THE DUAL OF THE JAMES TREE SPACE IS ASYMPTOTICALLY UNIFORMLY CONVEX

MARIA GIRARDI

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ABSTRACT. The dual of the James Tree space is asymptotically uniformly convex.

1. INTRODUCTION

In 1950, R. C. James [J1] constructed a Banach space which is now called the James space. This space, along with its many variants (such as the James tree space [J2]) and their duals and preduals, have been a rich source for further research and results (both positive ones and counterexamples), answering many questions, several of which date back to Banach [B, 1932]. See [FG] for a splendid survey of such spaces.

This paper’s main result, Theorem 5, shows that the dual $JT^*$ of the James tree space $JT$ is asymptotically uniformly convex. (See Section 2 for definitions.)

Schachermayer [S, Theorem 4.1] showed that $JT^*$ has the Kadec-Klee property. It follows from Theorem 5 of this paper that $JT^*$ enjoys the uniform Kadec-Klee property. Of course, the same can be said about the (unique) predual $JT_*$ of $JT$. In fact, Theorem 3 shows that the modulus of asymptotic convexity of $JT_*$ is of power type 3.

Johnson, Lindenstrauss, Preiss, and Schechtman [JLPS] showed that an asymptotically uniformly convex space has the point of continuity property and thus asked whether an asymptotically uniformly convex space has the Radon-Nikodým property. It is well-known that both $JT_*$ and $JT^*$ have the point of continuity property yet fail the Radon-Nikodým property. It follows from Theorem 5 of this paper that $JT^*$ is an asymptotically uniformly convex (dual) space without the Radon-Nikodým property. Thus $JT_*$ is a separable asymptotically uniformly convex space without the Radon-Nikodým property. To the best of the author’s knowledge, these are the first known examples of asymptotically uniformly convex spaces without the Radon-Nikodým property.

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2. Definitions and Notation

Throughout this paper $\mathfrak{X}$ denotes an arbitrary (infinite-dimensional real) Banach spaces. If $\mathfrak{X}$ is a Banach space, then $\mathfrak{X}^*$ is its dual space, $B(\mathfrak{X})$ is its (closed) unit ball, $S(\mathfrak{X})$ is its unit sphere, $\hat{\tau} : \mathfrak{X} \to \mathfrak{X}^{**}$ is the natural point-evaluation isometric embedding, $\hat{x} = \hat{\tau}(x)$ and $\hat{x} = \hat{\tau}(\mathfrak{X})$. If $Y$ is a subset of $\mathfrak{X}$, then $[Y]$ is the closed linear span of $Y$ and

$$\mathcal{N}(\mathfrak{X}) = \left\{ [x_i]_{1 \leq i \leq n}^T : x_i^* \in \mathfrak{X}^* \text{ and } n \in \mathbb{N} \right\}$$

$$\mathcal{W}(\mathfrak{X}^*) = \left\{ [x_i]_{1 \leq i \leq n}^\perp : x_i \in \mathfrak{X} \text{ and } n \in \mathbb{N} \right\} .$$

Thus $\mathcal{N}(\mathfrak{X})$ is the collection of (norm-closed) finite codimensional subspaces of $\mathfrak{X}$ while $\mathcal{W}(\mathfrak{X}^*)$ is the collection of weak-star closed finite codimensional subspaces of $\mathfrak{X}^*$. All notation and terminology, not otherwise explained, are as in [DU, LT1, LT2].

The \textit{modulus of convexity} $\delta_{\mathfrak{X}} : [0, 2] \to [0, 1]$ of $\mathfrak{X}$ is

$$\delta_{\mathfrak{X}}(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| : x, y \in S(\mathfrak{X}) \text{ and } \|x - y\| \geq \varepsilon \right\}$$

and $\mathfrak{X}$ is \textit{uniformly convex (UC)} if and only if $\delta_{\mathfrak{X}}(\varepsilon) > 0$ for each $\varepsilon \in (0, 2]$. The \textit{modulus of asymptotic convexity} $\overline{\delta}_{\mathfrak{X}} : [0, 1] \to [0, 1]$ of $\mathfrak{X}$ is

$$\overline{\delta}_{\mathfrak{X}}(\varepsilon) = \inf_{x \in S(\mathfrak{X})} \sup_{y \in \mathcal{N}(\mathfrak{X})} \inf_{y \in S(\mathfrak{U})} \left[ \|x + \varepsilon y\| - 1 \right]$$

and $\mathfrak{X}$ is \textit{asymptotically uniformly convex (AUC)} if and only if $\overline{\delta}_{\mathfrak{X}}(\varepsilon) > 0$ for each $\varepsilon$ in $(0, 1]$.

A space $\mathfrak{X}$ has the \textit{Kadec-Klee (KK) property} provided the relative norm and weak topologies on $B(\mathfrak{X})$ coincide on $S(\mathfrak{X})$. A space $\mathfrak{X}$ has the \textit{uniform Kadec-Klee (UKK) property} provided for each $\varepsilon > 0$ there exists $\delta > 0$ such that every $\varepsilon$-separated weakly convergent sequence $\{x_n\}$ in $B(\mathfrak{X})$ converges to an element of norm less than $1 - \delta$.

Related to the above geometric isometric properties are the following geometric isomorphic properties.

- $\mathfrak{X}$ has the \textit{Radon-Nikodým property (RNP)} provided each bounded subset of $\mathfrak{X}$ has non-empty slices of arbitrarily small diameter.
- $\mathfrak{X}$ has the \textit{point of continuity property (PCP)} provided each bounded subset of $\mathfrak{X}$ has non-empty relatively weakly open subsets of arbitrarily small diameter.
- $\mathfrak{X}$ has the \textit{complete continuity property (CCP)} provided each bounded subset of $\mathfrak{X}$ is Bocce denteable.
Implications between these various properties are summarized in the diagram below.

\[
\begin{array}{cccc}
\text{UC} & \rightarrow & \text{AUC} & \rightarrow \\
\downarrow & & \downarrow & \rightarrow \\
\text{RNP} & \rightarrow & \text{PCP} & \rightarrow \text{CCP}
\end{array}
\]

Helpful notation is
\[
\delta_X(\varepsilon) = \inf_{x \in S(\mathcal{X})} \delta_x(\varepsilon, x)
\]
where
\[
\delta_x(\varepsilon, x) = \sup_{y \in \mathcal{X}(\mathcal{X})} \inf_{y \in \mathcal{Y}} \left[ \|x + \varepsilon y\| - 1 \right].
\]

Note that, for each \(x \in S(\mathcal{X})\),
\[
\delta_x(\varepsilon, x) = \sup_{y \in \mathcal{Y}} \inf_{y \in \mathcal{Y}} \left[ \|x + y\| - 1 \right]
\]
and so \(\delta_x(\varepsilon, x)\) is a non-decreasing function of \(\varepsilon\). Thus \(\delta_X\) is a non-decreasing Lipschitz functions with Lipschitz constant at most one. For any space \(\mathcal{X}\) and \(\varepsilon \in [0, 1]\)
\[
\delta_X(\varepsilon) \leq \varepsilon = \delta_{\ell_1}(\varepsilon);
\]
thus, \(\ell_1\) is, in some sense, the most asymptotically uniformly convex space.

Uniform convexity, the KK property, and the UKK property have been extensively studied (for example, see [DGZ, LT2]). Asymptotic uniform convexity has been examined explicitly in [JLPS, M] and implicitly in [GKL, KOS]. The RNP, PCP, and CCP have also been extensively studied (for example, see [DU, GGMS, G1, G2]).

The \(JT\) space is construction on a (binary) tree
\[
\mathcal{T} = \bigcup_{n=0}^{\infty} \Delta_n
\]
where \(\Delta_n\) is the \(n\)th-level of the tree; thus,
\[
\Delta_0 = \{\emptyset\} \quad \text{and} \quad \Delta_n = \{-1, +1\}^n
\]
for each \(n \in \mathbb{N}\). The finite tree \(\mathcal{T}_N\) up through level \(N \in \mathbb{N} \cup \{0\}\) is
\[
\mathcal{T}_N = \bigcup_{n=0}^{N} \Delta_n
\]
The tree \(\mathcal{T}\) is equipped with its natural (tree) ordering: if \(t_1\) and \(t_2\) are elements of \(\mathcal{T}\), then \(t_1 < t_2\) provided one of the follow holds:

(1) \(t_1 = \emptyset\) and \(t_2 \neq \emptyset\)
(2) for some $n, m \in \mathbb{N}$

$$t_1 = (\varepsilon_1^1, \varepsilon_2^1, \ldots, \varepsilon_n^1) \quad \text{and} \quad t_2 = (\varepsilon_1^2, \varepsilon_2^2, \ldots, \varepsilon_m^2)$$

with $n < m$ and $\varepsilon_i^1 = \varepsilon_i^2$ for each $1 \leq i \leq n$.

A (finite) segment of $T$ is a linearly order subset $\{t_n, t_{n+1}, \ldots, t_{n+k}\}$ of $T$ where $t_i \in \Delta_i$ for each $n \leq i \leq n + k$. A branch of $T$ is a linearly order subset $\{t_0, t_1, t_2, \ldots\}$ of $T$ where $t_i \in \Delta_i$ for each $i \in \mathbb{N} \cup \{0\}$.

The James-Tree space $JT$ is the completion of the space of finitely supported functions $x : T \to \mathbb{R}$ with respect to the norm

$$\|x\|_{JT} = \sup \left\{ \left[ \sum_{i=1}^{n} \left| \sum_{t \in S_i} x_t \right|^2 \right]^{\frac{1}{2}} : S_1, S_2, \ldots, S_n \text{ are disjoint segments of } T \right\}.$$ 

By lexicographically ordering $T$, the sequence $\{\eta_t\}_{t \in T}$ in $JT$, where

$$\eta_t(s) = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \neq s \end{cases},$$

forms a monotone boundedly complete monotone (Schauder) basis of $JT$ with biorthogonal functions $\{\eta_t^*\}_{t \in T}$ in $JT^*$. Thus $JT^* = [\eta_t^*]_{t \in T}$.

For $N, M \in \mathbb{N} \cup \{0\}$ with $N \leq M$, the restriction maps from $JT$ to $JT$ given by

$$\pi_N(x) = \sum_{t \in \Delta_N} \eta_t^*(x) \eta_t$$

$$\pi_{[N,M]}(x) = \sum_{t \in \bigcup_{i=N}^{M} \Delta_i} \eta_t^*(x) \eta_t$$

$$\pi_{[N,\infty)}(x) = \sum_{t \in \bigcup_{i=N}^{\infty} \Delta_i} \eta_t^*(x) \eta_t$$

are each contractive projections (by the nature of the norm on $JT$); thus, so are their adjoints.

Let $\Gamma$ be the set of all branches of $T$. Then [LS, Theorem 1] the mapping

$$\pi_\infty : JT^* \to \ell_2(\Gamma)$$

given by

$$\pi_\infty(x^*) = \left\{ \lim_{t \in B} x^*(\eta_t) \right\}_{B \in \Gamma}.$$
is an isometric quotient mapping with kernel $JT$. Also, for each $x^* \in JT^*$,
\[
\|x^*\| = \lim_{N \to \infty} \|\pi_{[0,N]}^* x^*\|
\]
\[
\|\pi_{\infty}^* x^*\| = \lim_{N \to \infty} \|\pi_{[N,N]}^* x^*\| = \lim_{N \to \infty} \|\pi_{N}^* x^*\|
\]
by the weak-star lower semicontinuity of the norm on $JT^*$.

To show that $JT^*$ has the Kadec-Klee property, Schachermayer calculated the below two quantitative bounds.

**Fact 1.** [S, Lemma 3.8] Let
\[
f_1 : (0,1) \to (0,\infty)
\]
be a continuous strictly increasing function satisfying $f_1(t) < 2^{-10t^3}$ for each $t \in (0, 1)$. Let $N \in \mathbb{N}$ and $z^* \in JT^*$. If
\[
[1 - f_1(t)] \|z^*\| < \|\pi_{[0,N]}^* z^*\|
\]
then
\[
\|\pi_{[N,N]}^* z^*\| < \|\pi_{N}^* z^*\| + t \|z^*\| .
\]

**Fact 2.** [S, Lemma 3.11] Let
\[
f_2 : (0,1) \to (0,\infty)
\]
be a continuous strictly increasing function satisfying $f_2(t) < 2^{-26t^5}$ for each $t \in (0, 1)$. Let $N \in \mathbb{N}$ and $\varepsilon_0 \in (0, 1)$ and $\tilde{x}^*, \tilde{u}^* \in JT^*$. If
\[
\begin{align*}
(2.1) & \quad \|\pi_{[N,N]}^* \tilde{x}^*\| \leq 1 \\
(2.2) & \quad \|\pi_{N}^* \tilde{x}^*\| > 1 - f_2(\varepsilon_0) \\
(2.3) & \quad \|\pi_{\infty} \tilde{x}^*\| > 1 - f_2(\varepsilon_0) \\
(2.4) & \quad \|\pi_{[N,N]}^* (\tilde{x}^* + \tilde{u}^*)\| \leq 1 \\
(2.5) & \quad \|\pi_{N}^* \tilde{u}^*\| < f_2(\varepsilon_0) \\
(2.6) & \quad \|\pi_{\infty} \tilde{u}^*\| < f_2(\varepsilon_0) .
\end{align*}
\]
Then
\[
(2.7) \quad \|\pi_{[N,N]}^* \tilde{u}^*\| < \varepsilon_0 .
\]

3. Results

Theorem 3 shows that the modulus of asymptotic convexity of $JT_*$ is of power type 3. Its proof uses Fact 1.

**Theorem 3.** There exists a positive constant $k$ so that
\[
\delta_{JT_*}(\varepsilon) \geq k\varepsilon^3
\]
for each $\varepsilon \in (0, 1)$. Thus $JT_*$ is asymptotically uniformly convex.
Proof. Fix \( c \in (0, 2^{-10}) \) and find \( k \) so that
\[
0 < k (1 + k)^2 \leq c .
\] (1)

Fix \( \varepsilon \in (0, 1) \) and a finitely supported \( x_* \in S(JT_*) \). It suffices to show that
\[
\overline{\delta}_{JT_*}(\varepsilon, x_*) \geq k\varepsilon^3 .
\] (2)

Find \( N \in \mathbb{N} \) so that
\[
\pi^*_{[0,N-1]} \tilde{x}_* = \tilde{x}_*
\]
and let
\[
\mathcal{Y} = \left[ \eta_t \right]_{t \in T_N} .
\]

Fix \( y_* \in S(Y) \).
Assume that
\[
\varepsilon < \left[ 1 + k\varepsilon^3 \right] f_1^{-1}\left( \frac{k\varepsilon^3}{1 + k\varepsilon^3} \right) .
\] (3)

But inequality (3) is equivalent to
\[
c^{1/3} < k^{1/3} \left( 1 + k\varepsilon^3 \right)^{2/3} ,
\]
which contradicts (1). Thus \( \|x_* + \varepsilon y_*\| - 1 \geq k\varepsilon^3 \) and so (2) holds. \( \square \)

A modification of the proof of Theorem 3 shows that, for each \( \varepsilon \in (0, 1) \), the \( \overline{\delta}_{JT_*}(\varepsilon, x^*) \) stays uniformly bounded below from zero for \( x^* \in S(JT^*) \) whose \( \|\pi_{\infty} x^*\| \) is small. Recall that if \( x_* \in JT_* \) then \( \|\pi_{\infty} \tilde{x}_*\| = 0 \).

**Lemma 4.** For each \( \varepsilon \in (0, 1) \) there exists \( \eta = \eta(\varepsilon) > 0 \) so that
\[
\inf_{x^* \in S(JT^*)} \sup_{\|\pi_{\infty} x^*\| \leq \eta} \inf_{\mathcal{Y} \in \mathcal{W}(JT^*)} \left[ \|x^* + \varepsilon y^*\| - 1 \right] > 0 .
\]

Proof. Fix \( \varepsilon \in (0, 1) \). Keeping with the notation in Fact 1, find \( \delta, \eta_2 > 0 \) so that
\[
4\eta_2 + \frac{\delta}{1 - f_1(\delta)} < \varepsilon .
\]

Fix \( x^* \in S(JT^*) \) with
\[
\|\pi_{\infty} x^*\| \equiv b \leq \eta_2 .
\]
It suffices to show that
\[
\sup_{y \in W(JT^*)} \inf_{y^* \in S(Y)} \|x^* + \varepsilon y^*\| \geq \frac{1}{1 - f_1(\delta)}. \tag{4}
\]

Fix \(\eta_1 \in (0, 1)\). Find \(N \in \mathbb{N}\) so that
\[
1 - \eta_1 \leq \left\| \pi_{[0,N]} x^* \right\| \quad \text{and} \quad \left\| \pi_{[N,\omega]}^* x^* \right\| < b + \eta_2
\]
and let
\[
\mathcal{Y} = [\eta_1]_{t \in \mathcal{T}N}.
\]
Fix \(y^* \in S(\mathcal{Y})\).
Assume that
\[
\|x^* + \varepsilon y^*\| < \frac{1 - \eta_1}{1 - f_1(\delta)}.
\]
Then
\[
[1 - f_1(\delta)] \|x^* + \varepsilon y^*\| < \left\| \pi_{[0,N]}^* x^* \right\| = \left\| \pi_{[0,N]}^* (x^* + \varepsilon y^*) \right\|.
\]
Thus by Fact 1
\[
\left\| \pi_{[N,\omega]}^* (x^* + \varepsilon y^*) \right\| < \left\| \pi_N^* (x^* + \varepsilon y^*) \right\| + \delta \|x^* + \varepsilon y^*\|
\]
and so
\[
\varepsilon - (b + \eta_2) < (b + \eta_2) + \frac{\delta}{1 - f_1(\delta)}.
\]
But \(b \leq \eta_2\) and so
\[
\varepsilon < 4\eta_2 + \frac{\delta}{1 - f_1(\delta)}.
\]
A contradiction, thus
\[
\|x^* + \varepsilon y^*\| \geq \frac{1 - \eta_1}{1 - f_1(\delta)}.
\]
Since \(\eta_1 > 0\) was arbitrary, inequality (4) holds. \(\square\)

Thus to show that \(JT^*\) is asymptotically uniformly convex, one just needs to examine \(\delta_{JT^*}(\varepsilon, x^*)\) for \(x^* \in S(JT^*)\) whose \(\|\pi_{\infty} x^*\|\) is not small. Fact 2 is used for this case.

**Theorem 5.** \(JT^*\) is asymptotically uniformly convex.

**Proof.** Fix \(\varepsilon \in (0, 1)\) and let \(\varepsilon_0 = \varepsilon/4\). Let \(f_1: (0, 1) \rightarrow (0, 2^{12})\) be given by \(f_1(t) = 2^{-12} \cdot t^3\) and \(f_2\) be a function satisfying the hypothesis in Fact 2. Find \(\delta, \eta_2 > 0\) so that
\[
4\eta_2 + \frac{\delta}{1 - f_1(\delta)} < \varepsilon.
\]
Next find $\gamma_i > 0$ and $\tau > 1$ so that

\begin{align*}
\gamma_3 &< \gamma_2 < \frac{1}{2} \\
\tau &\leq \frac{(1 - \gamma_1)(1 - \gamma_2)}{1 - f_2(\varepsilon_0)} \\
\tau &< \frac{1 - \gamma_2}{\sqrt{1 - f_2^2(\varepsilon_0)}} \\
\tau &\leq \frac{\eta_2^3 \gamma_3^3}{2^{15}(1 - \gamma_2)^3} - \gamma_4 + 1
\end{align*}

(5) \hspace{1cm} (6) \hspace{1cm} (7) \hspace{1cm} (8)

\begin{align*}
\frac{\tau - 1 + \gamma_4}{\tau} &< f_1(1) \\
\tau &\leq \frac{1}{1 - f_1(\delta)}.
\end{align*}

(9) \hspace{1cm} (10)

Fix $x^* \in S(JT^*)$. It suffices to show that

$$\sup_{\mathcal{Y} \in \mathfrak{m}(JT^*)} \inf_{y \in S(\mathcal{Y})} \|x^* + \varepsilon y^*\| \geq \tau.$$  \hspace{1cm} (11)

Let

$$\|\pi_\infty x^*\| \equiv b.$$  

If $b \leq \eta_2$, then by the proof of Lemma 4 and (10), inequality (11) holds. So let $b > \eta_2$. Find $N \in \mathbb{N}$ so that

$$\begin{align*}
(1 - \gamma_1) b &< \|\pi_N^* x^*\| \leq \|\pi_{[N,\infty)}^* x^*\| < b \left( \frac{1 - \gamma_3}{1 - \gamma_2} \right) < \frac{b}{1 - \gamma_2} \\
1 - \gamma_4 &< \|\pi_{[0,N]}^* x^*\|.
\end{align*}$$  \hspace{1cm} (12) \hspace{1cm} (13)

Let $g_{x^*} \in JT^{**}$ be the functional given by

$$g_{x^*}(z^*) = \langle \pi_\infty z^*; \pi_\infty x^* \rangle_{H_2}$$

where the inner product in the natural inner product on $\ell_2(\Gamma)$. Let

$$\mathcal{Y} = [\eta_{[1]}^\perp \cap [g_{x^*}]^\top]$$

and fix $y^* \in S(\mathcal{Y})$.

Assume that

$$\|x^* + \varepsilon y^*\| < \tau.$$  \hspace{1cm} (14)

It suffices to find a contradiction to (14). Towards this, let

$$\tilde{x}^* = \frac{1 - \gamma_2}{\tau b} x^* \quad \text{and} \quad \tilde{y}^* = \frac{1 - \gamma_2}{\tau b} y^*.$$
It suffices to show (keeping with the same notation but with \( \bar{u}^* = \varepsilon \bar{y}^* \)) that conditions (2.1) through (2.6) of Fact 2 hold; for then condition (2.7) holds and so by (5)

\[
\varepsilon_0 > \left\| \pi^*_{[N,\omega]} \varepsilon \bar{y}^* \right\| = \frac{1 - \gamma_2}{\tau b} \varepsilon \geq \frac{\varepsilon}{4} = \varepsilon_0.
\]

Condition (2.1) follows from (12) since

\[
\left\| \pi^*_{[N,\omega]} \bar{x}^* \right\| \leq \frac{1 - \gamma_2}{\tau b} \frac{b}{1 - \gamma_2} \leq 1.
\]

Condition (2.2) follows from (12) and (6) since

\[
\left\| \pi^*_N \bar{x}^* \right\| > \frac{1 - \gamma_2}{\tau b} (1 - \gamma_1) b = \frac{(1 - \gamma_1)(1 - \gamma_2)}{\tau} \geq 1 - f_2(\varepsilon_0).
\]

Towards condition (2.3), note that by (7)

\[
\left\| \pi^*_\infty \bar{x}^* \right\| = \frac{1 - \gamma_2}{\tau b} b = \frac{1 - \gamma_2}{\tau} > \sqrt{1 - f_2^2(\varepsilon_0)} \quad (15)
\]

and so

\[
\left\| \pi^*_\infty \bar{x}^* \right\| > 1 - f_2(\varepsilon_0).
\]

Towards condition (2.4), note that by (14) and (13)

\[
\left\| x^* + \varepsilon y^* \right\| < \frac{\tau}{1 - \gamma_4} \left\| \pi^*_{[0,N]} (x^* + \varepsilon y^*) \right\|.
\]

Thus by Fact 1 and (9)

\[
\left\| \pi^*_{[N,\omega]} (x^* + \varepsilon y^*) \right\|
\leq b \frac{1 - \gamma_3}{1 - \gamma_2} + \tau^4 \left( \frac{\tau - 1 + \gamma_4}{\tau} \right)^{1/3}.
\]

Thus condition (2.4) holds provided

\[
b \frac{1 - \gamma_3}{1 - \gamma_2} + \tau^4 \left( \frac{\tau - 1 + \gamma_4}{\tau} \right)^{1/3} \leq \frac{\tau b}{1 - \gamma_2},
\]

or equivalently

\[
\tau^{2/3} \left( \tau - 1 + \gamma_4 \right)^{1/3} \leq \frac{b(\tau - 1 + \gamma_3)}{2^4(1 - \gamma_2)}.
\]
But by (8) and that $b > \eta_2$

$$\frac{\tau^{2/3}}{(\tau - 1 + \gamma_4)^{1/3}} \leq 2 (\tau - 1 + \gamma_4)^{1/3} \leq \frac{2\eta_2\gamma_3}{2(1 - \gamma_2)}$$

$$\leq \frac{b\gamma_3}{2^4(1 - \gamma_2)} \leq \frac{b(\tau - 1 + \gamma_3)}{2^4(1 - \gamma_2)}.$$  

Thus condition (2.4) holds.

Condition (2.5) follows from the fact that $y^* \in [\eta; \epsilon]$, Towards con-
dition (2.6), since $y^* \in [g; \epsilon]$, the vectors $\pi_x y^*$ and $\pi_x x^*$ are orthogonal
in $\ell_2(\Gamma)$ and so

$$\|\pi_x y^*\| = \|\pi_x (\perp y^*)\|^2 - \|\pi_x x^*\|^2;$$

but $\pi_x = \pi_x \pi_{x[N; \epsilon]}$ and so by condition (2.4) and (15)

$$\|\pi_x y^*\|^2 \leq \|\pi_x (\perp y^*)\|^2 - \|\pi_x x^*\|^2$$

$$< 1 - \left(1 - f_2^2(\epsilon_0)\right) = f_2^2(\epsilon_0).$$

Thus condition (2.6). \hfill \Box

The proof in [JLPS] that an asymptotically uniformly convex space has
the PCP show that if $\delta_X(\epsilon) > 0$ for each $\epsilon \in (0, 1]$ then $X$ has the PCP. A
bit more can be said.

**Proposition 6.** If $\delta_X(\frac{1}{2}) > 0$ then $X$ has the PCP.

The proof of Proposition 6 uses the following (essentially known) lemma.

**Lemma 7.** Let $X$ be a space without the PCP and $0 < \epsilon < 1$. Then there
is a closed subset $A$ of $X$ so that

1. each (nonempty) relatively weakly open subset of $A$ has diameter
larger than $1 - \epsilon$

2. $\sup\{\|a\| : a \in A\} = 1.$

**Proof of Lemma 7.** Let $X$ fail the PCP and $0 < \epsilon < 1$. By a standard
argument (e.g., see [SSW, Prop. 4.10]), there is a closed subset $\widetilde{A}$ of $X$ of
diameter one such that each (nonempty) relatively weakly open subset of
$\widetilde{A}$ has diameter larger than $1 - \epsilon$. Without loss of generality $0 \in \widetilde{A}$ (just
consider a translate of $\widetilde{A}$). Let

$$b = \sup\{\|x\| : x \in \widetilde{A}\} \quad \text{and} \quad A = \frac{\widetilde{A}}{b}.$$ 

Note that $0 < b \leq 1$. If $V$ is (nonempty) relatively weakly open subset of $A$, then $bV$ is a relatively weakly open subset of $\widetilde{A}$ and so

$$\text{diam } V = \frac{1}{b} \text{ diam } bV > 1 - \epsilon.$$
Thus $A$ does the job. 

**Proof of Proposition 6.** Let $\mathcal{X}$ be a Banach space without the PCP. Fix $t \in \left(0, \frac{1}{2}\right)$ and $\delta \in (0, t)$. It suffices to show that $\delta(\mathcal{X}) \leq 2\delta$.

Find a subset $A$ of $\mathcal{X}$ which satisfies the conditions of Lemma 7 with $\varepsilon = 1 - 2t$ and find $a \in A$ so that

$$\left\| \frac{a}{\|a\|} - a \right\| < \delta.$$ 

Let $\mathcal{Y} \in \mathcal{M}(\mathcal{X})$. It suffices to show that

$$\inf_{y \in \mathcal{Y}} \left( \sup_{\|y\| \geq t} \left[ \left\| \frac{a}{\|a\|} + y \right\| - 1 \right] \right) \leq 2\delta.$$ 

By condition (1) of Lemma 7 there exists $x \in A$ so that $\|x - a\| \geq t$ and $x - a$ is almost in $\mathcal{Y}$; thus, by a standard perturbation argument (e.g., see [GJ, Lemma 2]) there exists $y \in \mathcal{Y}$ so that

$$\|y\| \geq t \quad \text{and} \quad \|y - (x - a)\| < \delta.$$ 

Thus

$$\left\| \frac{a}{\|a\|} + y \right\| \leq \left\| \frac{a}{\|a\|} - a \right\| + \|y - x + a\| + \|x\| < 1 + 2\delta.$$ 

Thus $\delta(\mathcal{X}) = 0$. 

The observation below formalizes an essentially known fact, which to the best of the author’s knowledge, has not appeared in print as such. Recall that the **modulus of asymptotic smoothness** $\overline{\rho}_\mathcal{X} : [0, 1] \to [0, 1]$ of $\mathcal{X}$ is

$$\overline{\rho}_\mathcal{X}(\varepsilon) = \sup_{x \in S(\mathcal{X})} \inf_{\mathcal{Y} \in \mathcal{M}(\mathcal{X})} \sup_{y \in S(\mathcal{Y})} \left[ \|x + \varepsilon y\| - 1 \right]$$

and $\mathcal{X}$ is **asymptotically uniformly smooth** if and only if $\lim_{\varepsilon \to 0^+} \overline{\rho}_\mathcal{X}(\varepsilon)/\varepsilon = 0$. Also, $L_p(\mathcal{X})$ is the Lebesgue-Bochner space of strongly measurable $\mathcal{X}$-valued functions defined on a separable non-atomic probability space, equipped with its usual norm.

**Observation 8.** Let $1 < p < \infty$. For a Banach space $\mathcal{X}$, the following are equivalent.

1. $\mathcal{X}$ is uniformly convexifiable.
2. $L_p(\mathcal{X})$ is uniformly convexifiable.
3. $L_p(\mathcal{X})$ is asymptotically uniformly convexifiable.
4. $L_p(\mathcal{X})$ admits an equivalent UKK norm.
5. $L_p(\mathcal{X})$ is asymptotically uniformly smoothable.
Proof. Let $1 < p < \infty$ and $X$ be a Banach space.

That (1) though (4) are equivalent and that (2) implies (5) follows easily from the below known facts about a Banach space $Y$.

(i) $Y$ is uniformly convex if and only if $L_p(Y)$ is $[Mc]$.
(ii) $Y$ is uniformly convexifiable if and only if $L_p(Y)$ admits an equivalent UKK norm [DGK, Theorem 4].
(iii) $Y$ is uniformly convexifiable if and only if $Y$ is uniformly smoothable (cf. [DU, page 144]).

Towards showing that (5) implies (1), let $L_p(X)$ be asymptotically uniformly smoothable and $X_0$ be a separable subspace of $X$. It suffices to show that $X_0$ is uniformly convexifiable (cf. [DGZ, Remark IV.4.4]).

It follows from [GKL, Proposition 2.6] that if $Y$ is separable, then $Y$ is asymptotically uniformly smooth if and only if $Y^*$ has the UKK$^*$ property. Thus $[L_p(X_0)]^*$ admits an equivalent UKK$^*$ norm. But $\ell_1$ cannot embed into $L_p(X_0)$ since $L_p(X_0)$ is asymptotically uniformly smoothable and so $[L_p(X_0)]^*$ is asymptotically weak$^*$ uniformly convexifiable and so is also asymptotically uniformly convexifiable. Thus $L_q(X_0^*)$ is asymptotically uniformly convexifiable where $1/p + 1/q = 1$. From (3) implies (1) it follows that $X_0$ is uniformly convexifiable and so is $X_0$. \hfill $\square$

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Dept. of Math., University of South Carolina, Columbia, SC 29208, U.S.A.

E-mail address: girardi@math.sc.edu