THE DUAL OF THE JAMES TREE SPACE IS ASYMPTOTICALLY UNIFORMLY CONVEX

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Abstract. The dual of the James Tree space is asymptotically uniformly convex.

1. Introduction

In 1950, R. C. James [J1] constructed a Banach space which is now called the James space. This space, along with its many variants (such as the James tree space [J2]) and their duals and preduals, have been a rich source for further research and results (both positive ones and counterexamples), answering many questions, several of which date back to Banach [B, 1932]. See [FG] for a splendid survey of such spaces.

This paper's main result, Theorem 5, shows that the dual JT^* of the James tree space JT is asymptotically uniformly convex. (See Section 2 for definitions.)

Schachermayer [S, Theorem 4.1] showed that JT^* has the Kadec-Klee property. It follows from Theorem 5 of this paper that JT^* enjoys the *uniform* Kadec-Klee property. Of course, the same can be said about the (unique) predual JT_* of JT. In fact, Theorem 3 shows that the modulus of asymptotic convexity of JT_* is of power type 3.

Johnson, Lindenstrauss, Preiss, and Schechtman [JLPS] showed that an asymptotically uniformly convex space has the point of continuity property and thus asked whether an asymptotically uniformly convex space has the Radon-Nikodým property. It is well-known that both JT_* and JT^* have the point of continuity property yet fail the Radon-Nikodým property. It follows from Theorem 5 of this paper that JT^* is an asymptotically uniformly convex (*dual*) space without the Radon-Nikodým property. Thus JT_* is a *separable* asymptotically uniformly convex space without the Radon-Nikodým property. To the best of the author's knowledge, these are the first known examples of asymptotically uniformly convex spaces without the Radon-Nikodým property.

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2. Definitions and Notation

Throughout this paper $\mathfrak{X}$ denotes an arbitrary (infinite-dimensional real) Banach spaces. If $\mathfrak{X}$ is a Banach space, then $\mathfrak{X}^*$ is its dual space, $B(\mathfrak{X})$ is its (closed) unit ball, $S(\mathfrak{X})$ is its unit sphere, $\widehat{\imath} : \mathfrak{X} \to \mathfrak{X}^{**}$ is the natural point-evaluation isometric embedding, $\widehat{x} = \widehat{\imath}\,(x)$ and $\widehat{\mathfrak{X}} = \widehat{\imath}\,(\mathfrak{X})$. If Y is a subset of $\mathfrak{X}$, then $[Y]$ is the closed linear span of Y and

$$\begin{aligned} \mathfrak{N}\,(\mathfrak{X}) &= \left\{ [x_i^*]^{\top}_{1\le i\le n} : x_i^* \in \mathfrak{X}^* \text{ and } n \in \mathbb{N} \right\} \\ \mathcal{W}\,(\mathfrak{X}^*) &= \left\{ [x_i]^{\perp}_{1\le i\le n} : x_i \in \mathfrak{X} \text{ and } n \in \mathbb{N} \right\} . \end{aligned}$$

Thus $\mathfrak{N}\,(\mathfrak{X})$ is the collection of (norm-closed) finite codimensional subspaces of $\mathfrak{X}$ while $\mathcal{W}\,(\mathfrak{X}^*)$ is the collection of weak-star closed finite codimensional subspaces of $\mathfrak{X}^*$. All notation and terminology, not otherwise explained, are as in [DU, LT1, LT2].

The *modulus of convexity* $\delta_{\mathfrak{X}}\colon [0,2] \to [0,1]$ of $\mathfrak{X}$ is

$$\delta_{\mathfrak{X}}(\varepsilon) \;=\; \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| \;:\; x, y \in S(\mathfrak{X}) \text{ and } \quad \|x-y\| \ge \varepsilon \right\}$$

and $\mathfrak{X}$ is *uniformly convex (UC)* if and only if $\delta_{\mathfrak{X}}(\varepsilon) > 0$ for each $\varepsilon \in (0,2]$. The *modulus of asymptotic convexity* $\overline{\delta}_{\mathfrak{X}}\colon [0,1] \to [0,1]$ of $\mathfrak{X}$ is

$$\overline{\delta}_{\mathfrak{X}}(\varepsilon) \;=\; \inf_{x\in S(\mathfrak{X})} \; \sup_{\mathcal{Y}\in\mathfrak{N}(\mathfrak{X})} \; \inf_{y\in S(\mathcal{Y})} \; [\ \|x+\varepsilon y\| - 1\ ]$$

and $\mathfrak{X}$ is *asymptotically uniformly convex (AUC)* if and only if $\overline{\delta}_{\mathfrak{X}}(\varepsilon) > 0$ for each ε in $(0,1]$.

A space $\mathfrak{X}$ has the *Kadec-Klee (KK) property* provided the relative norm and weak topologies on $B(\mathfrak{X})$ coincide on $S(\mathfrak{X})$. A space $\mathfrak{X}$ has the *uniform Kadec-Klee (UKK) property* provided for each $\varepsilon > 0$ there exists $\delta > 0$ such that every ε-separated weakly convergent sequence $\{x_n\}$ in $B(\mathfrak{X})$ converges to an element of norm less than $1 - \delta$.

Related to the above geometric isometric properties are the following geometric isomorphic properties.

- $\mathfrak{X}$ has the *Radon-Nikodým property (RNP)* provided each bounded subset of $\mathfrak{X}$ has non-empty slices of arbitrarily small diameter.
- $\mathfrak{X}$ has the *point of continuity property (PCP)* provided each bounded subset of $\mathfrak{X}$ has non-empty relatively weakly open subsets of arbitrarily small diameter.
- $\mathfrak{X}$ has the *complete continuity property (CCP)* provided each bounded subset of $\mathfrak{X}$ is Bocce dentable.

Implications between these various properties are summarized in the diagram below.

$$\begin{array}{ccccccc} \text{UC} & \rightarrow & \text{AUC} & \rightarrow & \text{UKK} & \rightarrow & \text{KK} \\ \downarrow & & \downarrow & & \downarrow & & \\ \text{RNP} & \rightarrow & \text{PCP} & \rightarrow & \text{CCP} & & \end{array}$$

Helpful notation is

$$\overline{\delta}_{\mathfrak{X}}(\varepsilon) \;=\; \inf_{x\in S(\mathfrak{X})} \overline{\delta}_{\mathfrak{X}}(\varepsilon, x)$$

where

$$\overline{\delta}_{\mathfrak{X}}(\varepsilon, x) \;=\; \sup_{\mathcal{Y}\in\mathfrak{N}(\mathfrak{X})} \inf_{y\in S(\mathcal{Y})} \left[\ \|x+\varepsilon y\| - 1\ \right] .$$

Note that, for each $x \in S(\mathfrak{X})$,

$$\overline{\delta}_{\mathfrak{X}}(\varepsilon, x) \;=\; \sup_{\mathcal{Y}\in\mathfrak{N}(\mathfrak{X})} \inf_{\substack{y\in\mathcal{Y} \\ \|y\|\geq\varepsilon}} \left[\ \|x+y\| - 1\ \right]$$

and so $\overline{\delta}_{\mathfrak{X}}(\varepsilon, x)$ is a non-decreasing function of ε. Thus $\overline{\delta}_{\mathfrak{X}}$ is a non-decreasing Lipschitz functions with Lipschitz constant at most one. For any space $\mathfrak{X}$ and $\varepsilon \in [0, 1]$

$$\overline{\delta}_{\mathfrak{X}}(\varepsilon) \;\leq\; \varepsilon \;=\; \overline{\delta}_{\ell_1}(\varepsilon)\ ;$$

thus, ℓ_1 is, in some sense, the most asymptotically uniformly convex space.

Uniform convexity, the KK property, and the UKK property have been extensively studied (for example, see [DGZ, LT2]). Asymptotic uniform convexity has been examined explicitly in [JLPS, M] and implicitly in [GKL, KOS]. The RNP, PCP, and CCP have also been extensively studied (for example, see [DU, GGMS, G1, G2]).

The JT space is construction on a (binary) *tree*

$$\mathcal{T} \;=\; \bigcup_{n=0}^{\infty} \Delta_n$$

where Δ_n is the n^{th}-level of the tree; thus,

$$\Delta_0 \;=\; \{\emptyset\} \qquad \text{and} \qquad \Delta_n \;=\; \{-1, +1\}^n$$

for each $n \in \mathbb{N}$. The *finite tree* $\mathcal{T}_N$ up through level $N \in \mathbb{N} \cup \{0\}$ is

$$\mathcal{T}_N \;=\; \bigcup_{n=0}^{N} \Delta_n$$

The tree $\mathcal{T}$ is equipped with its natural (tree) ordering: if t_1 and t_2 are elements of $\mathcal{T}$, then $t_1 < t_2$ provided one of the follow holds:

(1) $t_1 = \emptyset$ and $t_2 \neq \emptyset$

(2) for some $n, m \in \mathbb{N}$

$$t_1 = (\varepsilon_1^1, \varepsilon_2^1, \ldots, \varepsilon_n^1) \qquad \text{and} \qquad t_2 = (\varepsilon_1^2, \varepsilon_2^2, \ldots, \varepsilon_m^2)$$

with $n < m$ and $\varepsilon_i^1 = \varepsilon_i^2$ for each $1 \le i \le n$.

A (finite) *segment* of $\mathcal{T}$ is a linearly order subset $\{t_n, t_{n+1}, \ldots, t_{n+k}\}$ of $\mathcal{T}$ where $t_i \in \Delta_i$ for each $n \le i \le n+k$. A *branch* of $\mathcal{T}$ is a linearly order subset $\{t_0, t_1, t_2, \ldots\}$ of $\mathcal{T}$ where $t_i \in \Delta_i$ for each $i \in \mathbb{N} \cup \{0\}$.

The *James-Tree space* JT is the completion of the space of finitely supported functions $x \colon \mathcal{T} \to \mathbb{R}$ with respect to the norm

$$\|x\|_{JT} = \sup \left\{ \left[\sum_{i=1}^{n} |\sum_{t \in S_i} x_t|^2 \right]^{\frac{1}{2}} : S_1, S_2, \ldots, S_n \text{ are disjoint segments of } \mathcal{T} \right\} .$$

By lexicographically ordering $\mathcal{T}$, the sequence $\{\eta_t\}_{t \in \mathcal{T}}$ in JT, where

$$\eta_t(s) = \begin{cases} 1 & \text{if } t = s \\ 0 & \text{if } t \neq s \end{cases} ,$$

forms a monotone boundedly complete monotone (Schauder) basis of JT with biorthogonal functions $\{\eta_t^*\}_{t \in \mathcal{T}}$ in JT^*. Thus $\widehat{JT_*} = [\eta_t^*]_{t \in \mathcal{T}}$.

For $N, M \in \mathbb{N} \cup \{0\}$ with $N \le M$, the restriction maps from JT to JT given by

$$\pi_N(x) = \sum_{t \in \Delta_N} \eta_t^*(x) \eta_t$$

$$\pi_{[N,M]}(x) = \sum_{t \in \cup_{i=N}^{M} \Delta_i} \eta_t^*(x) \eta_t$$

$$\pi_{[N,\omega)}(x) = \sum_{t \in \cup_{i=N}^{\infty} \Delta_i} \eta_t^*(x) \eta_t$$

are each contractive projections (by the nature of the norm on JT); thus, so are their adjoints.

Let Γ be the set of all branches of $\mathcal{T}$. Then [LS, Theorem 1] the mapping

$$\pi_\infty \colon JT^* \to \ell_2(\Gamma)$$

given by

$$\pi_\infty(x^*) = \left\{ \lim_{t \in B} x^*(\eta_t) \right\}_{B \in \Gamma}$$

is an isometric quotient mapping with kernel $\widehat{JT_*}$. Also, for each $x^* \in JT^*$,

$$\|x^*\| \;=\; \lim_{N\to\infty} \left\|\pi^*_{[0,N]}\, x^*\right\|$$
$$\|\pi_\infty x^*\| \;=\; \lim_{N\to\infty} \left\|\pi^*_{[N,\omega)}\, x^*\right\| \;=\; \lim_{N\to\infty} \|\pi^*_N x^*\|$$

by the weak-star lower semicontinuity of the norm on JT^*.

To show that JT^* has the Kadec-Klee property, Schachermayer calculated the below two quantitative bounds.

Fact 1. [S, Lemma 3.8] *Let*

$$f_1\colon (0,1) \to (0,\infty)$$

be a continuous strictly increasing function satisfying $f_1(t) < 2^{-10}t^3$ *for each* $t \in (0,1)$. *Let* $N \in \mathbb{N}$ *and* $z^* \in JT^*$. *If*

$$[1 - f_1(t)]\;\; \|z^*\| \;<\; \left\|\pi^*_{[0,N]}\;\, z^*\right\|$$

then

$$\left\|\pi^*_{[N,\omega)}\;\, z^*\right\| \;<\; \|\pi^*_N\;\, z^*\| \;+\; t\,\|z^*\| \;\;.$$

Fact 2. [S, Lemma 3.11] *Let*

$$f_2\colon (0,1) \to (0,\infty)$$

be a continuous strictly increasing function satisfying $f_2(t) < 2^{-26}t^5$ *for each* $t \in (0,1)$. *Let* $N \in \mathbb{N}$ *and* $\varepsilon_0 \in (0,1)$ *and* $\widetilde{x}^*, \widetilde{u}^* \in JT^*$. *If*

(2.1) $\left\|\pi^*_{[N,\omega)}\;\widetilde{x}^*\right\| \;\le\; 1$
(2.2) $\|\pi^*_N\;\widetilde{x}^*\| \;>\; 1 - f_2(\varepsilon_0)$
(2.3) $\|\pi_\infty\;\widetilde{x}^*\| \;>\; 1 - f_2(\varepsilon_0)$
(2.4) $\left\|\pi^*_{[N,\omega)}\;(\widetilde{x}^* + \widetilde{u}^*)\right\| \;\le\; 1$
(2.5) $\|\pi^*_N\,\widetilde{u}^*\| \;<\; f_2(\varepsilon_0)$
(2.6) $\|\pi_\infty\,\widetilde{u}^*\| \;<\; f_2(\varepsilon_0)$.

Then

(2.7) $\left\|\pi^*_{[N,\omega)}\,\widetilde{u}^*\right\| \;<\; \varepsilon_0$.

3. Results

Theorem 3 shows that the modulus of asymptotic convexity of JT_* is of power type 3. Its proof uses Fact 1.

Theorem 3. *There exists a positive constant* k *so that*

$$\overline{\delta}_{JT_*}(\varepsilon) \;\ge\; k\varepsilon^3$$

for each $\varepsilon \in (0,1]$. *Thus* JT_* *is asymptotically uniformly convex.*

Proof. Fix $c \in (0, 2^{-10})$ and find k so that

$$0 \ < \ k(1+k)^2 \ \leq \ c \ . \tag{1}$$

Fix $\varepsilon \in (0,1)$ and a finitely supported $x_* \in S\left(JT_*\right)$. It suffices to show that

$$\overline{\delta}_{JT_*}(\varepsilon, x_*) \ \geq \ k\varepsilon^3 \ . \tag{2}$$

Find $N \in \mathbb{N}$ so that

$$\pi^*_{[0,N-1]}\, \widehat{x}_* \ = \ \widehat{x}_*$$

and let

$$\mathcal{Y} \ = \ [\eta_t]^{\top}_{t \in \mathcal{T}_N} \ .$$

Fix $y_* \in S(\mathcal{Y})$.

Assume that

$$\|x_* + \varepsilon y_*\| \ - \ 1 \ < \ k\,\varepsilon^3 \ .$$

Then

$$\left[1 - \frac{k\varepsilon^3}{1+k\varepsilon^3}\right] \|\widehat{x}_* + \varepsilon \widehat{y}_*\| \ < \ 1 \ = \ \left\|\pi^*_{[0,N]}\ (\widehat{x}_* + \varepsilon \widehat{y}_*)\right\| \ .$$

Thus by Fact 1, with $f_1(t) = ct^3$,

$$\left\|\pi^*_{[N,\omega)}\ (\widehat{x}_* + \varepsilon \widehat{y}_*)\right\| \ < \ \|\pi^*_N\ (\widehat{x}_* + \varepsilon \widehat{y}_*)\| \ + \ f_1^{-1}\left(\frac{k\varepsilon^3}{1+k\varepsilon^3}\right) \|\,(\widehat{x}_* + \varepsilon \widehat{y}_*)\|$$

and so

$$\varepsilon \ < \ \left[1 + k\varepsilon^3\right]\ f_1^{-1}\left(\frac{k\varepsilon^3}{1+k\varepsilon^3}\right) \ . \tag{3}$$

But inequality (3) is equivalent to

$$c^{1/3} \ < \ k^{1/3}\left(1+k\varepsilon^3\right)^{2/3} \ ,$$

which contradicts (1). Thus $\|x_* + \varepsilon y_*\| \ - \ 1 \ \geq \ k\,\varepsilon^3$ and so (2) holds. □

A modification of the proof of Theorem 3 shows that, for each $\varepsilon \in (0,1)$, the $\overline{\delta}_{JT^*}(\varepsilon, x^*)$ stays uniformly bounded below from zero for $x^* \in S(JT^*)$ whose $\|\pi_\infty\, x^*\|$ is small. Recall that if $x_* \in JT_*$ then $\|\pi_\infty\, \widehat{x}_*\| = 0$.

Lemma 4. *For each $\varepsilon \in (0,1)$ there exists $\eta = \eta(\varepsilon) > 0$ so that*

$$\inf_{\substack{x^* \in S(JT^*) \\ \|\pi_\infty x^*\| \leq \eta}} \ \sup_{\mathcal{Y} \in \mathcal{W}(JT^*)} \ \inf_{y^* \in S(\mathcal{Y})} \ [\ \|x^* + \varepsilon y^*\| - 1\ ] \ > \ 0 \ .$$

Proof. Fix $\varepsilon \in (0,1)$. Keeping with the notation in Fact 1, find $\delta, \eta_2 > 0$ so that

$$4\eta_2 \ + \ \frac{\delta}{1 - f_1(\delta)} \ < \ \varepsilon \ .$$

Fix $x^* \in S(JT^*)$ with

$$\|\pi_\infty x^*\| \ \equiv \ b \leq \eta_2 \ .$$

It suffices to show that

$$\sup_{\mathcal{Y}\in\mathcal{W}(JT^*)} \inf_{y^*\in S(\mathcal{Y})} \|x^*+\varepsilon y^*\| \;\geq\; \frac{1}{1-f_1(\delta)} \;. \tag{4}$$

Fix $\eta_1 \in (0,1)$. Find $N \in \mathbb{N}$ so that

$$1-\eta_1 \leq \left\|\pi^*_{[0,N]}\, x^*\right\| \qquad \text{and} \qquad \left\|\pi^*_{[N,\omega)}\, x^*\right\| \;<\; b+\eta_2$$

and let

$$\mathcal{Y} \;=\; [\eta_t]^{\perp}_{t\in\mathcal{T}_N} \;.$$

Fix $y^* \in S(\mathcal{Y})$.

Assume that

$$\|x^*+\varepsilon y^*\| \;<\; \frac{1-\eta_1}{1-f_1(\delta)} \;.$$

Then

$$[1-f_1(\delta)]\; \|x^*+\varepsilon y^*\| \;<\; \left\|\pi^*_{[0,N]}\, x^*\right\| \;=\; \left\|\pi^*_{[0,N]}\; (x^*+\varepsilon y^*)\right\| \;.$$

Thus by Fact 1

$$\left\|\pi^*_{[N,\omega)}\; (x^*+\varepsilon y^*)\right\| \;<\; \|\pi^*_N\; (x^*+\varepsilon y^*)\| \;+\; \delta\, \|(x^*+\varepsilon y^*)\|$$

and so

$$\varepsilon-(b+\eta_2) \;<\; (b+\eta_2)+\frac{\delta}{1-f_1(\delta)} \;.$$

But $b \leq \eta_2$ and so

$$\varepsilon \;<\; 4\eta_2+\frac{\delta}{1-f_1(\delta)} \;.$$

A contradiction, thus

$$\|x^*+\varepsilon y^*\| \;\geq\; \frac{1-\eta_1}{1-f_1(\delta)} \;.$$

Since $\eta_1 > 0$ was arbitrary, inequality (4) holds. □

Thus to show that JT^* is asymptotically uniformly convex, one just needs to examine $\overline{\delta}_{JT^*}(\varepsilon, x^*)$ for $x^* \in S(JT^*)$ whose $\|\pi_\infty x^*\|$ is not small. Fact 2 is used for this case.

Theorem 5. *JT^* is asymptotically uniformly convex.*

Proof. Fix $\varepsilon \in (0,1)$ and let $\varepsilon_0 = \varepsilon/4$. Let $f_1\colon (0,1) \to (0, 2^{-12})$ be given by $f_1(t) = 2^{-12}t^3$ and f_2 be a function satisfying the hypothesis in Fact 2. Find $\delta, \eta_2 > 0$ so that

$$4\eta_2+\frac{\delta}{1-f_1(\delta)} \;<\; \varepsilon \;.$$

Next find $\gamma_i > 0$ and $\tau > 1$ so that

$$\gamma_3 \;<\; \gamma_2 \;<\; \frac{1}{2} \tag{5}$$

$$\tau \leq \frac{(1-\gamma_1)\,(1-\gamma_2)}{1-f_2(\varepsilon_0)} \tag{6}$$

$$\tau < \frac{1-\gamma_2}{\sqrt{1-f_2^2(\varepsilon_0)}} \tag{7}$$

$$\tau \leq \frac{\eta_2^3\gamma_3^3}{2^{15}(1-\gamma_2)^3} \;-\; \gamma_4 \;+\; 1 \tag{8}$$

$$\frac{\tau-1+\gamma_4}{\tau} \;<\; f_1(1) \tag{9}$$

$$\tau \;\leq\; \frac{1}{1-f_1(\delta)}\ . \tag{10}$$

Fix $x^* \in S(JT^*)$. It suffices to show that

$$\sup_{\mathcal{Y}\in\mathfrak{N}(JT^*)}\ \inf_{y\in S(\mathcal{Y})}\ \|x^*+\varepsilon y^*\| \;\geq\; \tau\ . \tag{11}$$

Let

$$\|\pi_\infty x^*\| \;\equiv\; b\ .$$

If $b \leq \eta_2$, then by the proof of Lemma 4 and (10), inequality (11) holds. So let $b > \eta_2$. Find $N \in \mathbb{N}$ so that

$$(1-\gamma_1)\,b \;<\; \|\pi_N^*\,x^*\| \;\leq\; \left\|\pi_{[N,\omega)}^*\,x^*\right\| \;<\; b\left(\frac{1-\gamma_3}{1-\gamma_2}\right) \;<\; \frac{b}{1-\gamma_2} \tag{12}$$

$$1-\gamma_4 \;<\; \left\|\pi_{[0,N]}^*\,x^*\right\|\ . \tag{13}$$

Let $g_{x^*} \in JT^{**}$ be the functional given by

$$g_{x^*}(z^*) = \langle \pi_\infty z^*, \pi_\infty x^*\rangle_{H_2}$$

where the inner product in the natural inner product on $\ell_2(\Gamma)$. Let

$$\mathcal{Y} \;=\; [\eta_t]_{t\in\mathcal{T}_N}^{\perp} \;\cap [g_{x^*}]^{\top}$$

and fix $y^* \in S(\mathcal{Y})$.

Assume that

$$\|x^*+\varepsilon y^*\| \;<\; \tau\ . \tag{14}$$

It suffices to find a contradiction to (14). Towards this, let

$$\widetilde{x}^* \;=\; \frac{1-\gamma_2}{\tau b}\,x^* \qquad \text{and} \qquad \widetilde{y}^* \;=\; \frac{1-\gamma_2}{\tau b}\,y^*\ .$$

It suffices to show (keeping with the same notation but with $\widetilde{u}^* = \varepsilon \widetilde{y}^*$) that conditions (2.1) through (2.6) of Fact 2 hold; for then condition (2.7) holds and so by (5)

$$\varepsilon_0 \ > \ \left\| \pi^*_{[N,\omega)} \, \varepsilon \widetilde{y}^* \right\| \ = \ \frac{1-\gamma_2}{\tau b} \, \varepsilon \ \geq \ \frac{\varepsilon}{4} \ = \ \varepsilon_0 \ .$$

Condition (2.1) follows from (12) since

$$\left\| \pi^*_{[N,\omega)} \ \widetilde{x}^* \right\| \ \leq \ \frac{1-\gamma_2}{\tau b} \, \frac{b}{1-\gamma_2} \ \leq \ 1 \ .$$

Condition (2.2) follows from (12) and (6) since

$$\| \pi^*_N \ \widetilde{x}^* \| \ > \ \frac{1-\gamma_2}{\tau b} \, (1-\gamma_1) \, b \ = \ \frac{(1-\gamma_1)\,(1-\gamma_2)}{\tau} \ \geq \ 1 - f_2(\varepsilon_0) \ .$$

Towards condition (2.3), note that by (7)

$$\| \pi_\infty \ \widetilde{x}^* \| \ = \ \frac{1-\gamma_2}{\tau b} \ b \ = \ \frac{1-\gamma_2}{\tau} \ > \ \sqrt{1 - f_2^2(\varepsilon_0)} \tag{15}$$

and so

$$\| \pi_\infty \ \widetilde{x}^* \| \ > \ 1 - f_2(\varepsilon_0) \ .$$

Towards condition (2.4), note that by (14) and (13)

$$\| x^* + \varepsilon y^* \| \ < \ \frac{\tau}{1-\gamma_4} \ \left\| \pi^*_{[0,N]} \ (x^* + \varepsilon y^*) \right\| \ .$$

Thus by Fact 1 and (9)

$$\begin{aligned} &\left\| \pi^*_{[N,\omega)} \, (x^* + \varepsilon y^*) \right\| \\ &\qquad < \ \| \pi^*_N \, (x^* + \varepsilon y^*) \| \ + \ f_1^{-1} \left(\frac{\tau - 1 + \gamma_4}{\tau} \right) \ \| (x^* + \varepsilon y^*) \| \\ &\qquad \leq \ b \, \frac{1-\gamma_3}{1-\gamma_2} \ + \ \tau 2^4 \left(\frac{\tau - 1 + \gamma_4}{\tau} \right)^{1/3} \ . \end{aligned}$$

Thus condition (2.4) holds provided

$$b \, \frac{1-\gamma_3}{1-\gamma_2} \ + \ \tau 2^4 \left(\frac{\tau - 1 + \gamma_4}{\tau} \right)^{1/3} \ \leq \ \frac{\tau b}{1-\gamma_2} \ ,$$

or equivalently

$$\tau^{2/3} \ (\tau - 1 + \gamma_4)^{1/3} \ \leq \ \frac{b\,(\tau - 1 + \gamma_3)}{2^4\,(1-\gamma_2)} \ .$$

But by (8) and that $b > \eta_2$

$$\begin{aligned} \tau^{2/3}\ (\tau - 1 + \gamma_4)^{1/3} \ &\leq\ 2\ (\tau - 1 + \gamma_4)^{1/3} \ \leq\ \frac{2\eta_2\gamma_3}{2^5(1-\gamma_2)} \\ &\leq\ \frac{b\gamma_3}{2^4(1-\gamma_2)} \qquad\quad \leq\ \frac{b\,(\tau - 1 + \gamma_3)}{2^4\,(1-\gamma_2)}\ . \end{aligned}$$

Thus condition (2.4) holds.

Condition (2.5) follows from the fact that $y^* \in [\eta_t]^{\perp}_{t\in\mathcal{T}_N}$. Towards condition (2.6), since $y^* \in [g_{x^*}]^{\top}$, the vectors $\pi_\infty \widetilde{y}^{\,*}$ and $\pi_\infty \widetilde{x}^{\,*}$ are orthogonal in $\ell_2(\Gamma)$ and so

$$\|\pi_\infty \varepsilon \widetilde{y}^{\,*}\|^2 \ =\ \|\pi_\infty(\widetilde{x}^{\,*} + \varepsilon\widetilde{y}^{\,*})\|^2 \ -\ \|\pi_\infty \widetilde{x}^{\,*}\|^2\ ;$$

but $\pi_\infty = \pi_\infty \pi^*_{[N,\omega)}$ and so by condition (2.4) and (15)

$$\begin{aligned} \|\pi_\infty \varepsilon \widetilde{y}^{\,*}\|^2 \ &\leq\ \left\|\pi^*_{[N,\omega)}(\widetilde{x}^{\,*} + \varepsilon\widetilde{y}^{\,*})\right\|^2 \ -\ \|\pi_\infty \widetilde{x}^{\,*}\|^2 \\ &<\ 1\ -\ \left[1 - f_2^2(\varepsilon_0)\right] \ =\ f_2^2(\varepsilon_0). \end{aligned}$$

Thus condition (2.6). □

The proof in [JLPS] that an asymptotically uniformly convex space has the PCP show that if $\overline{\delta}_{\mathfrak{X}}(\varepsilon) > 0$ for each $\varepsilon \in (0,1]$ then $\mathfrak{X}$ has the PCP. A bit more can be said.

Proposition 6. *If $\overline{\delta}_{\mathfrak{X}}\left(\frac{1}{2}\right) > 0$ then $\mathfrak{X}$ has the PCP.*

The proof of Proposition 6 uses the following (essentially known) lemma.

Lemma 7. *Let $\mathfrak{X}$ be a space without the PCP and $0 < \varepsilon < 1$. Then there is a closed subset A of $\mathfrak{X}$ so that*

1. *each (nonempty) relatively weakly open subset of A has diameter larger than $1 - \varepsilon$*
2. $\sup\{\, \|a\| : a \in A\} \ =\ 1.$

Proof of Lemma 7. Let $\mathfrak{X}$ fail the PCP and $0 < \varepsilon < 1$. By a standard argument (e.g., see [SSW, Prop. 4.10]), there is a closed subset $\widetilde{A}$ of $\mathfrak{X}$ of diameter one such that each (nonempty) relatively weakly open subset of $\widetilde{A}$ has diameter larger than $1 - \varepsilon$. Without loss of generality $0 \in \widetilde{A}$ (just consider a translate of $\widetilde{A}$). Let

$$b = \sup\{\, \|x\| : x \in \widetilde{A}\} \qquad \text{and} \qquad A = \frac{\widetilde{A}}{b}\ .$$

Note that $0 < b \leq 1$. If V is (nonempty) relatively weakly open subset of A, then bV is a relatively weakly open subset of $\widetilde{A}$ and so

$$\operatorname{diam} V = \frac{1}{b}\, \operatorname{diam} bV > 1 - \varepsilon\ .$$

Thus A does the job. ☐

Proof of Proposition 6. Let $\mathfrak{X}$ be a Banach space without the PCP. Fix $t \in \big(0, \frac{1}{2}\big)$ and $\delta \in (0, t)$. It suffices to show that $\overline{\delta}_{\mathfrak{X}}(t) \leq 2\delta$.

Find a subset A of $\mathfrak{X}$ which satisfies the conditions of Lemma 7 with $\varepsilon = 1 - 2t$ and find $a \in A$ so that

$$\left\| \frac{a}{\|a\|} \;-\; a \right\| \;<\; \delta \;.$$

Let $\mathcal{Y} \in \mathfrak{N}(\mathfrak{X})$. It suffices to show that

$$\inf_{\substack{y \in \mathcal{Y} \\ \|y\| \geq t}} \; \left[\; \left\| \frac{a}{\|a\|} + y \right\| - 1 \; \right] \;\leq\; 2\,\delta \;.$$

By condition (1) of Lemma 7 there exists $x \in A$ so that $\|x - a\| \geq t$ and $x - a$ is *almost* in $\mathcal{Y}$; thus, by a standard perturbation argument (e.g., see [GJ, Lemma 2]) there exists $y \in \mathcal{Y}$ so that

$$\|y\| \geq t \quad \text{and} \quad \|y - (x - a)\| \;<\; \delta \;.$$

Thus

$$\left\| \frac{a}{\|a\|} + y \right\| \;\leq\; \left\| \frac{a}{\|a\|} - a \right\| \;+\; \|y - x + a\| \;+\; \|x\| < \; 1 + 2\delta \;.$$

Thus $\overline{\delta}_{\mathfrak{X}}(\frac{1}{2}) = 0$. ☐

The observation below formalizes an essentially known fact, which to the best of the author's knowledge, has not appeared in print as such. Recall that the *modulus of asymptotic smoothness* $\overline{\rho}_{\mathfrak{X}} \colon [0,1] \to [0,1]$ of $\mathfrak{X}$ is

$$\overline{\rho}_{\mathfrak{X}}(\varepsilon) \;=\; \sup_{x \in S(\mathfrak{X})} \; \inf_{\mathcal{Y} \in \mathfrak{N}(\mathfrak{X})} \; \sup_{y \in S(\mathcal{Y})} \; [\; \|x + \varepsilon y\| - 1 \;]$$

and $\mathfrak{X}$ is *asymptotically uniformly smooth* if and only if $\lim_{\varepsilon \to 0^+} \rho_{\mathfrak{X}}(\varepsilon)/\varepsilon = 0$. Also, $L_p(\mathfrak{X})$ is the Lebesgue-Bochner space of strongly measurable $\mathfrak{X}$-valued functions defined on a separable non-atomic probability space, equipped with its usual norm.

Observation 8. Let $1 < p < \infty$. For a Banach space $\mathfrak{X}$, the following are equivalent.

(1) $\mathfrak{X}$ is uniformly convexifiable.
(2) $L_p(\mathfrak{X})$ is uniformly convexifiable.
(3) $L_p(\mathfrak{X})$ is asymptotically uniformly convexifiable.
(4) $L_p(\mathfrak{X})$ admits an equivalent UKK norm.
(5) $L_p(\mathfrak{X})$ is asymptotically uniformly smoothable.

Proof. Let $1 < p < \infty$ and $\mathfrak{X}$ be a Banach space.

That (1) though (4) are equivalent and that (2) implies (5) follows easily from the below known facts about a Banach space $\mathcal{Y}$.

(i) $\mathcal{Y}$ is uniformly convex if and only if $L_p(\mathcal{Y})$ is [Mc].
(ii) $\mathcal{Y}$ is uniformly convexifiable if and only if $L_p(\mathcal{Y})$ admits an equivalent UKK norm [DGK, Theorem 4].
(iii) $\mathcal{Y}$ is uniformly convexifiable if and only if $\mathcal{Y}$ is uniformly smoothable (cf. [DU, page 144]).

Towards showing that (5) implies (1), let $L_p(\mathfrak{X})$ be asymptotically uniformly smoothable and $\mathfrak{X}_0$ be a separable subspace of $\mathfrak{X}$. It suffices to show that $\mathfrak{X}_0$ is uniformly convexifiable (cf. [DGZ, Remark IV.4.4]).

It follows from [GKL, Proposition 2.6] that if $\mathcal{Y}$ is separable, then $\mathcal{Y}$ is asymptotically uniformly smooth if and only if $\mathcal{Y}^*$ has the UKK* property. Thus $[L_p(\mathfrak{X}_0)]^*$ admits an equivalent UKK* norm. But ℓ_1 cannot embed into $L_p(\mathfrak{X}_0)$ since $L_p(\mathfrak{X}_0)$ is asymptotically uniformly smoothable and so $[L_p(\mathfrak{X}_0)]^*$ is asymptotically weak* uniformly convexifiable and so is also asymptotically uniformly convexifiable. Thus $L_q(\mathfrak{X}_0^*)$ is asymptotically uniformly convexifiable where $1/p + 1/q = 1$. From (3) implies (1) it follows that $\mathfrak{X}_0^*$ is uniformly convexifiable and so so is $\mathfrak{X}_0$. □

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