) \) so that

\[
f(s) = \sum_{j=1}^\infty g_{kj}(s)
\]

\[
\left\|g_{kj}\right\|_{L_\infty(S_k,X)} \leq \varepsilon_j \|f\|_{L_\infty(S,X)}
\]

for each \( j \in \mathbb{N} \).

For \( n \in \mathbb{N} \), let

\[
g_n := \sum_{k<n} \left( g_{nk}^1 1_{S_k} \right) + \left( \sum_{j=1}^n g_{j,n}^2 \right) 1_{S_n}.
\]

Thus

\[
g_1 = g_1^1 1_{S_1}
\]

\[
g_2 = g_2^1 1_{S_1} + (g_2^2 + g_2^3) 1_{S_2}
\]

\[
g_3 = g_3^1 1_{S_1} + g_3^2 1_{S_2} + (g_3^3 + g_3^4 + g_3^5) 1_{S_3}
\]
Note that if \( s \in S_k \) then
\[
\sum_{n=1}^{\infty} g_n (s) = \sum_{j=1}^{\infty} g_j^k (s)
\]
and, by the triangle inequality,
\[
\sum_{n=1}^{\infty} \| g_n (s) \|_X \leq \sum_{j=1}^{\infty} \| g_j^k (s) \|_X.
\]
So clearly the \( g_n \)'s do as they should. \( \Box \)

Let \( 1 \leq p \leq \infty \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \). There is a natural isometric embedding of \( L_{p'} (T, Z^*) \) into \([L_p (T, Z)]^\ast\) given by
\[
\langle f, g \rangle := \int_T \langle f (t), g (t) \rangle \, d\mu (t) \quad \text{for} \quad g \in L_{p'} (T, Z^*) , \, f \in L_p (T, Z).
\]
There also is a natural isometric embedding of \( L_1 (T, Y^*) \) into \([L^w_\infty (T, Y^{**})]^\ast\); indeed, for \( g = \sum_{i=1}^{n} y_i^* 1_{B_i} \in L_1 (T, Y^*) \) and \( f \in L^w_\infty (T, Y^{**}) \) let
\[
\langle f, g \rangle := \int_T \langle g (t), f (t) \rangle \, d\mu (t) = \sum_{i=1}^{n} \int_{B_i} \langle y_i^*, f (t) \rangle \, d\mu (t)
\]
and observe that \( \| g \|_{[L^w_\infty (T, Y^{**})]^\ast} = \| g \|_{L_1 (T, Y^*)} \).

For a mapping
\[
k: T \times S \to \mathcal{B} (X, Y)
\]
the mapping
\[
k^*: T \times S \to \mathcal{B} (Y^*, X^*)
\]
is defined by \( k^* (t, s) := [k (t, s)]^\ast \).

Non-numerical subscripts on constants indicate dependency. All other notation and terminology, not otherwise explained, are as in [3, 9].

3. MAIN RESULTS

Several conditions on a kernel \( k: T \times S \to \mathcal{B} (X, Y) \) will be considered. The first one is a mild measurability condition.

**Definition 3.1.** \( k: T \times S \to \mathcal{B} (X, Y) \) satisfies condition \((C_0)\) provided that for each \( A \in \Sigma^\text{finite}_S \) and each \( x \in X \)
- there is \( T_{A,x} \in \Sigma^\text{null}_T \) so that if \( t \in T_{A,x} \) then the Bochner integral
  \[
  \int_A k (t, s) \, x \, d\nu (s)
  \]
exists
- the mapping
  \[
  T_{A,x} \ni t \mapsto \int_A k (t, s) \, x \, d\nu (s) \in X
  \]
**Remark 3.2.** Let $k: T \times S \to B(X, Y)$ satisfy condition (C$_0$). Then

for each $f \in E(S, X)$ there is $T_f \in \Sigma^\text{full}_T$ so that

\[
\text{if } t \in T_f \text{ then the Bochner integral } (Kf)(t) := \int_S k(t, s) [f(s)] \, d\nu(s) \text{ exists}
\]

(3.1)

and (3.1) defines a linear mapping

\[
K: E(S, X) \to L_0(T, Y)
\]

(3.2)

Next integrability conditions on $k$ are added to ensure that the mapping $K$ in (3.2) extends to the desired superspaces. \hfill \Box

Condition (C$_1$) will be used for the $L_1$-case in Theorem 3.4.

**Definition 3.3.** Let $k: T \times S \to B(X, Y)$ satisfies condition (C$_1$) provided there is a constant $C_1$ so that for each $x \in X$

- the mapping $T \times S \ni (t, s) \to \|k(t, s)x\|_Y \in \mathbb{R}$ is product measurable
- there is $S_x \in \Sigma^\text{full}_S$ so that

\[
\int_T \|k(t, s)x\|_Y \, d\mu(t) \leq C_1 \|x\|_X
\]

for each $s \in S_x$. \hfill \Box

Note that the first condition guarantees that the mapping $T \ni t \to \|k(t, s)x\|_Y \in \mathbb{R}$ is measurable for $\nu$-a.e. $s$. Also, the first condition is often satisfied even though the mapping $T \times S \ni (t, s) \to k(t, s)x \in Y$ may not be product measurable.

The $L_1$-case is a straightforward extension of the scalar-valued situation.

**Theorem 3.4.** Let $k: T \times S \to B(X, Y)$ satisfy conditions (C$_0$) and (C$_1$). Then the integral operator

\[
(Kf)(\cdot) := \int_S k(\cdot, s)f(s) \, d\nu(s) \quad \text{for } f \in E(S, X)
\]

extends to a bounded linear operator

\[
K: L_1(S, X) \to L_1(T, Y)
\]

of norm at most the constant $C_1$ from Definition 3.3.

**Proof.** Fix $f = \sum_{i=1}^n x_i 1_{A_i} \in E(S, X)$ with the $A_i$’s disjoint. By condition (C$_1$), for each $i$,

\[
\int_T \int_S \|k(t, s)x_i 1_{A_i}(s)\|_Y \, d\nu(s) \, d\mu(t) = \int_S \int_T \|k(t, s)x_i\|_Y \, d\mu(t) \, 1_{A_i}(s) \, d\nu(s)
\]

\[
\leq \int C_1 \|x_i\|_Y 1_{A_i}(s) \, d\nu(s)
\]

(3.3)
Condition \((C_0)\) gives that \(Kf \in L_0(T,Y)\) and also, combined with \((3.3)\), that
\[
\|Kf\|_{L^1(T,Y)} = \int_T \left\| \sum_{i=1}^n \int_S k(t,s) x_i 1_{A_i}(s) \, d\nu(s) \right\|_Y d\mu(t)
\leq \sum_{i=1}^n \int_T \int_S \|k(t,s) x_i 1_{A_i}(s)\|_Y \, d\nu(s) \, d\mu(t)
\leq \sum_{i=1}^n C_1 \|x_i\|_X \nu(A_i) = C_1 \|f\|_{L^1(S,X)}.
\]
This completes the proof. \(\Box\)

Condition \((C_0)\) will be used for the \(L_0^0\)-case in Theorem 3.6.

**Definition 3.5.** Let \(Z\) be a subspace of \(Y^*\). Then \(k: T \times S \to \mathcal{B}(X,Y)\) satisfies condition \((C_0)\), with respect to \(Z\), provided there is a constant \(C_0^0\) so that for each \(y^* \in Z\) there is \(T_{y^*} \in \Sigma_T^{\text{full}}\) so that for each \(t \in T_{y^*}\),

- the mapping \(S \ni s \mapsto \|k^* (t,s) y^*\|_{X^*} \in \mathbb{R}\) is measurable
- \(\int_S \|k^* (t,s) y^*\|_{X^*} \, d\nu(s) \leq C_0^0 \|y^*\|_{Y^*} \). \(\Box\)

**Theorem 3.6.** Let \(Z\) be a subspace of \(Y^*\) that \(\tau\)-norms \(Y\). Let \(k: T \times S \to \mathcal{B}(X,Y)\) satisfy conditions \((C_0)\) and \((C_0^0)\) with respect to \(Z\). Then the integral operator
\[
(Kf)(\cdot) := \int_S k(\cdot,s) f(s) \, d\nu(s) \quad \text{for } f \in \mathcal{E}(S,X)
\]
extends to a bounded linear operator
\[
K: L_0^0(S,X) \to L_0^\infty(T,Y)
\]
of norm at most \(\tau \cdot C_0^0\) where the constant \(C_0^0\) is from Definition 3.5.

**Proof.** Fix \(f \in \mathcal{E}(S,X)\). Fix \(y^* \in Z\). Find the corresponding sets \(T_f, T_{y^*} \in \Sigma_T^{\text{full}}\) from the definitions of conditions \((C_0)\) and \((C_0^0)\). If \(t \in T_f \cap T_{y^*}\) then
\[
|\langle (Kf)(t), y^* \rangle| = \left| \left\langle \int_S k(t,s) f(s) \, d\nu(s), y^* \right\rangle \right|
\leq \int_S \left| \langle f(s), k^*(t,s) y^* \rangle \right| \, d\nu(s)
\leq \int_S \|k^*(t,s) y^*\|_{X^*} \|f(s)\|_X \, d\nu(s)
\leq \|f\|_{L_0^\infty(S,X)} C_0^0 \|y^*\|_{Y^*}.
\]
Since \(T_f \cap T_{y^*} \in \Sigma_T^{\text{full}}\) and \(Z \tau\)-norms \(Y\),
\[
\|Kf\|_{L_0^\infty(T,Y)} \leq \tau C_0^0 \|f\|_{L_0^\infty(S,X)}.
\]
Remark 3.7 (on Theorem 3.6). Note that $K$ maps $L^0_\infty (S, X)$ into $L^0_\infty (T, Y)$ provided that for each $x \in X$ and $A \in \Sigma_S^{\text{finite}}$
\begin{equation}
T_f \ni t \mapsto \int_A k(t, s) x \, d\nu(s) \in Y
\end{equation}
defines a function in $L^0_\infty (T, Y)$. This will be the case, for example, if $\mu$ is a Radon measure on a locally compact Hausdorff space $T$ (e.g., $T$ is a Borel subset of $\mathbb{R}^N$, endowed with the Lebesgue measure) and (3.4) defines a function in $C^0_0 (T, Y)$: =
\begin{equation*}
\sup_{B \in \mathcal{B} \cap \text{finite}} \mathcal{B} \ni g \mapsto \inf \left\{ \|g 1_{T \setminus B}\|_{L^\infty} : B \text{ is compact} \right\} = 0
\end{equation*}
indeed, conditions (1) and (2) of Proposition 2.2 are then fulfilled.
\end{proof}
Interpolating between Theorems 3.4 and 3.6 gives the $L^p$-case for $1 < p < \infty$.

\begin{theorem}
Let $Z$ be a subspace of $Y^*$ that $\tau$-norms $Y$ and $1 < p < \infty$. Let $k : T \times S \to \mathcal{B} (X, Y)$ satisfy conditions $(C_0)$, $(C_1)$, and $(C^0_\infty)$ with respect to $Z$. Then the integral operator
\begin{equation*}
(Kf)(\cdot) := \int_S k(\cdot, s) f(s) \, d\nu (s) \quad \text{for } f \in \mathcal{E} (S, X)
\end{equation*}
extends to a bounded linear operator
\begin{equation*}
K : L^p (S, X) \to L^p (T, Y)
\end{equation*}
of norm at most $(C_1)^{1/p} (\tau : C^0_\infty)^{1/p'}$ where the constants $C_1$ and $C^0_\infty$ are from Definitions 3.3 and 3.5.
\begin{proof}
The proof follows directly from Theorems 3.4 and 3.6 and Lemma 3.9.
\end{proof}

The below interpolation lemma is a slight improvement on [1, Thm. 5.1.2].

\begin{lemma}
Let the linear mapping
\begin{equation*}
K : \mathcal{E} (S, X) \to L^1 (T, Y) + L^\infty (T, Y)
\end{equation*}
satisfy, for each $f \in \mathcal{E} (S, X)$,
\begin{align*}
\|Kf\|_{L^1 (T, Y)} & \leq c_1 \|f\|_{L^1 (S, X)} < \infty \\
\|Kf\|_{L^\infty (T, Y)} & \leq c_\infty \|f\|_{L^\infty (S, X)} < \infty .
\end{align*}
Then, for each $1 < p < \infty$, the mapping $K$ extends to a bounded linear operator
\begin{equation*}
K : L^p (S, X) \to L^p (T, Y)
\end{equation*}
of norm at most $(c_1)^{1/p} (c_\infty)^{1/p'}$.
\begin{proof}
Fix $B \in \Sigma_T^{\text{finite}}$ and a finite measurable partition $\pi$ of $B$. Let $\Sigma_0 := \sigma_B (\pi)$ be the $\sigma$-algebra of subsets of $B$ that is generated by $\pi$. Then the linear mapping
given by $K_0 f := \mathbb{E} ((K f) 1_B \mid \Sigma_0)$ where $\mathbb{E} (\cdot \mid \Sigma_0)$ is the conditional expectation operator relative to $\Sigma_0$, satisfies

$$\|K_0 f\|_{L_1(T,Y)} \leq c_1 \|f\|_{L_1(S,X)} < \infty$$

$$\|K_0 f\|_{L_\infty(T,Y)} \leq c_\infty \|f\|_{L_\infty(S,X)} < \infty$$

for each $f \in \mathcal{E}(S,X)$. Furthermore, $K_0 : L^0_\infty(S,X) \to L^0_\infty(T,Y)$. Thus, by [1, Thm. 5.1.2], for each $p \in (1,\infty)$, the linear mapping $K_0$ extends to a bounded linear operator from $L_p(S,X)$ to $L_p(T,Y)$ of norm at most $(c_1)^{1/p} (c_\infty)^{1/p'}$.

Next, fix $f \in \mathcal{E}(S,X)$ and $p \in (1,\infty)$. By assumption, $Kf \in L_1(T,Y) \cap L_\infty(T,Y)$; thus, $Kf \in L_p(T,Y)$. Fix $B \in \Sigma^\text{finite}_T$. Since $T$ is $\sigma$-finite, it suffices to show

$$\|(Kf) 1_B\|_{L_p(T,Y)} \leq (c_1)^{1/p} (c_\infty)^{1/p'} \|f\|_{L_p(S,X)} \tag{3.5}$$

Find a sequence $\{g_n\}_{n=1}^\infty$ of functions from $\mathcal{E}(T,Y)$ that are supported on $B$ and, for $\mu$-a.e. $t$,

$$g_n \rightarrow (Kf)(t) \quad 1_B(t)$$

$$\|g_n(t)\|_Y \leq \|(Kf)(t)\|_{1_B} \tag{3.6}$$

Let $\Sigma_n := \sigma(g_1, \ldots, g_n)$ be the $\sigma$-algebra of subsets of $B$ that is generated by $\{g_1, \ldots, g_n\}$. Note that $(Kf) 1_B$ is the limit in $L_p(T,Y)$ of $\{g_n\}_{n=1}^\infty$ (by (3.6)) and thus also of $\{\mathbb{E}((Kf) 1_B \mid \Sigma_n)\}_{n=1}^\infty$ since

$$\|(Kf) 1_B - \mathbb{E}((Kf) 1_B \mid \Sigma_n)\|_{L_p(T,Y)} \leq \|(Kf) 1_B - g_n\|_{L_p(T,Y)} + \|\mathbb{E}(g_n - (Kf) 1_B \mid \Sigma_n)\|_{L_p(T,Y)} .$$

But by the previous paragraph, for each $n \in \mathbb{N}$,

$$\|\mathbb{E}((Kf) 1_B \mid \Sigma_n)\|_{L_p(T,Y)} \leq (c_1)^{1/p} (c_\infty)^{1/p'} \|f\|_{L_p(S,X)} .$$

Thus (3.5) holds. \qed

Condition $(C_\infty)$, a strengthening of condition $(C_\infty^0)$, will be used for the $L_\infty$-case in Theorem 3.11.

**Definition 3.10.** Let $Z$ be a subspace of $Y^\ast$. Then $k : T \times S \rightarrow \mathcal{B}(X,Y)$ satisfies condition $(C_\infty)$, with respect to $Z$, provided there is a constant $C_\infty$ and $T_0 \in \Sigma^\text{full}_T$ so that for each $t \in T_0$ and $y^\ast \in Z$

- the mapping $S \ni s \rightarrow k^\ast(t,s)y^\ast \in X^\ast$ is measurable
- $\int_S \|k^\ast(t,s)y^\ast\|_{X^\ast} \, d\nu(s) \leq C_\infty \|y^\ast\|_{Y^\ast}$ \hfill \qed

The $L_\infty$-case is more delicate since, in general, $\mathcal{E}(S,X)$ is not norm dense in $L_\infty(S,X)$.

**Theorem 3.11.** Let $Z$ be a subspace of $Y^\ast$ that $\tau$-norms $Y$. Let $k : T \times S \rightarrow \mathcal{B}(X,Y)$ satisfy conditions $(C_0)$ and $(C_\infty)$ with respect to $Z$. In particular, $(C_\infty)$ gives that for each $t \in T_0 \in \Sigma^\text{full}_T$ and each $y^\ast \in Z$

$$\int_S \|k^\ast(t,s)y^\ast\|_{X^\ast} \, d\nu(s) \leq C_\infty \|y^\ast\|_{Y^\ast} .$$
defined by
\[(Kf)(\cdot) := \int_S k(\cdot, s) f(s) \, dv(s) \quad \text{for } f \in \mathcal{E}(S, X)\] (3.7)
extends (identifying \(Y\) as a subspace of \(Z^*\)) to a bounded (of norm at most \(C_1\)) linear operator
\[K : L_\infty(S, X) \to L_\infty^w(T, Z^*)\] (3.8)
that is given by
\[\langle y^*, (Kf)(t) \rangle := \int_S \langle k(t, s) f(s), y^* \rangle \, dv(s) \quad \text{for } f \in L_\infty(S, X) \text{ and } t \in T_0 \text{ and } y^* \in Z.\] (3.9)
Furthermore, the \(K\) of (3.8) maps \(L_1(S, X)\) into \(L_w^1(T, Z)\) provided either
(i) \(Y\) does not (isomorphically) contain \(c_0\)

or
(ii) for each \(t \in T_0\) the subset \(\{\|k^*(t, \cdot) y^*\|_{X^*} : y^* \in B(Z)\}\) of \(L_0(S, \mathbb{R})\) is equi-integrable.

Recall that the subset in (ii) is equi-integrable provided if \(\{A_n\}_{n=1}^\infty\) is a sequence from \(\Sigma_S\) with \(A_n \supseteq A_{n+1}\) and \(\nu(\cap_{n=1}^\infty A_n) = 0\) then
\[\lim_{n \to \infty} \sup_{y^* \in B(Z)} \int_{A_n} \|k^*(t, s) y^*\|_{X^*} \, dv(s) = 0.\] (3.10)

Remark 3.12 (on Theorem 3.11).

(a) There can be advantages in taking a proper norming subspace \(Z \subseteq Y^*\) over taking \(Z = Y^*\).

First, it eases the assumptions of \(K\). Second, \(Z^*\) may be much smaller than \(Y^{**}\) and so the conclusion \(K(L_\infty(S, X)) \subseteq L_\infty^w(T, Z^*)\) may be more useful than \(K(L_\infty(S, X)) \subseteq L_\infty^w(T, Y^{**})\).

For example, if \(Y = C[0, 1]\) then \(Z := L_1[0, 1] \subseteq Y^*\) 1-norms \(Y\); furthermore, \(Z^* \simeq L_\infty[0, 1]\) is nicer than \(Y^{**} \simeq (M[0, 1])^*\), which is very large.

(b) If \(Y = (Y_*)^*\) is a separable dual space, then \(Z := Y_* \subseteq Y^*\) 1-norms \(Y\) and, by Pettis’s measurability theorem (Fact 2.1), one has that \(L_\infty^w(T, Z^*) = L_\infty(T, Y)\).

(c) If \(\nu(S) < \infty\) and for each \(t \in T_0\) there exists \(q_t \in (1, \infty]\) and \(C_t \in (0, \infty)\) such that
\[\sup_{y^* \in B(Z)} \|k^*(t, \cdot) y^*\|_{L_{q_t}(S, X)} \leq C_t,\]
then the equi-integrability condition in (ii) holds; indeed, just apply Hölder’s inequality.

(d) Remark 3.7 is valid in this setting also.

Proof of Theorem 3.11. Fix \(f \in L_\infty(S, X)\). Fix \(t \in T_0\). For each \(y^* \in Z\) the function
\[\langle k(t, \cdot) f(\cdot), y^* \rangle : S \to \mathbb{K}\] is the condition (3.8).
(b) in $L_1(S, \mathbb{K})$ with
\[
\int_S |\langle k(t, s) f(s), y^* \rangle| \, d\nu(s) \leq \int_S \|k^*(t, s) y^*\|_{X^*} \|f(s)\|_X \, d\nu(s) 
\leq C_\infty \|y^*\|_Z \|f\|_{L_\infty(S, X)} .
\] (3.11)

Thus by the Closed Graph Theorem, applied to the mapping
\[
Z \ni y^* \mapsto \langle k(t, \cdot) f(\cdot), y^* \rangle \in L_1(S, \mathbb{K}) ,
\]
the mapping
\[
Z \ni y^* \mapsto \int_S \langle k(t, s) f(s), y^* \rangle \, d\nu(s) \in \mathbb{K}
\]
defines an element $(Kf)(t)$ of $Z^*$ that satisfies (3.9).

Let $f \in \mathcal{E}(S, X)$. By condition $(C_0)$, there is $T_f \in \Sigma_T^{\text{full}}$ such that if $t \in T_f$ then $k(t, \cdot) f(\cdot) \in L_1(S, Y)$. For each $t \in T_f \cap T_0 \in \Sigma_T^{\text{full}}$ and each $y^* \in Z$
\[
\left\langle \int_S k(t, s) f(s) \, d\nu(s), y^* \right\rangle = \int_S \langle k(t, s) f(s), y^* \rangle \, d\nu(s) = \langle y^*, (Kf)(t) \rangle .
\]

Hence (3.7) holds. Thus, by Theorem 3.6, $K$ maps $\mathcal{E}(S, X)$ into $L_\infty(T, Y)$.

Fix $f \in L_\infty(S, X)$. To see that $Kf$ is $\sigma(Z^*, Z)$-measurable, fix a sequence $\{f_n\}_{n \in \mathbb{N}}$ from $\mathcal{E}(S, X)$ that converges a.e. to $f$ and $\|f_n\|_{L_\infty(S, X)} \leq \|f\|_{L_\infty(S, X)}$ for each $n \in \mathbb{N}$. Then, by the Lebesgue Dominated Convergence Theorem, for each $y^* \in Z$ and for a.e. $t \in T$
\[
\langle y^*, (Kf)(t) \rangle = \int_S \langle f(s), k^*(t, s) y^* \rangle \, d\nu(s) 
= \lim_{n \to \infty} \int_S \langle f_n(s), k^*(t, s) y^* \rangle \, d\nu(s) = \lim_{n \to \infty} \langle y^*, (Kf_n)(t) \rangle
\]
and the latter functions $\{\langle y^*, (Kf_n)(\cdot) \rangle\}$ are $\mu$-measurable functions by condition $(C_0)$. Furthermore, by (3.11)
\[
\left| \sup_{t \in T_0} \| (Kf)(t) \|_{Z^*} = \sup_{t \in T_0} \sup_{y^* \in B(Z)} \left| \int_S \langle k(t, s) f(s), y^* \rangle \, d\nu(s) \right| \right| \leq C_\infty \|f\|_{L_\infty(S, X)}
\]
thus, $Kf \in L_\infty^w(T, Z^*)$ and the $K$ of (3.8) is of norm at most $C_\infty$.

**Proof of (i)** Assume that $c_0$ does not isomorphically embed into $Y$.

Fix $f \in L_\infty(S, X)$. By Lemma 2.3, there is a sequence $\{f_n\}_{n=1}^\infty$ from $\mathcal{E}(S, X)$ so that
\[
f(s) = \sum_{n=1}^\infty f_n(s)
\]
for a.e. $s \in S$. Since each $f_n$ is in $\mathcal{E}(S, X)$, by (3.7) and condition $(C_0)$, there is $T_1 \in \Sigma_T^{\text{full}}$, with $T_1 \subseteq T_0$, so that for each $t \in T_1$ and each $n \in \mathbb{N}$
\[
(Kf_n)(t) = \int k(t, s) f_n(s) \, d\nu(s) \in Y,
\]
Fix \( t \in T_1 \). By the Lebesgue Dominated Convergence Theorem,
\[
(Kf)(t) = \lim_{m \to \infty} \sum_{n=1}^{m} (Kf_n)(t) \quad \text{in the } \sigma(Z^*,Z)\text{-topology (3.13)}
\]
since, for each \( y^* \in Z \),
\[
\langle y^*, (Kf)(t) \rangle = \int_{S} \langle f(s), k^*(t,s) y^* \rangle \, d\nu(s)
\]
\[
= \lim_{m \to \infty} \int_{S} \left( \sum_{n=1}^{m} f_n(s), k^*(t,s) y^* \right) \, d\nu(s) = \lim_{m \to \infty} \left( \sum_{n=1}^{m} (Kf_n)(t), y^* \right).
\]
It suffices to show that the series in (3.13) converges also in the \( Z^*\)-norm topology; thus, since \( Y \) does not contain \( c_0 \), it suffices to show that
\[
\sum_{n=1}^{\infty} \langle (Kf_n)(t), y^* \rangle < \infty \quad \text{for each } y^* \in Y^* \quad (3.14)
\]
by a theorem of Bessaga and Pełczyński (cf., e.g., [4, Thm. V.8]).

For each \( y^* \in Z \),
\[
\sum_{n=1}^{\infty} \langle (Kf_n)(t), y^* \rangle = \sum_{n=1}^{\infty} \left| \int_{S} \langle k(t,s) f_n(s), y^* \rangle \, d\nu(s) \right|
\]
\[
\leq \sum_{n=1}^{\infty} \int_{S} \left| \langle f_n(s), k^*(t,s) y^* \rangle \right| \, d\nu(s)
\]
\[
\leq \sum_{n=1}^{\infty} \int_{S} \| k^*(t,s) y^* \|_{X^*} \| f_n(s) \|_{X} \, d\nu(s)
\]
\[
\leq \int_{S} \| k^*(t,s) y^* \|_{X^*} \left( \sum_{n=1}^{\infty} \| f_n(s) \|_{X} \right) \, d\nu(s)
\]
\[
\leq 2 \| f \|_{L_\infty(S,X)} C_{\infty} \| y^* \|_{Y^*}.
\]
Thus the mapping
\[
Z \ni y^* \overset{U}{\longrightarrow} \{ \langle (Kf_n)(t), y^* \rangle \}_{n=1}^{\infty} \in \ell_1
\]
is a bounded linear operator. Fix \( \{ \alpha_n \}_{n=1}^{\infty} \in B(\ell_\infty) \) and \( m \in \mathbb{N} \). If \( y^* \in B(Z) \) then
\[
\left| \langle \sum_{n=1}^{m} \alpha_n (Kf_n)(t), y^* \rangle \right| = \sum_{n=1}^{m} \alpha_n \langle (Kf_n)(t), y^* \rangle \leq \| U \|_{B(Z,\ell_1)}
\]
and so
\[
\left\| \sum_{n=1}^{m} \alpha_n (Kf_n)(t) \right\|_{Y} \leq \tau \| U \|_{B(Z,\ell_1)}.
\]
Fix \( f \in L_\infty(S, X) \). Choose a sequence \( \{f_n\}_{n=1}^\infty \) from \( \mathcal{E}(S, X) \) such that
\[
\lim_{n \to \infty} f_n(s) = f(s) \quad \text{for a.e. } s \in S
\]
\[
\|f_n\|_{L_\infty(S, X)} \leq \|f\|_{L_\infty(S, X)} \quad \text{for each } n \in \mathbb{N}.
\]
As in the proof of (i), since the \( f_n \)'s are in \( \mathcal{E}(S, X) \), there is \( T_1 \in \Sigma_T^{\text{full}} \), with \( T_1 \subseteq T_0 \), so that (3.12) holds for each \( t \in T_1 \) and each \( n \in \mathbb{N} \). It suffices to show that \( \{(Kf_n)(t)\}_{n=1}^\infty \) converges in \( Z^* \)-norm to \((Kf)(t)\) for each \( t \in T_1 \).

Fix \( t \in T_1 \). Fix \( \delta > 0 \) and let \( B_n := \{s \in S : \|f(s) - f_n(s)\|_X > \delta\} \) and \( A_n := \bigcup_{k=n}^\infty B_k \). Then
\[
\|(Kf)(t) - (Kf_n)(t)\|_{Z^*} = \sup_{y^* \in B(Z)} \left| \int_S \langle k(t, s)(f(s) - f_n(s)), y^* \rangle \, d\nu(s) \right|
\]
\[
\leq \sup_{y^* \in B(Z)} \int_S \|k^*(t, s)y^*\|_{X^*} \|f(s) - f_n(s)\|_X \, d\nu(s)
\]
\[
\leq \sup_{y^* \in B(Z)} \left[ \delta \int_{B_n^c} \|k^*(t, s)y^*\|_{X^*} \, d\nu(s) + 2 \|f\|_{L_\infty(S, X)} \int_{B_n} \|k^*(t, s)y^*\|_{X^*} \, d\nu(s) \right]
\]
\[
\leq \delta C_\infty + 2 \|f\|_{L_\infty(S, X)} \sup_{y^* \in B(Z)} \int_{A_n} \|k^*(t, s)y^*\|_{X^*} \, d\nu(s).\]
Note that \( \cap_{n=1}^\infty A_n \subseteq \{s \in S : f_n(s) \text{ does not converge to } f(s)\} \). Thus, by the equi-integrability assumption, \( \{(Kf_n)(t)\}_{n=1}^\infty \) converges in \( Z^* \)-norm to \((Kf)(t)\).

\[\square\]

The following example illustrates the limitations on the conclusions in Theorems 3.6 and 3.11.

**Example 3.13.** Let \( X = \mathbb{C} \) and \( Y = c_0 \). Thus
\[\mathcal{B}(X, Y) \simeq c_0 \quad \text{and} \quad \mathcal{B}(Y^*, X^*) \simeq \ell_\infty.\]

Let \( S = T = \mathbb{R} \). Define
\[
k_0 : \mathbb{R} \to c_0 \quad \quad k_0(\cdot) := \sum_{n=1}^\infty e_n 1_{I_n}(\cdot)
\]
\[
k : T \times S \to \mathcal{B}(X, Y) \quad \quad k(t, s) := k_0(t - s)
\]
where \( \{e_n\}_{n=1}^\infty \) is the standard unit vector basis of \( c_0 \) and \( I_n = [n - 1, n) \).

Since \( k_0 \in L_\infty(\mathbb{R}, c_0) \), for each \( f \in L_1(S, X) \) the Bochner integral
\[
(Kf)(t) := \int_S k(t, s)f(s) \, ds = \int_\mathbb{R} k_0(t - s)f(s) \, ds = (k_0 * f)(t)
\]
eexists for each \( t \in T \); furthermore, \( Kf \in L_\infty(T, Y) \) and \( Kf \) is uniformly continuous. From this it follows that \( k \) satisfies condition (C_0). The kernel \( k \) also satisfies condition (C_\infty) with \( Z = Y^* \) and \( T_0 = T \) since for each \( y^* \in Y^* \simeq \ell_1 \) and \( t \in T \)
\[
k^*(t, s)y^* = \sum_{n=1}^\infty y^*(e_n)1_{I_n}(t - s) \quad \text{for each } s \in S
\]
Theorem 3.6 gives that $K(L^0_\infty(S,X)) \subseteq L_\infty(T,Y)$. However, $K(E(S,X)) \not\subseteq L^0_\infty(T,Y)$. Indeed, $(K1_{I_1})(n) = e_n$ for each $n \in \mathbb{N}$ and so, since $K1_{I_1}$ is uniformly continuous, $K1_{I_1}$ does not satisfy (2) of Proposition 2.2 and so $K1_{I_1} \not\subseteq L^0_\infty(T,Y)$.

Theorem 3.11 gives that $K(L^1_\infty(S,X)) \subseteq L^{w^*_1}(T,Y^{**})$. However, $K(L^\infty(S,X)) \not\subseteq L_\infty(T,Y)$. Indeed, consider $f = 1_{(-\infty,0)} \in L_\infty(S,X)$. If $n \in \mathbb{N}$ and $0 < \delta \leq 1$ then

$$(Kf)(n-\delta) = \delta e_n + \sum_{j \in \mathbb{N}} e_{n+j} \in Y^{**} \setminus Y.$$ 

Thus $Kf \not\subseteq L_\infty(T,Y)$. \hfill \Box

4. REMARKS ON DUALITY AND WEAK CONTINUITY

The remarks in this section explore the duality and weak continuity of $K$. For this, dual versions of the four conditions in Section 3 are needed.

**Definition 4.1.** Let $(C)$ (possibly with respect to a subspace $Z$ of $Y^*$) be one of the four conditions in Section 3 on a kernel

$$k : T \times S \to B(X,Y).$$

Then $k$ satisfies condition $(C^*)$ (possibly with respect to a subspace $Z$ of $X^{**}$) provided the mapping

$$\tilde{k} : S \times T \to B(Y^*,X^*)$$

$$\tilde{k}(s,t) := [k(t,s)]^*$$

satisfies condition $(C)$ (possibly with respect to a subspace $Z$ of $X^{**}$).

For example, $k : T \times S \to B(X,Y)$ satisfies condition $(C^*_0)$ provided that for each $B \in \Sigma^\text{finite}_T$ and each $y^* \in Y^*$

- there is $S_{B,y^*} \in \Sigma^\text{full}_S$ so that if $s \in S_{B,y^*}$ then the Bochner integral
  $$\int_B k^*(t,s) y^* \, d\mu(t)$$

  exists

- the mapping
  $$S_{B,y^*} \ni s \to \int_B k^*(t,s) y^* \, d\mu(t) \in X^*$$

  defines a measurable function from $S$ into $X^*$.

\hfill \Box

**Remark 4.2.** Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Let $k : T \times S \to B(X,Y)$ be so that

(i) $k$ satisfies conditions $(C_0)$, $(C_1)$, $(C^*_0)$ with respect to $Z = Y^*$, and $(C^*_0)$

(ii) for each $y^* \in Y^*$ and each $x \in X$, the mapping $T \times S \ni (t,s) \to \langle k(t,s)x, y^* \rangle \in \mathbb{K}$ is
By Theorem 3.8 (with $Z = Y^*$), there is a bounded linear operator
\[ K : L_p(S, X) \to L_p(T, Y) \]
defined by
\[ (Kf)(\cdot) = \int_S k(\cdot, s) f(s) \, d\nu(s) \in L_p(T, Y) \quad \text{for } f \in \mathcal{E}(S, X). \]
Note that $k$ satisfies
(iv) $(C^*_T)$ by (iii) and the fact that $k$ satisfies $(C^*_0)$ with respect to $Z = Y^*$
(v) $(C^*_0)$ with respect to $Z = X$ since $k$ satisfies $(C_1)$.
So by Theorem 3.8 (with $Z = X$), there is a bounded linear operator
\[ \tilde{K} : L_{p'}(T, Y^*) \to L_{p'}(S, X^*) \]
defined by
\[ (\tilde{K}g)(\cdot) = \int_T k^*(t, \cdot) g(t) \, d\mu(t) \in L_{p'}(S, X^*) \quad \text{for } g \in \mathcal{E}(T, Y^*). \]
Note that
\[ K^*g = \tilde{K}g \quad \text{for each } g \in L_{p'}(T, Y^*) \]
since if $g = y^*1_B \in \mathcal{E}(T, Y^*) \subset L_{p'}(T, Y^*)$ and $f = x1_A \in \mathcal{E}(S, X) \subset L_p(S, X)$
\[ \langle f, K^*g \rangle = \langle Kf, g \rangle = \int_T \left( \int_S k(t, s) x1_A(s) \, d\nu(s), y^*1_B(t) \right) \, d\mu(t) = \int_T \left( \int_S \langle k(t, s) x1_A(s), y^*1_B(t) \rangle \, d\mu(t) \right) d\nu(s) \]
\[ = \int_S \left( \int_T k^*(t, s) y^*1_B(t) \, d\mu(t) \right) \, d\nu(s) = \langle f, \tilde{K}g \rangle \]
where assumption (ii) helps justify the use of Fubini’s theorem. Thus
\[ (K^*g)(\cdot) = \int_T k^*(t, \cdot) g(t) \, d\mu(t) \in L_{p'}(S, X^*) \quad \text{for } g \in \mathcal{E}(T, Y^*) \]
and $K^*$ maps $L_{p'}(T, Y^*)$ into $L_{p'}(S, X^*)$. Thus the $K$ in (4.1) is
\[ \sigma\left(L_p(S, X), L_{p'}(S, X^*)\right) \to \sigma\left(L_p(T, Y), L_{p'}(T, Y^*)\right) \]
continuous. \qed

Remark 4.3. Let $k : T \times S \to \mathcal{B}(X, Y)$ be so that
By Theorem 3.4, there is a bounded linear operator
\[ K : L_1(S, X) \to L_1(T, Y) \] (4.3)
defined by
\[ (Kf)(\cdot) = \int_S k(\cdot, s) f(s) \, d\nu(s) \in L_1(T, Y) \quad \text{for } f \in \mathcal{E}(S, X). \]
Since \( k \) satisfies condition (C_1), it satisfies condition (C_0) with respect to \( Z = X \); thus, by Theorem 3.6, there is a bounded linear operator
\[ \tilde{K} : L_0^0(T, Y^*) \to L_\infty(S, X^*) \]
defined by
\[ \left(\tilde{K}g\right)(\cdot) = \int_T k^*(t, \cdot) g(t) \, d\mu(t) \in L_\infty(S, X^*) \quad \text{for } g \in \mathcal{E}(T, Y^*). \]
Note that
\[ K^*g = \tilde{K}g \quad \text{for each } g \in L_0^0(T, Y^*) \]
since if \( g = y^*1_B \in \mathcal{E}(T, Y^*) \subset L_0^0(T, Y^*) \) and \( f = x1_A \in \mathcal{E}(S, X) \subset L_1(S, X) \) then the calculation in (4.2) shows that \( \langle f, K^*g \rangle = \langle f, \tilde{K}g \rangle \).
Thus
\[ (K^*g)(\cdot) = \int_T k^*(t, \cdot) g(t) \, d\mu(t) \in L_\infty(S, X^*) \quad \text{for } g \in \mathcal{E}(T, Y^*) \]
and \( K^* \) maps \( L_0^0(T, Y^*) \) into \( L_\infty(S, X^*) \). Thus the \( K \) in (4.3) is
\[ \sigma(L_1(S, X), L_\infty(S, X^*)) - \text{to} - \sigma(L_1(T, Y), L_0^0(T, Y^*)) \]
continuous. \( \square \)

Remark 4.4. Let \( k : T \times S \to \mathcal{B}(X, Y) \) be so that
\begin{itemize}
  \item \( k \) satisfies conditions (C_0), (C_1), (C^*_0), and (C_\infty) with respect to \( Z = X \)
  \item condition (ii) of Remark 4.2 holds
  \item \( X^* \) does not contain \( c_0 \).
\end{itemize}

By Theorem 3.4, there is a bounded linear operator
\[ K : L_1(S, X) \to L_1(T, Y) \] (4.4)
defined by
\[ (Kf)(\cdot) = \int_S k(\cdot, s) f(s) \, d\nu(s) \in L_1(T, Y) \quad \text{for } f \in \mathcal{E}(S, X). \]
By Theorem 3.11, there is a bounded linear operator
\[ \tilde{K} : L_\infty(T, Y^*) \to L_\infty(S, X^*) \]
defined by, for some \( S_0 \in \Sigma^\text{full}_S \),
\[ \langle x, (\tilde{K}g)(s) \rangle = \int \langle x, k^*(t, s) g(t) \rangle \, d\mu(t) \]
Note that
\[ K^* g = \overline{K} g \quad \text{for each } g \in L_\infty(T,Y^*) \]
since if \( g \in L_\infty(T,Y^*) \) and \( f = x 1_A \in \mathcal{E}(S,X) \subset L_1(S,X) \) then
\[
\langle f , K^* g \rangle = \langle K f , g \rangle = \\
\left( \int_T \int_S k(t,s) x 1_A(s) \, d\nu(s) , g(t) \right) \, d\mu(t) = \\
\left( \int_S \int_T \langle k(t,s) x , g(t) \rangle 1_A(s) \, d\nu(s) \, d\mu(t) \right) = \\
\left( \int_S \left( x , \mathcal{K}^*(t,s) g(t) \right) 1_A(s) \, d\nu(s) \right) = \\
\left( f , \mathcal{K} g \right)
\]
where assumption (ii) helps justify the use of Fubini’s theorem. Thus, by (3.7),
\[
(K^* g) (\cdot) = \int_T \mathcal{K}^*(t,\cdot) g(t) \, d\mu(t) \in L_\infty(S,X^*) \quad \text{for } g \in \mathcal{E}(T,Y^*)
\]
and \( K^* \) maps \( L_\infty(T,Y^*) \) into \( L_\infty(S,X^*) \). Thus the \( K \) in (4.4) is
\[
\sigma \left( L_1(S,X) , L_\infty(S,X^*) \right) \to \sigma \left( L_1(T,Y) , L_\infty(T,Y^*) \right)
\]
continuous. \( \square \)

**Remark 4.5.** Let \( k : T \times S \to \mathcal{B}(X,Y) \) be so that
- \( k \) satisfies conditions \((C_0)\), \((C_\infty)\) with respect to \( Z = Y^* \), and \((C_0^*)\)
- conditions (ii) and (iii) of Remark 4.2 hold.

Then (3.9) of Theorem 3.11, with \( Z = Y^* \), defines a bounded linear operator
\[
K : L_\infty(S,X) \to L_\infty^w(T,Y^{**}) \quad (4.5)
\]
Note that \( k \) satisfies condition \((C_1^*)\) by (iii) and the fact that \( k \) satisfies condition \((C_\infty)\) with respect to \( Z = Y^* \). So by Theorem 3.4
\[
\left( \overline{\mathcal{K}} g \right) (\cdot) := \int_T k^*(t,\cdot) g(t) \, d\mu(t) \in L_1(S,X^*) \quad \text{for } g \in \mathcal{E}(T,Y^*)
\]
defines a bounded linear operator
\[
\overline{K} : L_1(T,Y^*) \to L_1(S,X^*)
\]
Note that
\[
\overline{K} g = K^* g \quad \text{for each } g \in L_1(T,Y^*)
\]
since for \( g = y^*1_B \in \mathcal{E}(T,Y^*) \subset L_1(T,Y^*) \) and \( f \in L_\infty(S,X) \)
\[
\langle f , K^* g \rangle = \langle K f , g \rangle
\]
\[
\begin{align*}
&= \int_T \left[ \int_S \langle k(t,s)f(s), y^* \rangle \, d\nu(s) \right] 1_B(t) \, d\mu(t) \\
&= \int_S \int_T \langle f(s), k^*(t,s)y^* 1_B(t) \rangle \, d\mu(t) \, d\nu(s) \\
&= \int_S \left[ \int_T k^*(t,s)y^* 1_B(t) \, d\mu(t) \right] \, d\nu(s) \\
&= \int_S \left[ f(s), \left( \tilde{K}g \right)(s) \right] \, d\nu(s) \\
&= \left\langle f, \tilde{K}g \right\rangle
\end{align*}
\]

where assumption (ii) helps justify the use of Fubini’s theorem. Thus

\[
\left( K^* g \right)(\cdot) = \int_T k^*(t,\cdot) g(t) \, d\mu(t) \in L_1(S,X^*) \quad \text{for each } g \in \mathcal{E}(T,Y^*)
\]

and \( K^* \) maps \( L_1(T,Y^*) \) into \( L_1(S,X^*) \). So the \( K \) of (4.5) is

\[
\sigma(L_\infty(S,X), L_1(S,X^*)) \rightarrow \sigma \left( L^w_\infty(T,Y^{**}), L_1(T,Y^*) \right)
\]

continuous. Thus, since \( \mathcal{E}(S,X) \) (resp. the Schwartz class \( \mathcal{S}(\mathbb{R}^N, X) \) in the case \( S = \mathbb{R}^N \)) is \( \sigma(L_\infty(S,X), L_1(S,X^*)) \)-dense in \( L_\infty(S,X) \), many of the properties of \( K \) are determined by its restriction to \( \mathcal{E}(S,X) \) (resp. \( \mathcal{S}(\mathbb{R}^N, X) \)).

If furthermore \( Y \) does not contain \( c_0 \), then (3.9) of Theorem 3.11, with \( Z = Y^* \), defines a bounded linear operator

\[
K : L_\infty(S,X) \rightarrow L_\infty(T,Y)
\]

that is \( \sigma(L_\infty(S,X), L_1(S,X^*)) \rightarrow \sigma(L_\infty(T,Y), L_1(T,Y^*)) \) continuous.

5. CONVOLUTION OPERATORS

The results thus far are now applied to convolution operators on \( T = S = \mathbb{R}^N \) (endowed with the Lebesgue measure).

**Corollary 5.1.** Let \( Z \) be a subspace of \( Y^* \) that \( \tau \)-norms \( Y \). Let \( k : \mathbb{R}^N \rightarrow \mathcal{B}(X,Y) \) be strongly measurable on \( X \) and \( k^* : \mathbb{R}^N \rightarrow \mathcal{B}(Y^*,X^*) \) be strongly measurable on \( Z \) and

\[
\begin{align*}
\int_{\mathbb{R}^N} \| k(s)x \|_Y \, ds &\leq C_1 \| x \|_X < \infty \quad \text{for each } x \in X \quad (5.1) \\
\int_{\mathbb{R}^N} \| k^*(s)y^* \|_{X^*} \, ds &\leq C_2 \| y^* \|_{Y^*} < \infty \quad \text{for each } y^* \in Z \quad (5.2)
\end{align*}
\]

Then the convolution operator

\[
K : \mathcal{E}(\mathbb{R}^N, X) \rightarrow L_0(\mathbb{R}^N, Y)
\]

defined by

\[
(Kf)(t) := \int_{\mathbb{R}^N} k(t-s)f(s) \, ds \quad \text{for } f \in \mathcal{E}(\mathbb{R}^N, X) \quad (5.3)
\]
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\* K_0^0: L_0^0 (\mathbb{R}^N, X) \rightarrow L_0^0 (\mathbb{R}^N, Y),

and, if \( Y \) does not contain \( c_0 \), then to \( K_\infty: L_\infty (\mathbb{R}^N, X) \rightarrow L_\infty (\mathbb{R}^N, Y) \) satisfying

\[
\langle y^*, (K_\infty f) (t) \rangle = \int_{\mathbb{R}^N} \langle k(t-s) f(s), y^* \rangle \, dv(s)
\]

for \( f \in L_\infty (\mathbb{R}^N, X) \) and \( t \in \mathbb{R}^N \) and \( y^* \in Z \).

Furthermore,

\[
\|K_p\|_{L_p \rightarrow L_p} \leq (C_1)^{\frac{1}{p}} (\tau C_\infty)^{\frac{1}{p}}
\]

for \( 1 \leq p \leq \infty \) and \( \|K_\infty^0\|_{L_0^0 \rightarrow L_0^0} \leq \tau C_\infty \).

Remark 5.2. In Corollary 5.1, if \( Z = Y^* \) and either

- \( 1 < p < \infty \)
- \( p = 1 \) and \( X^* \) does not contain \( c_0 \)
- \( p = \infty \) and \( Y \) does not contain \( c_0 \),

then the dual operator

\[
K_p^*: [L_p (\mathbb{R}^N, Y)]^* \rightarrow [L_p (\mathbb{R}^N, X)]^*
\]

has the form

\[
(K_p^* \hat{g})(s) = \int_{\mathbb{R}^N} k^* (t-s) g(t) \, dt \in L_{p'} (\mathbb{R}^N, X^*)
\]

for \( g \in \mathcal{E} (\mathbb{R}^N, Y^*) \) and \( K_p^* \) maps \( L_{p'} (\mathbb{R}^N, Y^*) \) into \( L_{p'} (\mathbb{R}^N, X^*) \) and thus \( K_p \) is

\[
\sigma (L_p (\mathbb{R}^N, X), L_{p'} (\mathbb{R}^N, X^*)) \rightarrow \sigma (L_p (\mathbb{R}^N, Y), L_{p'} (\mathbb{R}^N, Y^*))
\]

continuous.

Proof of Corollary 5.1 and Remark 5.2. If \( f = x1_A \in \mathcal{E} (\mathbb{R}^N, X) \), then \( Kf = [k(\cdot) x] * 1_A \) with \( k(\cdot) x \in L_1 (\mathbb{R}^N, Y) \) and \( 1_A \in L_\infty (\mathbb{R}^N, \mathbb{R}) \). Thus for each \( f \in \mathcal{E} (\mathbb{R}^N, X) \): the Bochner integral in (5.3) exists for each \( t \in \mathbb{R}^N \), \( Kf \) is a uniformly continuous function from \( \mathbb{R}^N \) to \( Y \), and \( Kf \) vanishes at infinity. Thus \( K (\mathcal{E} (\mathbb{R}^N, X)) \subset L_0^0 (\mathbb{R}^N, Y) \).

It is straightforward to verify that the kernel

\[
k_0: \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathcal{B} (X, Y)
\]

satisfies conditions: \( (C_0), (C_1), (C_\infty) \) with respect to \( Z \) with \( T_0 = \mathbb{R}^N \), \( (C_0^*), (C_\infty^*) \) with respect to \( Z = X \), and (ii) and (iii) of Remark 4.2.

The corollary now follows from: Theorems 3.4, 3.6, 3.8, 3.11 and Remarks 4.2, 4.4, 4.5. \( \square \)

Remark 5.3 (on Corollary 5.1).

(a) The proof shows that if \( k \) is strongly measurable on \( X \) and (5.1) holds, then one has the
(b) Under the stronger assumption that \( k \in L_1(\mathbb{R}^N, \mathcal{B}(X,Y)) \), for \( 1 \leq p \leq \infty \) the Bochner integrals

\[
(K_p f)(t) := \int_{\mathbb{R}^N} k(t-s) f(s) \, ds
\]

exist and define a bounded linear operator

\[
K_p : L_p(\mathbb{R}^N, X) \to L_p(\mathbb{R}^N, Y)
\]

This fact is well-known and easy to show; indeed, for \( f \in L_p(\mathbb{R}^N, X) \cap L_\infty(\mathbb{R}^N, X) \) and \( f_s(t) := f(t-s) \),

\[
\int_{\mathbb{R}^N} \|k(t-s) f(s)\|_Y \, ds = \int_{\mathbb{R}^N} \|k(s) f_s(t)\|_Y \, ds \leq \|k\|_{L_1(\mathbb{R}^N, \mathcal{B}(X,Y))} \|f\|_{L_\infty(\mathbb{R}^N, X)}
\]

for each \( t \in \mathbb{R}^N \) and

\[
\|(Kf)(\cdot)\|_{L_p(\mathbb{R}^N,Y)} \leq \int_{\mathbb{R}^N} \|k(s) f_s(\cdot)\|_{L_p(\mathbb{R}^N,Y)} \, ds
\]

\[
\leq \int_{\mathbb{R}^N} \|k(s)\|_{\mathcal{B}(X,Y)} \|f_s(\cdot)\|_{L_p(\mathbb{R}^N,X)} \, ds
\]

\[
= \|k\|_{L_1(\mathbb{R}^N, \mathcal{B}(X,Y))} \|f\|_{L_p(\mathbb{R}^N,X)}
\]

thus, \( \|K_p\|_{L_p \to L_p} \leq \|k\|_{L_1(\mathbb{R}^N, \mathcal{B}(X,Y))} \).

If, in addition, \( k \) satisfies (5.1) and \( k^* \) satisfies (5.2) with \( Z = Y^* \), then it was shown in [8, Lemma 4.5] that \( \|K_p\|_{L_p \to L_p} \leq (C_1)^{\frac{1}{p}} (C_\infty)^{\frac{1}{p'}} \).

References


