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Operator-valued Fourier multiplier theorems on Besov spaces

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Abstract. Presented is a general Fourier multiplier theorem for operator-valued multiplier functions on vector-valued Besov spaces where the required smoothness of the multiplier functions depends on the geometry of the underlying Banach space (specifically, its Fourier type). The main result covers many classical multiplier conditions, such as Mihlin and Hörmander conditions.

1. INTRODUCTION

In recent years, Fourier multiplier theorems with operator-valued multiplier functions have found many applications in the theory of evolution equations, in particular, in connection with: maximal regularity of parabolic equations [2, 28, 27, 6, 18, 4], stability theory [29, 15, 20], elliptic operators on infinite dimensional state spaces [1], and pseudo differential operators on manifolds with singularities [25]. Of particular interest are versions of Mihlin's multiplier theorem for functions m, from $\mathbb{R}^N$ into the space $\mathcal{B}(X)$ of bounded operators on a Banach space X, which satisfy that

$$\text{(1.1)}\quad \text{the set} \qquad \left\{ (1+|t|)^{|\alpha|}\ D^{\alpha} m(t) : t \in \mathbb{R}^N \ ,\ |\alpha| \le l \right\} \qquad \text{is norm bounded ;}$$

specifically, one would like to know for which X-valued function spaces $E\left(\mathbb{R}^N, X\right)$ can one extend the operator

$$T_m \colon \mathcal{S}\left(\mathbb{R}^N, X\right) \to \mathcal{S}'\left(\mathbb{R}^N, X\right) \qquad \text{given by} \qquad T_m f \ := \ \left[m(\cdot)\, \widehat{f}(\cdot) \right]^{\vee} \ ,$$

defined on the Schwartz class, to a bounded operator from $E\left(\mathbb{R}^N, X\right)$ into itself.

For the Bochner spaces $L_p\left(\mathbb{R}^N, X\right)$, additional hypotheses are needed, specifically, extension is possible only when the underlying Banach space X has the UMD property, $1 < p < \infty$, and the set in (1.1) is R-bounded (see [9, 30, 21] for scalar-valued m and [28, 26] for operator-valued m; see also [10, 14, 4]).

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In sharp contrast to these results for $L_p\left(\mathbb{R}^N, X\right)$, it was discovered independently by H. Amann and L. Weis that for Besov spaces $B^s_{q,r}\left(\mathbb{R}^N, X\right)$, additional restrictions on X and m are not needed and all indices $s \in \mathbb{R}$ and $q, r \in [1, \infty]$ are allowed (for a detailed proof of the Mihlin case with $l = N + 1$ see [2], in [29] a more general result was announced).

This paper presents a more elementary proof of these multiplier theorems (avoiding the theory of vector-valued distributions), which has rather general assumptions on m (again expressed in terms of Besov spaces) and which also allows for the determination of the *optimal* order l of derivatives in (1.1) in terms of the geometry of the underlying Banach space X. Indeed, Corollary 4.11 gives that for a Banach space X of Fourier type $p \in [1, 2]$ one may choose $l = [N/p]+1$. Recall that a Banach space X has Fourier type $p \in [1, 2]$ provided the Fourier transform defines a bounded operator from $L_p(X)$ into $L_{p'}(X)$, i.e. the Hausdorff Young inequality holds for the exponent p. Since a Hilbert space has Fourier type 2, one recovers the classical result of J. Schwartz (cf. [5, Section 6.1]). Since each Banach space has Fourier type 1, one obtains the main result in [2]. It follows from a result of Bourgain [9] that each uniformly convex space has Fourier type p for some $p > 1$; thus, for uniformly convex spaces one may choose $l = N$. If X is a subspace of an $L_q(\Omega)$ space, then X has Fourier type $p = \min(q, q')$; thus, the l in (1.1) improves (i.e. decreases) as q tends towards 2. Furthermore, the order $l = [N/p] + 1$ is optimal with respect to the scale of Besov spaces. In the case of Bochner spaces $L_p\left(\mathbb{R}^N, X\right)$ on can obtain analogous results to the ones described above (see [13]); however, the method of proof is quite different.

The paper is organized as follows. Section 2 collects definitions and basic properties of vector-valued function spaces, in particular, Besov spaces. Section 3 contains basic estimates for the Fourier transform on Besov spaces $B^s_{q,r}\left(\mathbb{R}^N, X\right)$ for a Banach space X with Fourier type p. Section 4 presents the multiplier theorems. Theorem 4.3 gives a multiplier theorem on $L_q\left(\mathbb{R}^N, X\right)$; its assumption is much stronger than (1.1). Applying this result to the *blocks* of the Paley Littlewood decomposition gives a multiplier theorem for Besov spaces $B^s_{q,r}\left(\mathbb{R}^N, X\right)$, Theorem 4.8, whose proof is almost trivial if $q, r \in [1, \infty)$. The case $q = \infty$ or $r = \infty$ requires a bit more care. The assumptions in Theorem 4.8 are stated in a very general form. From this general formulation follows as immediate corollaries (see Corollaries 4.11, 4.13, 4.14) vector-valued generalizations of several classical multiplier theorems conditions (Mihlin's condition, Hörmanders condition, Lipschitz condition).

Theorem 4.8 was announced in [29]; the present expanded paper would have never seen the light of day without the collaboration of the first named author.

2. BASICS

Notation is standard. Throughout this paper X, Y, Z are Banach spaces over the field $\mathbb{C}$ and X^* is the (topological) dual space of X. The space $\mathcal{B}(X, Y)$ of bounded linear operators from X to Y is endowed with the usual uniform operator topology, unless otherwise stated. The Bochner-Lebesgue space $L_p\left(\mathbb{R}^N, X\right)$, where $1 \le p \le \infty$, is endowed with its usual norm topology. The conjugate exponent p' of $p \in [1, \infty]$ is given by $\frac{1}{p} + \frac{1}{p'} = 1$. Also, $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers while $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. If convenient and confusion seems unlikely, the various function spaces $E\left(\mathbb{R}^N, X\right)$ in this paper are denoted simply by just $E(X)$ or E, with the exception of the Schwartz class.

The Schwartz class $\mathcal{S}\left(\mathbb{R}^N, X\right)$, or simply $\mathcal{S}(X)$, is the space of X-valued rapidly decreasing smooth functions φ on $\mathbb{R}^N$, equipped with the locally convex topology generated by the seminorms

$$\|\varphi\|_{k,\alpha} \;:=\; \sup_{t\in\mathbb{R}^N} \left(1+|t|^2\right)^k \|(D^\alpha\varphi)(t)\|_X$$

for all $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{N}_0^N$. As customary, we often denote $\mathcal{S}\left(\mathbb{R}^N, \mathbb{C}\right)$ by just $\mathcal{S}$. Note that $\mathcal{S}(X)$ is norm (resp. $\sigma\left(L_p(X), L_{p'}(X^*)\right)$) dense in $L_p(X)$ when $1 \le p < \infty$ (resp. $1 \le p \le \infty$).

The space of X–valued tempered distributions $\mathcal{S}'\left(\mathbb{R}^N, X\right)$ is the space of continuous linear operators $L : \mathcal{S} \to X$, equipped with the bounded convergence topology. Each *sufficiently bounded* function $m : \mathbb{R}^N \to X$ (e.g. a measurable function which grows at most polynominally as $|x| \to \infty$ or an $L_p(X)$ function for $1 \le p \le \infty$) defines an $L_m \in \mathcal{S}'\left(\mathbb{R}^N, X\right)$ by

$$L_m(\varphi) := \int_{\mathbb{R}^N} \varphi(t) m(t)\, dt \;;$$

when convenient and confusion seems unlikely, we will identify such a function m with $L_m \in \mathcal{S}'(X)$.

It is well-known that the Fourier transform $\mathcal{F}\colon \mathcal{S}(X) \to \mathcal{S}(X)$ defined by

$$(\mathcal{F}\varphi)(t) \;\equiv\; \widehat{\varphi}(t) \;:=\; \int_{\mathbb{R}^N} e^{-it\cdot s}\varphi(s)\, ds$$

is an isomorphism whose inverse is given by

$$\left(\mathcal{F}^{-1}\varphi\right)(t) \;\equiv\; \check{\varphi}(t) \;=\; (2\pi)^{-N} \int_{\mathbb{R}^N} e^{it\cdot s}\varphi(s)\, ds\;,$$

where $\varphi \in \mathcal{S}(X)$ and $t \in \mathbb{R}^N$. Also, the Fourier transform $\mathcal{F}\colon \mathcal{S}'(X) \to \mathcal{S}'(X)$ defined by

$$(\mathcal{F}L)(\varphi) \;\equiv\; \widehat{L}(\varphi) \;:=\; L(\widehat{\varphi}) \qquad \text{where} \quad L \in \mathcal{S}'(X)\;\;,\; \varphi \in \mathcal{S} \tag{2.1}$$

is an isomorphism whose inverse is given by $\left(\mathcal{F}^{-1}L\right)(\varphi) \;\equiv\; \check{L}(\varphi) \;=\; L(\check{\varphi})$.

The derivative, translation, and dilation properties of $\mathcal{F}$ and $\mathcal{F}^{-1}$ which hold in the scalar-valued case also hold in the vector-valued case. However, the Hausdorff-Young inequality need not hold. Thus we have to consider the following class of Banach spaces that was introduced by Peetre [22].

Definition 2.1. Let X be a Banach space and $1 \le p \le 2$. Let $\mathcal{F}_{p,N}(X)$ be the smallest $C \in [0, \infty]$ so that

$$\|\mathcal{F}f\|_{L_{p'}(\mathbb{R}^N,X)} \;\le\; C\; \|f\|_{L_p(\mathbb{R}^N,X)} \qquad \text{for each} \quad f \in \mathcal{S}\left(\mathbb{R}^N, X\right). \tag{2.2}$$

X has *Fourier type* p provided the *Fourier type constant* $\mathcal{F}_{p,N}(X)$ is finite for some (and thus then, by [17], for each) $N \in \mathbb{N}$.

Remark 2.2. The simple estimate $\|\mathcal{F}f(t)\|_X \le \|f\|_{L_1(X)}$ shows that each Banach space X has Fourier type 1 with $\mathcal{F}_{1,N}(X) = 1$. The notion becomes more restrictive as p increases to 2. A Banach space has Fourier type 2 if and only if X is isomorphic to a Hilbert space [19]. A space $L_q((\Omega, \Sigma, \mu), \mathbb{R})$ has Fourier type $p = \min(q, q')$ [22]. Each closed subspace (by definition) and each quotient space (by duality) of a Banach space X has the same Fourier type as X. Bourgain [7, 9] has shown that each B–convex Banach space (thus, in particular, each uniformly convex Banach space) has some non-trivial Fourier type $p > 1$.

For completeness we include the proof the the following well-known proposition.

Proposition 2.3. *Let X have Fourier type $p \in [1, 2]$ and $p \le q \le p'$. Then X^* and $L_q(\mathbb{R}^N, X)$ also have Fourier type p. Specifically, $\mathcal{F}_{p,M}(X^*) = \mathcal{F}_{p,M}(X)$ and $\mathcal{F}_{p,M}\left(L_q\left(\mathbb{R}^N, X\right)\right) = \mathcal{F}_{p,M}(X)$ for each $M \in \mathbb{N}$.*

Proof. If X is an isometric subspace of Z then $\mathcal{F}_{p,M}(X) \le \mathcal{F}_{p,M}(Z)$; thus, it suffices to show just $\le$ in the two claimed equalities. The claim for X^* follows from the fact that the adjoint of $\mathcal{F}\colon L_p(X) \to L_{p'}(X)$, restricted to $L_p(X^*)$, is the Fourier transform map from $L_p(X^*)$ to $L_{p'}(X^*)$. Let h be a simple function with finite support in $L_p\left(\mathbb{R}^M, L_q\left(\mathbb{R}^N, X\right)\right)$; thus, $\left[\widehat{h}(t)\right](u) = [h(\cdot)(u)]^{\widehat{}}(t)$. By the general Minkowski-Jessen inequality, since $q \le p'$ and $p \le q$ (and wlog $p \ne 1$),

$$\begin{aligned}
&\left[\int_{\mathbb{R}^M}\left(\int_{\mathbb{R}^N}\left\|\left[\widehat{h}(t)\right](u)\right\|^q du\right)^{p'/q} dt\right]^{1/p'} \\
&\qquad\qquad \le \left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^M}\left\|[h(\cdot)(u)]^{\widehat{}}(t)\right\|^{p'} dt\right)^{q/p'} du\right]^{1/q} \\
&\qquad\qquad \le C\left[\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^M}\|h(s)(u)\|^p\, ds\right)^{q/p} du\right]^{1/q} \\
&\qquad\qquad \le C\left[\int_{\mathbb{R}^M}\left(\int_{\mathbb{R}^N}\|h(s)(u)\|^q\, du\right)^{p/q} ds\right]^{1/p},
\end{aligned}$$

where $C = \mathcal{F}_{p,M}(X)$. This shows the claim for $L_q(\mathbb{R}^N, X)$. □

In order to define Besov spaces we consider the *dyadic-like* subsets, $\{J_k\}_{k=0}^\infty$ and $\{I_k\}_{k=0}^\infty$, of $\mathbb{R}^N$. Let $\{J_k\}_{k=0}^\infty$ be the partitioning of $\mathbb{R}^N$ given by

(2.3) $$J_0 = \{t \in \mathbb{R}^N : |t| \le 1\} \quad , \quad J_k = \{t \in \mathbb{R}^N : 2^{k-1} < |t| \le 2^k\} \quad \text{for } k \in \mathbb{N} .$$

Enlarge each J_k to form a sequence $\{I_k\}_{k=0}^\infty$ of overlapping subsets defined by

(2.4) $$I_0 = \left\{t \in \mathbb{R}^N : |t| \le 2\right\} \quad , \quad I_k = \left\{t \in \mathbb{R}^N : 2^{k-1} < |t| \le 2^{k+1}\right\} \quad \text{for } k \in \mathbb{N} .$$

To simplify notation later, let $J_k = \emptyset$ for $k < 0$. Next define a partition of unity $\{\varphi_k\}_{k\in\mathbb{N}_0}$ of functions from $\mathcal{S}\left(\mathbb{R}^N, \mathbb{R}\right)$ as follows. Take a nonnegative function $\psi \in \mathcal{S}(\mathbb{R}, \mathbb{R})$ with support in $\left[2^{-1}, 2\right]$ which satisfies

$$\sum_{k=-\infty}^{\infty} \psi(2^{-k}s) = 1 \ \text{ for } \ \ s \in \mathbb{R} \setminus \{0\}$$

and let, for $t \in \mathbb{R}^N$,

$$\varphi_k(t) = \psi\left(2^{-k}|t|\right) \qquad \text{for } \ k \in \mathbb{N} \qquad\qquad \text{and} \qquad\qquad \varphi_0(t) = 1 - \sum_{k=1}^{\infty} \varphi_k(t) \ .$$

To simplify notation later on, let $\varphi_k \equiv 0$ if $k < 0$. Note the following useful properties:

$$\begin{aligned} \operatorname{supp} \varphi_k \subset \overline{I_k} & \qquad \text{for each } k \in \mathbb{N}_0 \\ \sum_{k=0}^{\infty} \varphi_k(s) \ = \ 1 & \qquad \text{for each } s \in \mathbb{R}^N \\ J_m \ \cap \ \operatorname{supp} \varphi_k \ = \ \emptyset & \qquad \text{if } \quad |m-k| > 1 \\ \varphi_{k-1}(s) \ + \ \varphi_k(s) \ + \ \varphi_{k+1}(s) \ = \ 1 & \qquad \text{for each } s \in \operatorname{supp} \varphi_k \ , \ k \in \mathbb{N}_0 \ . \end{aligned}$$

Among the many equivalent descriptions of Besov spaces, the most useful one for us is given in terms of the so called *Littlewood–Paley decomposition*. Roughly speaking this means that we consider $f \in \mathcal{S}'(X)$ as a distributional sum $f = \sum_k f_k$ of analytic functions f_k whose Fourier transforms have support in *dyadic-like* I_k and then define the Besov norm in terms of the f_k's.

Definition 2.4. Let $1 \le q, r \le \infty$ and $s \in \mathbb{R}$. The *Besov space* $B^s_{q,r}\left(\mathbb{R}^N, X\right)$ is the space of all $f \in \mathcal{S}'\left(\mathbb{R}^N, X\right)$ for which

$$\begin{aligned} \|f\|_{B^s_{q,r}(\mathbb{R}^N,X)} \ &:= \ \left\| \left\{ 2^{ks} \ (\check{\varphi}_k * f) \ \right\}_{k=0}^{\infty} \ \right\|_{\ell_r(L_q(X))} \\ &\equiv \ \begin{cases} \left[\displaystyle\sum_{k=0}^{\infty} 2^{ksr} \ \|\check{\varphi}_k * f\|^r_{L_q(X)} \right]^{1/r} & \text{if } r \neq \infty \\ \displaystyle\sup_{k\in\mathbb{N}_0} \left[\ 2^{ks} \ \|\check{\varphi}_k * f\|_{L_q(X)} \ \right] & \text{if } r = \infty \end{cases} \end{aligned} \tag{2.5}$$

is finite; q is the *main index* while s is the *smoothness index*. $B^s_{q,r}\left(\mathbb{R}^N, X\right)$, together with the norm in (2.5), is a Banach space. $\overset{\circ}{B}{}^s_{q,r}\left(\mathbb{R}^N, X\right)$ is the closure of $\mathcal{S}\left(\mathbb{R}^N, X\right)$ in $B^s_{q,r}\left(\mathbb{R}^N, X\right)$, with the induced norm.

Different choices of $\{\varphi_k\}$ lead to equivalent norms on $B^s_{q,r}\left(\mathbb{R}^N, X\right)$ [23, Lem. 3.2]. A well-known [23, Prop. 3.1] equivalent norm on the Besov spaces $B^s_{q,r}(\mathbb{R}, X)$, for $0 < s < 1$ and $1 \le q, r < \infty$, is

$$\|f\|'_{B^s_{q,r}(X)} \ = \ \|f\|_{L_q(X)} \ + \ B^s_{q,r}(f) \tag{2.6}$$

where $f_h(s) = f(s+h)$,

$$B^s_{q,r}(f) = \left(\int_0^\infty \left[t^{-s} w_q(f,t) \right]^r \frac{dt}{t} \right)^{1/r} \tag{2.7}$$
$$w_q(f,t) = \sup_{|h|\le t} \|f_h - f\|_{L_q(X)} ;$$

there are similar expressions for other values of s and $N > 1$. Besov spaces also can be introduced via the real interpolation method (cf. [2]): for $m \in \mathbb{N}_0$, $s \in (0,m)$, $q \in [1,\infty)$, and $r \in [1,\infty]$

$$B^s_{q,r}(\mathbb{R}^N, X) = \left(L_q\left(\mathbb{R}^N, X\right) \ , \ W^m_q\left(\mathbb{R}^N, X\right) \right)_{\frac{s}{m}, r}$$

where $W^m_q\left(\mathbb{R}^N, X\right)$ are usual Sobolev spaces, which are defined below.

Definition 2.5. Let $1 \le q \le \infty$ and $m \in \mathbb{N}_0$. The *Sobolev space* $W^m_q\left(\mathbb{R}^N, X\right)$ is

$$W^m_q\left(\mathbb{R}^N, X\right) = \left\{ f \in \mathcal{S}'\left(\mathbb{R}^N, X\right) : D^\alpha f \in L_q\left(\mathbb{R}^N, X\right) \quad \text{for each} \quad |\alpha| \le m \right\} ,$$

equipped with the norm

$$\|f\|_{W^m_q(\mathbb{R}^N, X)} = \sum_{0 \le |\alpha| \le m} \|D^\alpha f\|_{L_q(\mathbb{R}^N, X)} .$$

It is well-know that the Sobolev spaces are Banach spaces.

We collect some known facts about Besov and Sobolev spaces (cf. [2]).

Fact 2.6. Let $s \in \mathbb{R}$ and $1 \le q, r \le \infty$ and $m \in \mathbb{N}_0$. Here, $\hookrightarrow$ denotes that the natural injection is a continuous linear operator. Then:

$$\mathcal{S}(X) \hookrightarrow W^m_q(X) \hookrightarrow \mathcal{S}'(X)$$
$$\mathcal{S}(X) \hookrightarrow B^s_{q,r}(X) \hookrightarrow \mathcal{S}'(X) \tag{2.8}$$
$$W^{m+1}_q(X) \hookrightarrow B^s_{q,r}(X) \hookrightarrow W^m_q(X) \hookrightarrow L_q(X) \qquad \text{if} \quad m < s < m+1 \tag{2.9}$$
$$B^s_{q,r}(X) \hookrightarrow B^s_{q,r+\varepsilon}(X) \qquad \text{if} \quad \varepsilon > 0$$
$$B^s_{q,\infty}(X) \hookrightarrow B^{s-\varepsilon}_{q,1}(X) \qquad \text{if} \quad \varepsilon > 0$$
$$\overset{\circ}{B}{}^s_{q,r}(X) = B^s_{q,r}(X) \qquad \text{if} \quad q, r < \infty \ .$$

Also $B^{N/q_1}_{q_1,1}\left(\mathbb{R}^N, X\right) \hookrightarrow B^{N/q_2}_{q_2,1}\left(\mathbb{R}^N, X\right) \hookrightarrow L_\infty\left(\mathbb{R}^N, X\right)$ provided $1 \le q_1 \le q_2 < \infty$.

Let $(E(Z), E^*(Z^*))$ be one of the pairs, where $1 \le q, r \le \infty$ and $s \in \mathbb{R}$:

$$(L_q(Z), L_{q'}(Z^*)) \qquad \text{or} \qquad \left(B^s_{q,r}(Z), B^{-s}_{q',r'}(Z^*) \right) .$$

There is a natural embedding of $E^*(Z^*)$ into $[E(Z)]^*$ as a norming subspace for $E(Z)$. This embedding is given by the duality map

$$\langle \cdot \, , \, \cdot \rangle_{E(Z)} : E^*(Z^*) \times E(Z) \to \mathbb{C}$$

where

$$\langle\, g \,,\, f \,\rangle_{L_q(Z)} \;:=\; \int_{\mathbb{R}^N} \langle\, g\,(t) \;,\; f\,(t) \,\rangle_Z \; dt \;=\; \int_{\mathbb{R}^N} g\,(t)\,[f\,(t)]\; dt$$

in the Lebesgue space setting with $E = L_q$ and

$$\langle\, g \,,\, f \,\rangle_{E(Z)} \;:=\; \sum_{n,m\in\mathbb{N}_0} \langle\, \check{\varphi}_n * g \;,\; \check{\varphi}_m * f \,\rangle_{L_q(Z)} \tag{2.10}$$

in the Besov space setting with $E = B^s_{q,r}$. One can check that this definition of duality is independent of the choice of the φ_k's. Furthermore, as seen by [2] and the fact that $\overset{\circ}{B}{}^{s}_{q,r}(\mathbb{R}^N, X)$ norms $B^{-s}_{q',r'}(\mathbb{R}^N, X^*)$,

$$\overset{\circ}{B}{}^{s}_{q,r}(X) \quad \text{is} \quad \sigma(B^s_{q,r}(X),\; B^{-s}_{q',r'}(X^*))\text{–dense in} \quad B^s_{q,r}(X)\;. \tag{2.11}$$

Often $E^*\,(Z^*) = [E\,(Z)]^*$, for example, provided Z^* has the Radon-Nikodým property and either $E = L_q$ (cf. [11]) or $E = B^s_{q,r}$(cf. [2]) where $q, r \in [1,\infty)$ and $s \in \mathbb{R}$. Recall that if Z is reflexive or if Z^* is separable, then Z^* has the Radon-Nikodým property.

For more information regarding Besov spaces, see [2, 23, 24].

3. THE FOURIER TRANSFORM ON BESOV SPACES

As a consequence of the Hausdorff-Young inequality we get the following estimates for the Fourier transform on Besov spaces.

Theorem 3.1. *Let X be a Banach space with Fourier type $p \in [1,2]$. Let $1 \le q \le p'$ and $s \ge N\left(\frac{1}{q} - \frac{1}{p'}\right)$ and $1 \le r \le \infty$. Then there exists a constant C, depending only on $\mathcal{F}_{p,N}(X)$, so that if*

$$f \in B^s_{p,r}\left(\mathbb{R}^N, X\right)$$

then

$$\left\| \left\{ \widehat{f} \cdot \chi_{J_m} \right\}_{m=0}^{\infty} \right\|_{\ell_r(L_q(\mathbb{R}^N,X))} \;\le\; C \;\; \|f\|_{B^s_{p,r}(\mathbb{R}^N,X)} \;\;. \tag{3.1}$$

A variant of Theorem 3.1 for Besov spaces on the multidimensional torus and Lorentz sequence spaces is already contained in [17, Thm. 4]; the proof is based on interpolation theory and does not give the statements we need here. An immediate corollary of Theorem 3.1 follows by choosing $q = r = 1$ (for (3.2)) or $r = q = p'$ (for (3.3)).

Corollary 3.2. *Let X have Fourier type $p \in [1,2]$. Then the Fourier transform $\mathcal{F}$ defines bounded operators*

$$\mathcal{F} \;:\; B^{N/p}_{p,1}\left(\mathbb{R}^N, X\right) \;\longrightarrow\; L_1\left(\mathbb{R}^N, X\right) \tag{3.2}$$

$$\mathcal{F} \;:\; B^{0}_{p,p'}\left(\mathbb{R}^N, X\right) \;\longrightarrow\; L_{p'}\left(\mathbb{R}^N, X\right)\;\;. \tag{3.3}$$

The norms of the above maps $\mathcal{F}$ are bounded above by a constant depending only on $\mathcal{F}_{p,N}(X)$.

Theorem 3.1 and Corollary 3.2 remain valid if $\mathcal{F}$ is replaced with $\mathcal{F}^{-1}$.

Proof.[Proof of Theorem 3.1] Fix $f \in B^s_{p,r}\left(\mathbb{R}^N, X\right)$. Then, for each $k \in \mathbb{N}_0$, since $\check{\varphi}_k * f \in L_p(X)$ and X has Fourier type p,

$$\varphi_k \cdot \widehat{f} \ = \ \mathcal{F}\left(\check{\varphi}_k * f\right) \ \in \ L_{p'}(X) \ .$$

Thus

$$\widehat{f} \cdot \chi_{J_m} \ = \ \left(\sum_{k=m-1}^{m+1} \varphi_k \widehat{f} \right) \cdot \chi_{J_m} \ \in \ L_q(X)$$

for each $m \in \mathbb{N}_0$.

If there exists a constant C_1 so that

$$\left\| \widehat{f} \cdot \chi_{J_m} \right\|_{L_q(X)} \ \leq \ C_1 \sum_{k=m-1}^{m+1} 2^{ks} \left\| \widehat{f} \cdot \varphi_k \right\|_{L_{p'}(X)} \ \text{ for each } \ m \in \mathbb{N}_0 \tag{3.4}$$

then

$$\begin{aligned} \left\| \widehat{f} \cdot \chi_{J_m} \right\|_{L_q(X)} \ &\leq \ C_1 \sum_{k=m-1}^{m+1} 2^{ks} \left\| \mathcal{F}\left(\check{\varphi}_k * f\right) \right\|_{L_{p'}(X)} \\ &\leq \ C_1 \ \mathcal{F}_{p,N}(X) \sum_{k=m-1}^{m+1} 2^{ks} \left\| \check{\varphi}_k * f \right\|_{L_p(X)} \end{aligned}$$

and so

$$\left\| \left\{ \widehat{f} \cdot \chi_{J_m} \right\}_{m=0}^{\infty} \right\|_{\ell_r(L_q(X))} \ \leq \ 9 \ C_1 \ \mathcal{F}_{p,N}(X) \ \| f \|_{B^s_{p,r}(X)} \ .$$

Thus it suffices to show that there exists C_1 so that (3.4) holds.

First consider the case where $q \neq p'$. Choose $1 \leq u < \infty$ so that $\frac{1}{q} = \frac{1}{p'} + \frac{1}{u}$; thus, $\frac{N}{u} \leq s$. By the generalized Hölder's inequality, for each $m \in \mathbb{N}_0$,

$$\begin{aligned} \left\| \widehat{f} \cdot \chi_{J_m} \right\|_{L_q(X)} \ &\leq \ \sum_{k=m-1}^{m+1} \left\| \widehat{f} \cdot \varphi_k \cdot \chi_{J_m} \right\|_{L_q(X)} \\ &\leq \ \sum_{k=m-1}^{m+1} \left\| \widehat{f} \varphi_k \left[\frac{1+|\cdot|}{4} \right]^{N/u} \chi_{J_m} \right\|_{L_{p'}(X)} \left\| \left[\frac{1+|\cdot|}{4} \right]^{-N/u} \chi_{J_m} \right\|_{L_u(\mathbb{R})} \\ &\leq \ \sum_{k=m-1}^{m+1} \left\| \left[\frac{1+|\cdot|}{4} \right]^{N/u} \chi_{J_m} \right\|_{L_\infty(\mathbb{R})} \left\| \widehat{f} \varphi_k \right\|_{L_{p'}(X)} \left[\int_{J_m} \left[\frac{1+|t|}{4} \right]^{-N} dt \right]^{1/u} \qquad (3.5) \\ &\leq \ \sum_{k=m-1}^{m+1} \left(2^{m-1} \right)^{N/u} \left\| \widehat{f} \varphi_k \right\|_{L_{p'}(X)} \ \left[C_2 \right]^{\frac{1}{u}} \ \leq \ C_2 \sum_{k=m-1}^{m+1} 2^{ks} \left\| \widehat{f} \varphi_k \right\|_{L_{p'}(X)} \end{aligned}$$

for some universal constant $C_2 > 1$.

If $q = p'$, then

$$\left\| \widehat{f} \cdot \chi_{J_m} \right\|_{L_q(X)} \leq \sum_{k=m-1}^{m+1} \left\| \widehat{f} \cdot \varphi_k \cdot \chi_{J_m} \right\|_{L_{p'}(X)} \leq \sum_{k=m-1}^{m+1} 2^{ks} \left\| \widehat{f} \varphi_k \right\|_{L_{p'}(X)}$$

for each $m \in \mathbb{N}_0$. $\square$

Remark 3.3. The statement of Theorem 3.1 remains valid if $B^s_{p,r}$ is replaced by W^j_p where $s < j \in \mathbb{N}$ (with a new constant C which is just the product of the original constant C from Theorem 3.1 and the embedding constant from $W^j_p(\mathbb{R}^N, X) \hookrightarrow B^s_{p,r}(\mathbb{R}^N, X)$). Also, it follows from Corollary 3.2 that if X has Fourier type $p \in [1,2]$ and $N/p < j \in N$, then the Fourier transform $\mathcal{F}$ defines a bounded operator

$$\mathcal{F} \; : \; W^j_p\left(\mathbb{R}^N, X\right) \longrightarrow L_1\left(\mathbb{R}^N, X\right) . \tag{3.2'}$$

Note that (3.2') is the appropriate vector-valued version of the well-known Bernstein theorem. Furthermore, if X has Fourier type $p \in [1,2]$ and $N/p < j \in N$ then there is a constant C so that

$$\left\| \widehat{f} \right\|_{L_1(\mathbb{R}^N, X)} \leq C \; \|f\|^{1-\frac{N}{jp}}_{L_p(\mathbb{R}^N, X)} \left(\sum_{|\alpha|=j} \|D^\alpha f\|_{L_p(\mathbb{R}^N, X)} \right)^{\frac{N}{jp}} \tag{3.6}$$

for each $f \in W^j_p\left(\mathbb{R}^N, X\right)$. A short proof of (3.6) is obtained from the classical argument of Bernstein's theorem (see, e.g. [3, Lemma 8.2.1]) by replacing the Cauchy-Schwarz inequality by Hölder's inequality and using the fact that the appropriate Hausdorff-Young inequality holds in a space with Fourier type p.

4. FOURIER MULTIPLIERS ON BESOV SPACES

For a bounded measurable function $m : \mathbb{R}^N \to \mathcal{B}(X, Y)$, its corresponding *Fourier multiplier operator* T_0 is defined by the following (commutative) diagram

$$\begin{array}{ccc} \mathcal{S}\left(\mathbb{R}^N, X\right) & \xrightarrow{T_0} & \mathcal{S}'\left(\mathbb{R}^N, Y\right) \\ \mathcal{F}\Big\downarrow & & \Big\uparrow \mathcal{F}^{-1} \\ \mathcal{S}\left(\mathbb{R}^N, X\right) & \xrightarrow{M_m} & \mathcal{S}'\left(\mathbb{R}^N, Y\right) \end{array}$$

where M_m is the multiplication operator induced by m; thus,

$$T_0(f) = \mathcal{F}^{-1}[m(\cdot) \; (\mathcal{F}f)(\cdot)] \; . \tag{4.1}$$

Note that since m is bounded

$$(T_0 f)^{\widehat{}} \; (\cdot) \; = \; m(\cdot) \left[\widehat{f}(\cdot) \right] \; \in \; L_1(Y) \tag{4.2}$$

and T_0 maps $\mathcal{S}(X)$ into $L_\infty(Y)$.

In this section we identify conditions on m, generalizing the classical Mihlin condition, so that

$$\|T_0 f\|_{B^s_{q,r}(Y)} \;\leq\; C\;\|f\|_{B^s_{q,r}(X)} \qquad \text{for each} \quad f \in \mathcal{S}(X)\ . \tag{4.3}$$

Note that (4.3) implies that T_0 extends uniquely to a bounded operator

$$\widetilde{T}_0 \colon \overset{\circ}{B}{}^s_{q,r}(X) \to B^s_{p,r}(Y)\ .$$

If we can find a further extention

$$T_m \colon B^s_{q,r}(X) \to B^s_{p,r}(Y) \tag{4.4}$$

of $\widetilde{T}_0$ which is $\sigma\left(B^s_{q,r}(X),\ B^{-s}_{q',r'}(X^*)\right)$–to–$\sigma\left(B^s_{q,r}(Y),\ B^{-s}_{q',r'}(Y^*)\right)$ continuous then, by (2.11), T_m is uniquely determined by (4.1). This leads to the following definition of a Fourier multiplier.

Definition 4.1. Let $\left(E\left(\mathbb{R}^N, Z\right), E^*\left(\mathbb{R}^N, Z^*\right)\right)$ be one of the following dual systems, where $1 \leq q, r \leq \infty$ and $s \in \mathbb{R}$:

$$\left(L_q(Z), L_{q'}(Z^*)\right) \qquad \text{or} \qquad \left(B^s_{q,r}(Z), B^{-s}_{q',r'}(Z^*)\right)\ .$$

A bounded measurable function $m : \mathbb{R}^N \to \mathcal{B}(X, Y)$ is called a *Fourier multiplier* from $E(X)$ to $E(Y)$ if there is a bounded linear operator

$$T_m : E(X) \to E(Y)$$

such that

$$T_m(f) \;=\; \mathcal{F}^{-1}[m(\cdot)\ (\mathcal{F}f)(\cdot)] \quad \text{for each} \quad f \in \mathcal{S}(X) \tag{4.5}$$

$$T_m \quad \text{is} \quad \sigma(E(X), E^*(X^*))\text{–to–}\sigma(E(Y), E^*(Y^*)) \quad \text{continuous}\ . \tag{4.6}$$

The (uniquely determined) operator T_m is the *Fourier multiplier operator induced by* m.

Note that

$$(T_m f)\widehat{\ }\,(\cdot) \;=\; m(\cdot)\left[\widehat{f}(\cdot)\right] \;\in\; L_1(Y) \qquad \text{for each} \qquad f \in \mathcal{S}(X) \tag{4.7}$$

and T_m maps $\mathcal{S}(X)$ into $L_\infty(Y)$.

Remark 4.2. If $T_m \in \mathcal{B}(E(X), E(Y))$ and T^*_m maps $E^*(Y^*)$ into $E^*(X^*)$ then T_m satisfies the continuity condition (4.6).

We will first give rather general criteria for Fourier multipliers in terms of the Besov norm of the multiplier function; later we derive from these results analogues of the classical Mihlin and Hörmander conditions. To simplify the statements of our results, we let

$$\mathcal{M}_p(m) \;:=\; \inf\;\left\{\|m(a\cdot)\|_{B^{N/p}_{p,1}(\mathbb{R}^N, \mathcal{B}(X,Y))} \;:\; a > 0\right\}\ .$$

First we give a multiplier result on $L_q(X)$ in the spirit of Steklin's theorem which, in spite of its strong hypothesis, is still useful in many circumstances.

Theorem 4.3. *Let X and Y have Fourier type $p \in [1,2]$. Then there is a constant C, depending only on $\mathcal{F}_{p,N}(X)$ and $\mathcal{F}_{p,N}(Y)$, so that if*

$$m \in B^{N/p}_{p,1}\left(\mathbb{R}^N, \mathcal{B}\left(X,Y\right)\right) \tag{4.8}$$

then m is a Fourier multiplier from $L_q(\mathbb{R}^N, X)$ to $L_q(\mathbb{R}^N, Y)$ with

$$\|T_m\|_{L_q(X)\to L_q(Y)} \ \le\ C\ \mathcal{M}_p\left(m\right)$$

for each $q \in [1, \infty]$.

Remark 4.4. a) Since $W^m_p(\mathbb{R}^N, Z) \subset B^{N/p}_{p,1}(\mathbb{R}^N, Z)$ for $m > N/p$, one can estimate with Sobolev-norms in order to verify the assumption (4.8) (e.g. see [16, Prop. 6.4] or Lemma 4.10).
b) If $m(s) = n(s)A$, where n is a scalar-valued function and $A \in \mathcal{B}\left(X,Y\right)$, then the result holds if $n \in B^{N/2}_{2,1}(\mathbb{R}^N, \mathbb{C})$ without any Fourier type requirement on X and Y. See the proof of Corollary 4.15.
c) We can replace $m \in B^{N/p}_{p,1}(\mathbb{R}^N, \mathcal{B}\left(X,Y\right))$ by the *pointwise* conditions

$$\begin{aligned} \|m(\cdot)x\|_{B^{N/p}_{p,1}(Y)} \ &\le\ M\,\|x\|_X \quad \text{for} \quad x \in X \\ \|m(\cdot)^* y^*\|_{B^{N/p}_{p,1}(X^*)} \ &\le\ M\,\|y^*\|_{Y^*} \quad \text{for} \quad y^* \in Y^* \ . \end{aligned}$$

Combine the proof below with [12].

The proof of Theorem 4.3 uses the following lemma.

Lemma 4.5. *Let $k \in L_1\left(\mathbb{R}^N, \mathcal{B}\left(X,Y\right)\right)$ and $1 \le q \le \infty$. Assume that there exists constants C_i so that for each $x \in X$ and $y^* \in Y^*$*

$$\int_{\mathbb{R}^N} \|k\left(s\right)\, x\|_Y \ ds \ \le\ C_0 \,\|x\|_X \quad \textit{and} \quad \int_{\mathbb{R}^N} \left\|k\left(s\right)^* \ y^*\right\|_{X^*} ds \ \le\ C_1 \,\|y^*\|_{Y^*} \ . \tag{4.9}$$

Then the convolution operator $K\colon L_q\left(\mathbb{R}^N, X\right) \to L_q\left(\mathbb{R}^N, Y\right)$ defined by

$$\left(Kf\right)(t) \ =\ \int_{\mathbb{R}^N} k(t-s) f(s)\, ds \qquad \textit{for} \ \ t \in \mathbb{R}^N \tag{4.10}$$

satisfies that $\|K\|_{L_q\to L_q} \ \le\ C_0^{\frac{1}{q}}\ C_1^{1-\frac{1}{q}}$.

Proof.[Proof of Lemma 4.5] Since $k \in L_1\left(\mathbb{R}^N, \mathcal{B}\left(X,Y\right)\right)$ it is well-known that (4.10) defines a bounded operator on $L_q\left(X\right)$. Indeed, for $f \in L_q\left(X\right) \cap L_\infty\left(X\right)$ we have, where $f_s(t) := f(t-s)$,

$$\int_{\mathbb{R}^N} \|k\left(t-s\right) f\left(s\right)\|_Y \ ds \ =\ \int_{\mathbb{R}^N} \|k(s) f_s(t)\|_Y \ ds \ \le\ \|k\|_{L_1(\mathcal{B}(X,Y))}\ \|f\|_{L_\infty(X)}$$

for each $t \in \mathbb{R}^N$ and

$$\begin{aligned}\|Kf(\cdot)\|_{L_q(Y)} &\leq \int_{\mathbb{R}^N} \|k(s)\, f_s(\cdot)\|_{L_q(Y)}\, ds \\ &\leq \int_{\mathbb{R}^N} \|k(s)\|_{\mathcal{B}(X,Y)}\, \|f_s(\cdot)\|_{L_q(X)}\, ds \;=\; \|k\|_{L_1(\mathcal{B}(X,Y))}\, \|f\|_{L_q(X)} \;.\end{aligned}$$

But what is important for us is that, by using (4.9), we can get a better norm estimate on $\|K\|$ than $\|k\|_{L_1}$.

For $q = 1$ we have from (4.9)

$$\begin{aligned}\|Kf\|_{L_1(Y)} &\leq \int\int \|k(t-s)f(s)\|_Y \, dt\, ds \\ &\leq C_0 \int \|f(s)\|_X \, ds \;=\; C_0\, \|f\|_{L_1(X)} \;.\end{aligned}$$

Thus $\|K\|_{L_1 \to L_1} \leq C_0$. If $q = \infty$, then for each $f \in L_\infty(X)$ and $y^* \in Y^*$ and $t \in \mathbb{R}^N$

$$\begin{aligned}|\langle y^*, (Kf)(t)\rangle_Y| &\leq \int |\langle k(t-s)^* y^*, f(s)\rangle_X|\, ds \\ &\leq \int \|k(t-s)^* y^*\|_{X^*}\, \|f(s)\|_X\, ds \;\leq\; C_1\, \|y^*\|_{Y^*}\, \|f\|_{L_\infty(X)} \;.\end{aligned}$$

Hence $\|K\|_{L_\infty \to L_\infty} \leq C_1$. Let $L^0_\infty(X)$ denotes the closure, in the $L_\infty(X)$ norm, of the simple functions $\sum_{k=1}^m x_k \chi_{A_k}$, where $x_k \in X$ and $\mathrm{vol}(A_k) < \infty$ and $m \in \mathbb{N}$. Then one can check that K maps $L^0_\infty(X)$ into $L^0_\infty(Y)$. Indeed, for $f = x \cdot \chi_A$, we have $Kf(t) = \int_{t-A} k(s)x ds \to 0$ for $|t| \to \infty$ and Kf is a continuous function from $\mathbb{R}^N$ to Y. Now the Riesz-Thorin Theorem (cf. [5, Thm 5.1.2]) yields the claim for $1 < p < \infty$. □

Remark 4.6. The assumption in Lemma 4.5 that $k \in L_1\left(\mathbb{R}^N, \mathcal{B}(X,Y)\right)$ can be replaced by much weaker measurability conditions. This is explored in [12].

Proof.[Proof of Theorem 4.3] First assume in addition that $m \in \mathcal{S}(\mathcal{B}(X,Y))$.

Thus $\check{m} \in \mathcal{S}(\mathcal{B}(X,Y))$. Fix $x \in X$. For an appropriate choice of $a > 0$, we can apply Corollary 3.2 to the function

$$t \;\to\; m(at)\, x \quad \text{in} \quad B^{N/p}_{p,1}(Y)$$

and use that $[m(a\cdot)\,x]^\vee(s) \;=\; a^{-N}\, \check{m}\left(\frac{s}{a}\right) x$ to get

$$\begin{aligned}\|\check{m}(\cdot)\,x\|_{L_1(Y)} &= \left\|[m(a\cdot)\,x]^\vee\right\|_{L_1(Y)} \\ &\leq C_1\, \|m(a\cdot)\|_{B^{N/p}_{p,1}}\, \|x\|_X \;\leq\; 2\, C_1\, \mathcal{M}_p(m)\, \|x\|_X\end{aligned} \tag{4.11}$$

for some constant C_1 which depends on $\mathcal{F}_{p,N}(Y)$.

By the additional assumption on m

$$[m(\cdot)]^* \;\in\; \mathcal{S}(\mathcal{B}(Y^*, X^*)) \qquad \text{and} \qquad \left[m(\cdot)^*\right]^\vee \;=\; [\check{m}(\cdot)]^* \;\in\; \mathcal{S}(\mathcal{B}(Y^*, X^*)) \;.$$

Let $y^* \in Y^*$. Similarly, by applying Corollary 3.2 to an appropriate function

$$t \;\to\; [m\,(at)]^*\, y^* \qquad \text{in} \qquad B^{N/p}_{p,1}\,(X^*)$$

and using the fact that $\mathcal{M}_p(m) = \mathcal{M}_p(m^*)$, one has

$$\left\| \,[\check{m}\,(\cdot)]^*\, y^* \right\|_{L_1(X^*)} \;\le\; 2C_2\, \mathcal{M}_p(m)\; \|y^*\|_{Y^*} \tag{4.12}$$

for some constant C_2 which depends $\mathcal{F}_{p,N}\,(X^*)$.

By Lemma 4.5, the convolution operator

$$(T_m f)\,(t) \;:=\; \int\limits_{\mathbb{R}^N} \check{m}(t-s) f(s)\, ds$$

satisfies, where $C = 2\, \max\{C_1, C_2\}$,

$$\|T_m\|_{\mathcal{B}(L_q(X), L_q(Y))} \;\le\; C\, \mathcal{M}_p(m)\ .$$

Furthermore T_m satisfies (4.5) since $m \in L_1\,(\mathcal{B}\,(X,Y))$. Also, T_m satisfies (4.6); indeed, T_m^* restricted to $L_{q'}\,(Y^*)$ is just $T_{m(-\cdot)^*}$ and so Remark 4.2 holds.

For the general case where $m \in B^{N/p}_{p,1}\,(\mathcal{B}\,(X,Y))$, choose a sequence $\{m_n\}_n$ from $\mathcal{S}\,(\mathcal{B}\,(X,Y))$ that converges to m in the $B^{N/p}_{p,1}$–norm and obtain operators $T_{m_n} \in \mathcal{B}\,(L_q(X), L_q(Y))$ with $\|T_{m_n} - T_{m_l}\|$ at most $C\,\|m_n - m_l\|_{B^{N/p}_{p,1}}$. Then

$$T \;:=\; \|\cdot\|_{\mathcal{B}(L_q(X), L_q(Y))} - \lim_{n\to\infty} T_{m_n}$$

has the desired properties. Indeed, conditions (4.5) and (4.6) pass from T_{m_n} to T_m. One also has that $\|T_m\| \le C\,\|m\|_{B^{N/p}_{p,1}}$. To get the desired bound, fix $a > 0$ such that $m(a\cdot) \in B^{N/p}_{p,1}\,(\mathcal{B}\,(X,Y))$. Then $I_Y \circ T_{m(a\cdot)} = T_m \circ I_X$ where $I_Z \colon L_q\,(Z) \to L_q\,(Z)$ is the isometry $(Tf)\,(t) = a^{N/q}\, f\,(at)$. Thus

$$\|T_m\| \;=\; \left\|T_{m(a\cdot)}\right\| \;\le\; C\; \|m(a\cdot)\|_{B^{N/p}_{p,1}(\mathcal{B}(X,Y))}$$

and so $\|T_m\| \le C\mathcal{M}_p\,(m)$. □

The following remark collects some basic facts about the Fourier multiplier operators T_m given in Theorem 4.3 that will be used in the proof of Theorem 4.8. Recall that for a distribution $g \in \mathcal{S}'\left(\mathbb{R}^N, X\right)$ and a closed subset Ω of $\mathbb{R}^N$

$$\operatorname{supp} g \subset \Omega \qquad \text{if and only if} \qquad g\,(\varphi) = 0 \;\text{ for each }\; \varphi \in \mathcal{S} \;\text{ with }\; \Omega \cap \operatorname{supp} \varphi = \emptyset\ .$$

Remark 4.7. Assume that we are in the setting of Theorem 4.3. Let $f \in L_q\,(X)$ and Ω be a closed subset of $\mathbb{R}^N$.

a) Viewing f and $T_m f$ as distributions, if $\operatorname{supp} \widehat{f} \subset \Omega$ then $\operatorname{supp} \widehat{T_m f} \subset \Omega$.

b) $T_{m_1} + T_{m_2} = T_{m_1+m_2}$. If $\varphi \in \mathcal{S}$, then $\check{\varphi} * (T_m f) \;=\; T_m\,(\check{\varphi} * f) \;=\; T_{\varphi m}\,(f)$.

c) If $\varphi \in \mathcal{S}$ is 1 on $\operatorname{supp}\, \widehat{f}$, then $T_{\varphi m} f = T_m f$.

d) T_m^* restricted to $L_{q'}(Y^*)$ is $T_{m(-\cdot)^*}$.

Proof. a) If $f \in \mathcal{S}(X)$ then the claim follows from the fact that T_m has the form (4.7). In general, choose a sequence $\{f_n\}$ from $\mathcal{S}(X)$ that converges to f in $\sigma(L_q(X), L_{q'}(X^*))$ and with supp $\widehat{f}_n \subset \Omega$. Then $\{T_m f_n\}$ converges to $T_m f$ in $\sigma(L_q(Y), L_{q'}(Y^*))$. If $\varphi \in \mathcal{S}$ with $\Omega \cap \operatorname{supp}\varphi = \emptyset$ and $y^* \in Y^*$ then

$$\begin{aligned} y^*\left[(T_m f)\widehat{\ }(\varphi)\right] &= (y^* \otimes \widehat{\varphi})(T_m f) \\ &= \lim_{n\to\infty}(y^* \otimes \widehat{\varphi})(T_m f_n) \;=\; \lim_{n\to\infty} y^*\left[(T_m f_n)\widehat{\ }(\varphi)\right] \;. \end{aligned}$$

Thus $(T_m f)\widehat{\ }(\varphi) = 0$.
b) The claim is trivial if T_m is a convolution operator with kernel $\check{m} \in \mathcal{S}(\mathcal{B}(X,Y))$. But according to the proof of Theorem 4.3, T_m can be approximated by such operators in operator norm.
c) Follows directly from b).
d) The proof is contained in the proof of Theorem 4.3. □

By applying this Theorem 4.3 to the *blocks* of the Littlewood–Paley decomposition of $B^s_{q,r}$ we will now get the main result of this section.

Theorem 4.8. *Let X and Y be Banach spaces with Fourier type $p \in [1,2]$. Then there is a constant C, depending only on $\mathcal{F}_{p,N}(X)$ and $\mathcal{F}_{p,N}(Y)$, so that if $m : \mathbb{R}^N \to \mathcal{B}(X,Y)$ satisfy*
(4.13)
$$\varphi_k \cdot m \in B^{N/p}_{p,1}\left(\mathbb{R}^N, \mathcal{B}(X,Y)\right) \qquad \textit{and} \qquad \mathcal{M}_p(\varphi_k \cdot m) \;\le\; A \qquad \textit{for each } k \in \mathbb{N}_0$$

then m is a Fourier multiplier from $B^s_{q,r}(\mathbb{R}^N, X)$ to $B^s_{q,r}(\mathbb{R}^N, Y)$ and $\|T_m\| \le CA$ for each $s \in \mathbb{R}$ and $q, r \in [1,\infty]$.

Remark 4.9. a) Note that each Banach space has Fourier type 1 and each uniformly convex Banach space has Fourier type p for some $p > 1$. Our result shows that the required smoothness of the multiplier function m depends not only on the dimension of $\mathbb{R}^N$ but also on the geometry of the Banach spaces X and Y.
b) Our results are sharp in the following sense. Note that the spaces $B^{N/p}_{p,1}(\mathbb{R}^N, Z)$ form a scale since

$$B^{N/p}_{p,1}(\mathbb{R}^N, Z) \subset B^{N/q}_{q,1}(\mathbb{R}^N, Z)$$

if $p < q$. In general it is not possible to replace $B^{N/p}_{p,1}$ in Theorem 4.8 by a larger space $B^{N/q}_{q,1}$ for some $q > p$. Indeed, it was shown in [29, Remark 3.7] that counterexamples for such a stronger statement can be found in the context of stability theory for semigroups on L_p-spaces.
c) We can replace (4.13) by a weaker *pointwise* condition:

$$\begin{aligned} \|\varphi_k m(\cdot)\, x\|_{B^{N/p}_{p,1}(Y)} &\le A\,\|x\| \quad \text{for} \quad x \in X \\ \left\|\varphi_k m(\cdot)^*\, y^*\right\|_{B^{N/p}_{p,1}(X^*)} &\le A\,\|y^*\| \quad \text{for} \quad y^* \in Y^* \;. \end{aligned}$$

See the proof below and Remark 4.4c.

The heuristic idea behind the proof of Theorem 4.8 is to *formally* decompose the desired Fourier multiplier operator T_m as

$$\begin{aligned} T_m f \;&=\; \left[m\widehat{f}\,\right]^{\vee} \;=\; \sum_{k\in\mathbb{N}_0} \left[(\varphi_{k-1}+\varphi_k+\varphi_{k+1})\;\, m\;\,(\check{\varphi}_k * f)^{\frown}\,\right]^{\vee} \\ &=\; \sum_{k\in\mathbb{N}_0} T_{(\varphi_{k-1}+\varphi_k+\varphi_{k+1})m}\,(\check{\varphi}_k * f) \end{aligned}$$

where $T_{(\varphi_{k-1}+\varphi_k+\varphi_{k+1})m}$ is the Fourier multiplier operator on $L_q(X)$ given by Theorem 4.3. It then follows easily that such a decomposition yields a bounded operator from $\overset{\circ}{B}{}^s_{q,r}(X)$ to $B^s_{q,r}(Y)$. The case when $\overset{\circ}{B}{}^s_{q,r}(X) \neq B^s_{q,r}(X)$ (i.e. $q=\infty$ or $r=\infty$) requires further consideration.

Proof.[Proof of Theorem 4.8] Theorem 4.3 gives that $\varphi_k \cdot m$ induces a Fourier multiplier operator $T_{m\varphi_k}$ with

$$\|T_{m\varphi_k}\|_{\mathcal{B}(L_q(X),L_q(Y))} \;\le\; C\mathcal{M}_p(\varphi_k\cdot m) \;\le\; CA$$

for some constant C depending only on $\mathcal{F}_{p,N}(X)$ and $\mathcal{F}_{p,N}(Y)$. Let $\psi_k = \varphi_{k-1} + \varphi_k + \varphi_{k+1}$; note that ψ_k is 1 on $\operatorname{supp}\varphi_k$. Then $\psi_k \cdot m$ induces the Fourier multiplier operator $T_{m\psi_k}$ with

$$T_{m\psi_k} \;=\; T_{m\varphi_{k-1}} + T_{m\varphi_k} + T_{m\varphi_{k+1}} \;\in\; \mathcal{B}\left(L_q(X), L_q(Y)\right) \tag{4.14}$$

and $\|T_{m\psi_k}\| \le 3CA$.

As in the introduction of Section 4, define $T_0 : \mathcal{S}(X) \to \mathcal{S}'(Y)$ by (4.1); namely,

$$T_0(f) = \mathcal{F}^{-1}[m(\cdot)\;(\mathcal{F}f)(\cdot)]\;.$$

If $f \in \mathcal{S}(X)$, then

$$\check{\varphi}_k * T_0 f \;=\; T_{m\psi_k}(\check{\varphi}_k * f)$$

for each $k \in \mathbb{N}_0$ since

$$\begin{aligned} \left[T_{m\psi_k}(\check{\varphi}_k * f)\right]^{\frown}(\cdot) \;&=\; \psi_k(\cdot)\, m(\cdot) \left[(\check{\varphi}_k * f)^{\frown}(\cdot)\right] \;=\; \psi_k(\cdot)\, m(\cdot) \left[\varphi_k(\cdot)\,\widehat{f}(\cdot)\right] \\ &=\; \varphi_k(\cdot)\left[m(\cdot)\,\widehat{f}(\cdot)\right] \;=\; \varphi_k(\cdot)\,(T_0 f)^{\frown}(\cdot) \;=\; (\check{\varphi}_k * T_0 f)^{\frown}(\cdot)\;; \end{aligned}$$

so, by the definition of the Besov norm

$$\|T_0 f\|_{B^s_{q,r}(Y)} \;\le\; 3CA\;\|f\|_{B^s_{q,r}(X)}\;.$$

Thus T_0 extends to a bounded linear operator from $\overset{\circ}{B}{}^s_{q,r}(X)$ to $B^s_{q,r}(Y)$. If $q, r < \infty$, then $\overset{\circ}{B}{}^s_{q,r}(X) = B^s_{q,r}(X)$ and so all that would remain is to verify the weak continuity condition (4.6). However, we continue with the proof in order to also cover the case $q=\infty$ or $r=\infty$.

We shall show that

$$T_m \colon B^s_{q,r}(X) \to B^s_{q,r}(Y)$$

defined by

$$T_m f \ := \ \sum_{k=0}^{\infty} f_k \qquad \text{where} \qquad f_k \ := \ T_{m\psi_k}(\check{\varphi}_k * f) \in L_q(Y) \tag{4.15}$$

is indeed a (norm) continuous operator. Towards this, fix $f \in B^s_{q,r}(X)$.

First we show that the formal series $T_m f$ in (4.15) defines an element in $\mathcal{S}'(Y)$. Towards this, fix $\varphi \in \mathcal{S}$. Remark 4.7 gives that supp $\widehat{f_k} \subset \overline{I_k}$. Thus

$$f_k(\varphi) \ = \ \widehat{f_k}(\check{\varphi}) \ = \ \widehat{f_k}(\psi_k(-\cdot)\,\check{\varphi}) \ = \ f_k\left((2\pi)^{-N}\ \psi_k(-\cdot)^{\widehat{}} \ * \ \varphi\right) \ = \ f_k\left(\check{\psi}_k * \varphi\right)$$

and so

$$\begin{aligned}
\sum_{k=0}^{\infty} \|f_k(\varphi)\|_Y \ &\le \ \sum_{k=0}^{\infty} \|f_k\|_{L_q(Y)} \ \left\|\check{\psi}_k \ * \ \varphi\right\|_{L_{q'}(\mathbb{C})} \\
&\le \ 3AC \sum_{k=0}^{\infty} \left(2^{ks}\,\|\check{\varphi}_k * f\|_{L_q(X)}\right) \ \left(2^{-ks}\,\left\|\check{\psi}_k \ * \ \varphi\right\|_{L_{q'}(\mathbb{C})}\right) \\
&\le \ 27AC\,2^{|s|}\ \|f\|_{B^s_{q,r}(X)}\ \|\varphi\|_{B^{-s}_{q',r'}(\mathbb{C})} \ .
\end{aligned}$$

Thus

$$\|\cdot\|_Y \ - \ \lim_{n\to\infty} \sum_{k=0}^{n} f_k(\varphi) \ := \ (T_m f)(\varphi) \qquad \text{for} \qquad \varphi \in \mathcal{S} \tag{4.16}$$

defines a linear map $T_m f$ from $\mathcal{S}$ into Y which is continuous by (2.8).

By Remark 4.7, for each $j, k \in \mathbb{N}_0$

$$\check{\varphi}_j * T_{m\psi_k}(\check{\varphi}_k * f) \ = \ T_{m\psi_k}(\check{\varphi}_j * \check{\varphi}_k * f) \ = \ \check{\varphi}_k * T_{m\psi_k}(\check{\varphi}_j * f) \ .$$

Thus, since the support of φ_k intersects the support of φ_j only for $|k-j| \le 1$, applying Remark 4.7 further gives

$$\begin{aligned}
\check{\varphi}_k * T_m f \ &= \ \sum_{j=k-1}^{k+1} \ \check{\varphi}_k * T_{m\psi_j}(\check{\varphi}_j * f) \ = \ \sum_{j=k-1}^{k+1} \ \check{\varphi}_j * T_{m\psi_j}(\check{\varphi}_k * f) \\
&= \ \sum_{j=k-1}^{k+1} \ T_{\varphi_j \psi_j m}\ (\check{\varphi}_k * f) \ = \ T_{m\psi_k}(\check{\varphi}_k * f) \ .
\end{aligned} \tag{4.17}$$

Hence $\check{\varphi}_k * T_m f \in L_q(Y)$ and

$$\|\check{\varphi}_k * T_m f\|_{L_q(Y)} \ \le \ 3CA\ \|\check{\varphi}_k * f\|_{L_q(X)} \ ,$$

from which it follows that range of T_m is contained in $B^s_{q,r}(X)$ and that norm of T_m as an operator from $B^s_{q,r}(X)$ to $B^s_{q,r}(Y)$ is bounded by a constant depending on the items claimed.

Furthermore, T_m extends T_0; indeed, if $f \in \mathcal{S}(X)$ then (viewing equality as pointwise in $\mathcal{S}'(Y)$)

$$\begin{aligned}(T_m f)\widehat{\ } &= \sum_{k=0}^{\infty} [T_{m\psi_k}(\check{\varphi}_k * f)]\widehat{\ } = \sum_{k=0}^{\infty} \left(\psi_k \, m \, \varphi_k \, \widehat{f}\right) \\ &= \sum_{k=0}^{\infty} \left(\varphi_k \, m \, \widehat{f}\right) = \left(\sum_{k=0}^{\infty} \varphi_k\right) m \, \widehat{f} = (T_0 f)\widehat{\ } ,\end{aligned}$$

where the first equality follows from (2.1) and (4.16) while the fourth equality follows from viewing each $\varphi_k m \widehat{f}$ as an element of $L_1(Y)$.

It remains to show only that T_m satisfies (4.6). Since $[m(-\cdot)]^* : \mathbb{R}^N \to \mathcal{B}(Y^*, X^*)$ also satisfies condition (4.13), the Fourier multiplier operator $T_{m(-\cdot)^*}$, defined by (4.5), extends to $T_{m(-\cdot)^*} \in \mathcal{B}(E^*(Y^*), E^*(X^*))$ where $E = B^s_{q,r}$. By Remark 4.2, it suffices to show that T_m^* restricted to $E^*(Y^*)$ is $T_{m(-\cdot)^*}$. Towards this, fix $g \in E^*(Y^*)$ and $f \in E(X)$. By (4.17) and (2.10)

$$\begin{aligned}\langle T_m^* g, f \rangle_{E(X)} &= \sum_{n,k \in \mathbb{N}_0} \langle \check{\varphi}_n * g, \check{\varphi}_k * T_m f \rangle_{L_q(Y)} \\ &= \sum_{n,k \in \mathbb{N}_0} \langle \check{\varphi}_n * g, T_{m\psi_k}(\check{\varphi}_k * f) \rangle_{L_q(Y)}\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}\left\langle T_{m(-\cdot)^*} g, f \right\rangle_{E(X)} &= \sum_{n,k \in \mathbb{N}_0} \left\langle \check{\varphi}_n * T_{m(-\cdot)^*} g, \check{\varphi}_k * f \right\rangle_{L_q(X)} \\ &= \sum_{n,k \in \mathbb{N}_0} \left\langle T_{m(-\cdot)^* \psi_n(\cdot)}(\check{\varphi}_n * g), \check{\varphi}_k * f \right\rangle_{L_q(X)} .\end{aligned} \tag{4.19}$$

Fix $K_0 \in \mathbb{N}_0$ and choose a radial $\psi \in \mathcal{S}$ with compact support such that ψ is 1 on $\cup_{k=1}^{K_0+1} \operatorname{supp} \varphi_k$. If $n, k \in \{0, 1, \ldots, K_0\}$ then $\psi_k = \psi\psi_k$ and $\psi_n = \psi\psi_n$ and so, by Remark 4.7,

$$T_{m\psi_k}(\check{\varphi}_k * f) = T_{m\psi\psi_k}(\check{\varphi}_k * f) = T_{m\psi}(\check{\varphi}_k * f) \tag{4.20}$$

and

$$T_{m(-\cdot)^*\psi_n(\cdot)}(\check{\varphi}_n * g) = T_{m(-\cdot)^*\psi(\cdot)\psi_n(\cdot)}(\check{\varphi}_n * g) = T_{m(-\cdot)^*\psi(\cdot)}(\check{\varphi}_n * g) \tag{4.21}$$

since ψm and $\psi(\cdot)m(-\cdot)^*$ satisfy the assumptions of Theorem 4.3 (indeed, $\psi m = \sum_{k=0}^{l} \psi\varphi_k m$ for some l large enough, likewise for $\psi(\cdot)m(-\cdot)^*$). Thus $\langle T_m^* g, f \rangle = \left\langle T_{m(-\cdot)^*} g, f \right\rangle$ by (4.18),(4.19), (4.20), (4.21), and Remark 4.7. □

The next lemma gives a convenient way to verify the assumption of Theorem 4.8 in terms of derivatives.

Lemma 4.10. *Let $\frac{N}{p} < l \in \mathbb{N}$ and $u \in [p, \infty]$. If $m \in C^l(\mathbb{R}^N, \mathcal{B}(X, Y))$ satisfies, for some A,*

$$\| D^\alpha m \,|_{I_0} \|_{L_u(\mathcal{B}(X,Y))} \leq A \tag{4.22}$$

and for each $k \in \mathbb{N}$, with $m_k(\cdot) := m\left(2^{k-1}\cdot\right)$,

$$\left\| D^\alpha m_k \mid_{I_1} \right\|_{L_u(\mathcal{B}(X,Y))} \;\leq\; A \tag{4.23}$$

for each $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq l$, then m satisfies condition (4.13) *of Theorem* 4.8.

Proof. Since $l > \frac{N}{p}$ one has that $W_p^l\left(\mathbb{R}^N, \mathcal{B}(X,Y)\right) \subset B_{p,1}^{N/p}\left(\mathbb{R}^N, \mathcal{B}(X,Y)\right)$, say with embedding constant K (see (2.9)). Let $\widetilde{u} \in [p,\infty]$ be so that $1/u + 1/\widetilde{u} = 1/p$. Note that, for $i \in \mathbb{N}_0$,

$$C_{\widetilde{u}}^l(\varphi_i) \;:=\; \sum_{|\alpha|\leq l} \; \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \; \left\| D^\beta \varphi_i \right\|_{L_{\widetilde{u}}}$$

is finite. Thus, by (4.22),

$$\begin{aligned}
\mathcal{M}_p(\varphi_0 m) \;&\leq\; \|\varphi_0 m\|_{B_{p,1}^{N/p}} \;\leq\; K \; \|\varphi_0 m\|_{W_p^l} \\
&=\; K \sum_{|\alpha|\leq l} \left\| \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \; D^\beta \varphi_0 \; D^{\alpha-\beta} m \mid_{I_0} \right\|_{L_p} \;\leq\; K \; A \; C_{\widetilde{u}}^l(\varphi_0) \; .
\end{aligned}$$

For each $k \in \mathbb{N}$,

$$\mathcal{M}_p(\varphi_k m) \;\leq\; \left\| \varphi_k\left(2^{k-1}\cdot\right) m\left(2^{k-1}\cdot\right) \right\|_{B_{p,1}^{N/p}} \;=\; \|\varphi_1 m_k\|_{B_{p,1}^{N/p}} \;; \tag{4.24}$$

thus by (4.23),

$$\mathcal{M}_p(\varphi_k m) \;\leq\; K \; \|\varphi_1 m_k\|_{W_p^l} \;\leq\; K \; A \; C_{\widetilde{u}}^l(\varphi_1) \; .$$

Thus for each $k \in \mathbb{N}_0$

$$\mathcal{M}_p(\varphi_k m) \;\leq\; K\, A \left[C_{\widetilde{u}}^l(\varphi_0) \vee C_{\widetilde{u}}^l(\varphi_1) \right] \;;$$

so m does indeed satisfy condition (4.13) of Theorem 4.8. □

Now we show that the classical Mihlin conditions imply assumption (4.13) of Theorem 4.8.

Corollary 4.11. *Let $q, r \in [1,\infty]$ and $s \in \mathbb{R}$. If $m \in C^l(\mathbb{R}^N, \mathcal{B}(X,Y))$ satisfies, for some constant A,*

$$\sup_{t\in\mathbb{R}^N} \left\| (1+|t|)^{|\alpha|} \, D^\alpha m(t) \right\|_{\mathcal{B}(X,Y)} \;\leq\; A \tag{4.25}$$

for each multi-index α with $|\alpha| \leq l$, then m is a Fourier multiplier from $B_{q,r}^s(\mathbb{R}^N, X)$ to $B_{q,r}^s(\mathbb{R}^N, Y)$ provided one of the following conditions hold.

a) *X and Y are arbitrary Banach spaces and $l = N+1$.*

b) *X and Y are uniformly convex Banach spaces and $l = N$.*

c) *X and Y have Fourier type p and $l = [\frac{N}{p}] + 1$.*

Remark 4.12. Part a) was shown by H. Amann in [2]. By Remark 2.2, part c) can be applied to Banach spaces X and Y that are subspaces of $L_q\left((\Omega, \Sigma, \mu), \mathbb{R}\right)$ where $p = \min(q', q)$.

Proof. In case c), X and Y are assumed to have Fourier type p. In case b), since X and Y are uniformly convex, they have Fourier type p for some $p > 1$. In case a), let $p = 1$. Thus, in each of the three cases, X and Y have Fourier type p and, by design, $N/p < l \in \mathbb{N}$.

Keeping with the notation from Lemma 4.10, let $u = \infty$. Fix $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq l$. Clearly (4.25) implies (4.22) since

$$\|D^\alpha m\,|_{I_0}\|_{L_\infty} \;\leq\; \left\|(1+|\cdot|)^{|\alpha|}\, D^\alpha m\,(\cdot)\right\|_{L_\infty} \;\leq\; A\;.$$

For each $k \in \mathbb{N}$ and $t \in I_1$

(4.26)
$$\|D^\alpha m_k\,(t)\|_{\mathcal{B}(X,Y)} \;\leq\; \left\||t|^{|\alpha|}\; D^\alpha m_k\,(t)\right\|_{\mathcal{B}(X,Y)} \;=\; \left\|\left|2^{k-1}t\right|^{|\alpha|}\; D^\alpha m\left(2^{k-1}t\right)\right\|_{\mathcal{B}(X,Y)}\;;$$

thus (4.25) implies (4.23) since

$$\|D^\alpha m_k\,|_{I_1}\|_{L_\infty} \;\leq\; \left\|(1+|\cdot|)^{|\alpha|}\, D^\alpha m\,(\cdot)\right\|_{L_\infty} \;\leq\; A\;.$$

So by Lemma 4.10, m satisfies condition (4.13) of Theorem 4.8. □

The next results generalizes Hörmanders condition.

Corollary 4.13. *Let X and Y have Fourier type p and $l = \left[\frac{N}{p}\right] + 1$. Let $m \in C^l(\mathbb{R}^N, \mathcal{B}(X,Y))$ satisfy, for some constant A,*

$$\left[\int_{|t|\leq 2} \|D^\alpha m(t)\|^p\; dt\right]^{1/p} \;\leq\; A \tag{4.27}$$

and, for $1 \leq R < \infty$,

$$\left[R^{-N}\int_{R<|t|<4R} \|D^\alpha m(t)\|^p\; dt\right]^{1/p} \leq A R^{-|\alpha|} \tag{4.28}$$

for each multi-index $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq l$. Then m is a Fourier multiplier from $B^s_{q,r}\left(\mathbb{R}^N, X\right)$ to $B^s_{q,r}\left(\mathbb{R}^N, Y\right)$ for each $s \in \mathbb{R}$ and $q, r \in [1,\infty]$.

Proof. Keeping with the notation from Lemma 4.10, let $u = p$. Fix $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq l$. Then (4.27) is just (4.22). Also, (4.28) and (4.26) give that

$$\|D^\alpha m_k\,|_{I_1}\|_{L_p} \;\leq\; 4^l A$$

for each $k \in \mathbb{N}$. So by Lemma 4.10, m satisfies condition (4.13) of Theorem 4.8. □

Variants of Corollary 4.11, where the assumption (4.25) of bounds on derivatives is replaced by Lipschitz conditions, follow from the formal statement of Theorem 4.8. The proof of the next corollary shows how to obtain such variants.

Corollary 4.14. *Let X and Y have Fourier type $p \in (1,2]$. Assume that $m \colon \mathbb{R} \to \mathcal{B}(X,Y)$ satisfies, for some constant A and $l \in (1/p,1)$,*

$$\|m(t)\| \;\leq\; A \quad \textit{for} \quad t \in \mathbb{R} \tag{4.29}$$

$$(1+|t|)^l \left\| \frac{m(t+u)-m(t)}{|u|^l} \right\| \;\leq\; A \quad \textit{for} \quad u,t \in \mathbb{R}\ ,\ u \neq 0\ . \tag{4.30}$$

Then m is a Fourier multiplier from $B^s_{q,r}(X)$ to $B^s_{q,r}(Y)$ for each $s \in \mathbb{R}$ and $1 \leq q,r \leq \infty$.

Proof. It suffices to show that there is a constant C so that, following the notation from (2.7) and with $m_k(\cdot) := m\left(2^{k-1}\cdot\right)$,

$$B^{1/p}_{p,1}(\varphi_0 m) \leq C \quad \text{and} \quad B^{1/p}_{p,1}(\varphi_1 m_k) \leq C \quad \text{for each } k \in \mathbb{N}\ ; \tag{4.31}$$

indeed, for then m satisfies (4.13) from Theorem 4.8 by (4.24), (2.6), and (4.29).

Let $u_0 = 9$; thus, if $|h| \geq u_0$ then $I_j \cap (I_j - h) = \emptyset$ for $j = 0,1$.

Fix $k \in \mathbb{N}$. Note that if $t \in I_1$ and $h \in \mathbb{R} \setminus \{0\}$ then by (4.30)

$$\left\| \frac{m_k(t+h)-m_k(t)}{|h|^l} \right\| \;\leq\; |t|^l \left\| \frac{m_k(t+h)-m_k(t)}{|h|^l} \right\| \;\leq\; A\ . \tag{4.32}$$

If $|h| \leq u_0$, then by (4.32) and (4.29),

$$\begin{aligned}
\|\varphi_1 m_k(\cdot+h) - \varphi_1 m_k(\cdot)\|_{L_p} &= \Bigg\| \varphi_1(\cdot)\,|h|^l\, \frac{m_k(\cdot+h)-m_k(\cdot)}{|h|^l} \chi_{I_1}(\cdot) \quad + \\
&\qquad m_k(\cdot+h)\,|h|\, \frac{\varphi_1(\cdot+h)-\varphi_1(\cdot)}{|h|} \chi_{(I_1-h)\cup I_1}(\cdot) \Bigg\|_{L_p} \\
&\leq\; \|\varphi_1\|_{L_p} |h|^l A \;+\; A\,|h|\, \|\varphi_1'\|_{L_\infty} \left\|\chi_{(I_1-h)\cup I_1}\right\|_{L_p} \\
&\leq\; A \left[\|\varphi_1\|_{L_p} + u_0^{1-l} \|\varphi_1'\|_{L_\infty} (2\,|I_1|)^{1/p} \right] |h|^l\ .
\end{aligned} \tag{4.33}$$

If $|h| \geq u_0$, then by (4.29)

$$\|\varphi_1 m_k(\cdot+h) - \varphi_1 m_k(\cdot)\|_{L_p} \;=\; 2^{1/p}\, \|\varphi_1 m_k\|_{L_p} \;\leq\; 2^{1/p}\, \|\varphi_1\|_{L_p} A\ . \tag{4.34}$$

Thus, by (4.33) and (4.34), there is a constant C_1, dependent on φ_1 but independent of k, so that (following the notation from (2.7))

$$\begin{aligned}
w_p(\varphi_1 m_k, u) &:= \sup_{|h| \leq u} \|\varphi_1 m_k(\cdot+h) - \varphi_1 m_k(\cdot)\|_{L_p} \\
&\leq \begin{cases} C_1 u^l & \text{if } 0 \leq u \leq u_0 \\ C_1 & \text{if } u \geq u_0\ . \end{cases}
\end{aligned} \tag{4.35}$$

Similar calculations show that (4.35) holds for $\varphi_0 m$ with some constant C_0 depending on φ_0. Thus (4.31) holds. □

The proof of Theorem 4.8 also gives a result for essentially scalar valued multipliers which has the same smoothness requirement as in the classical theorems but without any Fourier type assumptions on the Banach spaces.

Corollary 4.15. *Let* $\{X_j\}_{j=1}^{n_1}$ *and* $\{Y_i\}_{i=1}^{n_2}$ *be Banach spaces and* $S_{ij} \in \mathcal{B}(X_j, Y_i)$ *for* $i = 1, \ldots, n_2$ *and* $j = 1, \ldots, n_1$. *Put* $X = X_1 \times \ldots \times X_{n_1}$ *and* $Y = Y_1 \times \ldots \times Y_{n_2}$ *and* $S = \{S_{ij}\}_{i,j}$. *Assume that* $n \colon \mathbb{R}^N \to \mathbb{C}^{n_2}$ *satisfies for each* $k \in \mathbb{N}_0$

$$\varphi_k n \in B_{2,1}^{N/2}\left(\mathbb{R}^N, \mathbb{C}^{n_2}\right) \qquad \text{and} \qquad \mathcal{M}_2(\varphi_k n) \ \le \ A \ . \tag{4.36}$$

Then $t \to m(t) := n(t)\, S \in \mathcal{B}(X, Y)$ *is a Fourier multiplier from* $B_{q,r}^s\left(\mathbb{R}^N, X\right)$ *to* $B_{q,r}^s\left(\mathbb{R}^N, Y\right)$ *for each* $s \in \mathbb{R}$ *and* $q, r \in [1, \infty]$.

Remark 4.16. Condition (4.36) follows from either condition (4.25) of Corollary 4.11 or conditions (4.27) and (4.28) of Corollary 4.13 or conditions (4.29) and (4.30) of Corollary 4.14 with $p = 2$ adjusted to this special setting.

Proof. Looking back at the proof of Theorem 4.8, one sees that one only needs that the $\varphi_k m$'s induce Fourier multiplier operators $T_{m\varphi_k}$'s on $L_q\left(\mathbb{R}^N, X\right)$ which satisfy the conditions in Remark 4.7 and

$$\|T_{m\varphi_k}\|_{L_q(X) \to L_q(Y)} \ \le \ C\mathcal{M}_2(\varphi_k n) \ .$$

But this follows from (4.36); indeed, just make minor modifications to the proof of Theorem 4.3, applying Corollary 3.2 to $\varphi_k n \in B_{2,1}^{N/2}\left(\mathbb{R}^N, \mathbb{C}^{n_2}\right)$, which is valid since $\mathbb{C}^{n_2}$ has Fourier type 2 (Plancherel's Theorem). Towards this, note that there exists a constant C_1, dependent on the choice of product topologies, so that

$$\|\alpha y\|_Y \le C_1 \|\alpha\|_{\mathbb{C}^{n_2}} \|y\|_Y$$

for each $\alpha \in \mathbb{C}^{n_2}$ and $y \in Y$. If $\varphi_k n \in \mathcal{S}(\mathbb{C}^{n_2})$, then (4.11) takes the form

$$\left\|(\varphi_k n)^\vee \ Sx\right\|_{L_1(Y)} \ \le \ 2\,C_1 \ \|S\|\, \mathcal{M}_2(\varphi_k n)\, \|x\|_X$$

for each $x \in X$ while (4.12) takes the form

$$\left\|(\varphi_k n)^\vee \ S^* y^*\right\|_{L_1(X^*)} \ \le \ 2\,C_1 \ \|S\|\, \mathcal{M}_2(\varphi_k n)\, \|y^*\|_{Y^*}$$

for each $y^* \in Y^*$. Thus by Lemma 4.5, $\varphi_k m$ induces the Fourier multiplier operator

$$(T_{\varphi_k m} f)(t) \ := \int\limits_{\mathbb{R}^N} [\varphi_k n]^\vee \ (t-s) S f(s)\, ds$$

with $\|T_{\varphi_k m}\|_{L_q \to L_q} \le 2\,C_1 \ \|S\|\, \mathcal{M}_2(\varphi_k n)$. The general case where $\varphi_k n \in B_{2,1}^{N/2}$ follows, as in the proof of Theorem 4.3, by approximation. □

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