ON VARIOUS MODES OF SCALAR CONVERGENCE IN $L_0(\mathfrak{X})$

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Abstract. A sequence $\{f_n\}$ of strongly-measurable functions taking values in a Banach space $\mathfrak{X}$ is scalarly null a.e. (resp. scalarly null in measure) if $x^* f_n \to 0$ a.e. (resp. $x^* f_n \to 0$ in measure) for every $x^* \in \mathfrak{X}^*$. Let $1 \leq p \leq \infty$. The main questions addressed in this paper are whether an $L_p(\mathfrak{X})$-bounded sequence that is scalarly null a.e. will converge weakly a.e. (or have a subsequence which converges weakly a.e.), and whether an $L_p(\mathfrak{X})$-bounded sequence that is scalarly null in measure will have a subsequence that is scalarly null a.e. The answers to these and other similar questions often depend upon p and upon the geometry of $\mathfrak{X}$.

1. INTRODUCTION

Consider the space $L_0(\mathfrak{X})$ of strongly-measurable functions defined on the usual Lebesgue measure space (Ω, Σ, μ) on $[0,1]$ and taking values in the Banach space $\mathfrak{X}$. Among the most important linear subspaces of $L_0(\mathfrak{X})$ are the Bochner-Lebesgue spaces $L_p(\mathfrak{X})$ for $1 \leq p \leq \infty$. When $\mathfrak{X} = \mathbb{R}$, the $L_p(\mathfrak{X})$ spaces are just the usual Lebesgue spaces, which we shall denote by L_p. A sequence $\{f_n\}$ in $L_0(\mathfrak{X})$ may converge to f in $L_0(\mathfrak{X})$ in a variety of modes. In this paper, we examine the implications going between the four modes described below.

The sequence $\{f_n\}$ converges *scalarly a.e.* (resp. *scalarly in L_p* for a fixed $1 \leq p \leq \infty$, resp. *scalarly in measure*) to f if for each x^* in the dual space $\mathfrak{X}^*$ of $\mathfrak{X}$,

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the corresponding sequence $\{x^* f_n\}$ in $L_0(\mathbb{R})$ converges almost everywhere (resp. in $L_p(\mathbb{R})$, resp. in measure) to $x^* f$. Thus, $\{f_n\}$ converges scalarly a.e. to f if for each x^* in $\mathfrak{X}^*$ there is a set A (which depends on x^*) of full measure such that $\{x^* f_n(\omega)\}$ converges to $x^* f(\omega)$ for each ω in A. If the sequence satisfies the stronger property that there is a set A (independent of x^*) of full measure such that $\{x^* f_n(\omega)\}$ converges to $x^* f(\omega)$ for each ω in A and each x^* in $\mathfrak{X}^*$, then we say that $\{f_n\}$ converges *weakly almost everywhere* (i.e. *weakly a.e.*) to f.

The following obvious positive and negative implications hold:

$$\begin{array}{ccc}
\text{weakly a.e.} & \longrightarrow & \text{scalarly a.e.} \\
\not\uparrow\ \not\downarrow & \not\nearrow\ \not\swarrow\quad \searrow\ \not\nwarrow & \not\uparrow\ \downarrow \\
\text{scalarly in } L_p & \begin{array}{c}\longrightarrow \\ \not\longleftarrow\end{array} & \text{scalarly in measure}
\end{array}$$

for each $1 \le p < \infty$. Similarly:

$$\begin{array}{ccc}
\text{weakly a.e.} & \longrightarrow & \text{scalarly a.e.} \\
\not\downarrow & \nearrow\ \not\swarrow\quad \searrow\ \not\nwarrow & \not\uparrow\ \downarrow \\
\text{scalarly in } L_p & \begin{array}{c}\longrightarrow \\ \not\longleftarrow\end{array} & \text{scalarly in measure}
\end{array}$$

for $p = \infty$.

Sets of the form

$$V_{x^*,\varepsilon}(0) = \{g \in L_0(\mathfrak{X})\colon \mu\{|x^*(g)| \ge \varepsilon\} < \varepsilon\} \ ,$$

where $x^* \in \mathfrak{X}^*$ and $\varepsilon > 0$, form a local subbasis at zero for the translation-invariant topology of scalar convergence in measure. Endowed with this topology, $L_0(\mathfrak{X})$ is a non-locally convex Hausdorff topological vector space. For a fixed $1 \le p \le \infty$, sets of the form

$$S_{x^*,\varepsilon}(0) = \left\{g \in L_p(\mathfrak{X})\colon \ \|x^* g\|_{L_p} < \varepsilon\right\} \ ,$$

where $x^* \in \mathfrak{X}^*$ and $\varepsilon > 0$, form a local subbasis at zero for the translation-invariant topology of scalar convergence in L_p. Endowed with this topology, $L_p(\mathfrak{X})$ is a locally convex Hausdorff topological vector space.

Let $\mathfrak{X}_0$ be a norm-closed subspace of $\mathfrak{X}$. The Hahn-Banach Theorem quickly gives two observations. First, a sequence of $L_0(\mathfrak{X}_0)$ functions converges to the null function in one of the above modes when viewed as a sequence in $L_0(\mathfrak{X}_0)$ if and only if it does so when viewed as a sequence in $L_0(\mathfrak{X})$. Secondly, the topology of scalar convergence in measure on $L_0(\mathfrak{X}_0)$ coincides with the subspace topology inherited from the topology of scalar convergence in measure on $L_0(\mathfrak{X})$. Let us show that under this identification $L_0(\mathfrak{X}_0)$ is in fact a *closed* subspace of $L_0(\mathfrak{X})$.

Proposition 1.1. *Let $\mathfrak{X}_0$ be a (norm-closed) subspace of $\mathfrak{X}$. Then $L_0(\mathfrak{X}_0)$ is a closed subspace of $L_0(\mathfrak{X})$ in the topology of scalar convergence in measure. In particular, if $\{f_n\}$ is a sequence of $\mathfrak{X}_0$-valued functions in $L_0(\mathfrak{X})$ that converges scalarly in measure to f in $L_0(\mathfrak{X})$, then f is also $\mathfrak{X}_0$-valued.*

Proof. Let $f \in L_0(\mathfrak{X})$ belong to the closure of $L_0(\mathfrak{X}_0)$ in the topology of scalar convergence in measure. Since the range of f is essentially separably-valued, there is a subset $Y \supset \mathfrak{X}_0$ such that $Y/\mathfrak{X}_0$ is separable and $f(\omega) \in Y$ a.e. By the Hahn-Banach theorem there exists a sequence $\{x_n^*\}$ in $\mathfrak{X}^*$ such that if $y \in Y$ then $y \in \mathfrak{X}_0$ if and only if $x_n^*(y) = 0$ for all $n \geq 1$. In particular, if $g \in L_0(\mathfrak{X}_0)$ then $x_n^*(g(\omega)) = 0$ for all $n \geq 1$ a.e., and it follows that $x_n^*(f(\omega)) = 0$ for all $n \geq 1$ a.e., which proves that $f \in L_0(\mathfrak{X}_0)$. ∎

For $1 \leq p \leq \infty$, it is easy to see that $L_p(\mathfrak{X})$ is *not* a closed subspace of $L_0(\mathfrak{X})$ in the topology of scalar convergence in measure for any Banach space $\mathfrak{X}$. So we consider the unit ball

$$B(L_p(\mathfrak{X})) = \{g \in L_0(\mathfrak{X})\colon \|g\|_{L_p(\mathfrak{X})} \leq 1\}$$

of $L_p(\mathfrak{X})$.

Proposition 1.2. *For $1 \leq p \leq \infty$, the unit ball $B(L_p(\mathfrak{X}))$ of $L_p(\mathfrak{X})$ is closed in $L_0(\mathfrak{X})$ in the topology of scalar convergence in measure.*

Proof. Consider $f \in L_0(\mathfrak{X}) \backslash B(L_p(\mathfrak{X}))$. It is sufficient to find an open neighborhood about f that does not meet $B(L_p(\mathfrak{X}))$.

Fix $\varepsilon > 0$ so that $(1-4\varepsilon)\,\|f\|_{L_p(\mathfrak{X})} > 1$. Since f is strongly-measurable, there is a countably-valued function $g \in L_0(\mathfrak{X})$ satisfying

$$\|f(\omega) - g(\omega)\|_{\mathfrak{X}} \;\leq\; \varepsilon\; \max\{\|f(\omega)\|_{\mathfrak{X}}\,, \|g(\omega)\|_{\mathfrak{X}}\}$$

for almost all ω. By making an appropriate choice of representative we may write $g = \sum_k x_k 1_{E_k}$, where $\{E_k\}_k$ partitions the support of g into sets of strictly positive measure. Now $\|g\|_{L_p(\mathfrak{X})} \geq (1-\varepsilon)\,\|f\|_{L_p(\mathfrak{X})}$, and so $(1-3\varepsilon)\,\|g\|_{L_p(\mathfrak{X})} > 1$. Hence we may choose $N \in \mathbb{N}$ so that

$$(1-3\varepsilon)\,\|\tilde{g}\|_{L_p(\mathfrak{X})} > 1\ ,$$

where $\tilde{g} = \sum_{k=1}^N x_k 1_{E_k}$. Now find $\{x_k^*\}_{k=1}^N$ in $S(\mathfrak{X}^*)$ with $x_k^*(x_k) = \|x_k\|$.

Consider the following neighborhood of f in the topology of scalar convergence in measure:

$$U = \bigcap_{k=1}^N \{h \in L_0(\mathfrak{X})\colon \mu\,\{|x_k^*(h-f)| \geq \varepsilon\,\|x_k\|\} < \varepsilon\mu(E_k)\}\ .$$

For $1 \le k \le N$, if $\omega \in E_k$ then $\|f(\omega) - \tilde{g}(\omega)\| \le \varepsilon \|x_k\|$; thus, for each $h \in U$

$$\mu\{\omega \in E_k : \|h(\omega)\| \ge (1-2\varepsilon)\|x_k\|\} \ge (1-\varepsilon)\mu(E_k) \ .$$

Hence

$$\|h\|_{L_p(\mathfrak{X})} \ge (1-\varepsilon)^{1/p}(1-2\varepsilon)\,\|\tilde{g}\|_{L_p(\mathfrak{X})} > (1-3\varepsilon)\,\|\tilde{g}\|_{L_p(\mathfrak{X})} > 1 \ ,$$

following the convention that $1/\infty = 0$. So U does not intersect $B(L_p(\mathfrak{X}))$, as required. ■

This suggests imposing the following natural boundedness conditions: a sequence $\{f_n\}$ of $L_p(\mathfrak{X})$ functions is said to be:

- *pointwise bounded a.e.* if $\sup_n \|f_n(\omega)\|_{\mathfrak{X}} < \infty$ for each ω in some set of full measure.
- bounded in $L_p(\mathfrak{X})$ (for short, $L_p(\mathfrak{X})$-*bounded*) if $\sup_n \|f_n\|_{L_p(\mathfrak{X})} < \infty$.

From Proposition 1.2, we see that sequences of the latter type are well-behaved in the following sense.

Corollary 1.3. *Let $1 \le p \le \infty$ and suppose that $\{f_n\}$ is bounded in $L_p(\mathfrak{X})$. If $\{f_n\}$ converges scalarly in measure to $f \in L_0(\mathfrak{X})$, then $f \in L_p(\mathfrak{X})$.*

Remarks. 1. Hence, in discussing the convergence (in any one of the above four modes) of a sequence in $L_0(\mathfrak{X})$ of functions valued in a subspace $\mathfrak{X}_0$ of $\mathfrak{X}$, there is no loss of generality in taking the limit function to be the null function and viewing the sequence as in $L_0(\mathfrak{X}_0)$.
2. Similarly, if we choose to restrict ourselves to the subset $L_p(\mathfrak{X})$ of $L_0(\mathfrak{X})$, in discussing scalar convergence in measure for an $L_p(\mathfrak{X})$-bounded sequence, there will be no loss of generality in taking the limit function to be the null function.
3. The question of the *existence* of a limit for an $L_p(\mathfrak{X})$-bounded *Cauchy sequence* in the topology of scalar convergence of measure is more problematic and will be deferred until Section 5.

We will also use the following elementary facts without further comment. Fact 1.4 provides a useful necessary condition for weak a.e. convergence while Fact 1.5 will be used to prove scalar convergence a.e.

Fact 1.4. A weakly convergent sequence in a Banach space is norm-bounded. Thus, if for a given sequence $\{f_n\}$ in $L_0(\mathfrak{X})$, there exists a subset B of strictly positive μ-measure such that $\limsup \|f_n(\omega)\| = \infty$ for each $\omega \in B$, then $\{f_n\}$ does not converge weakly a.e.

Fact 1.5. A sequence $\{f_n\}$ in L_1 converges to the null function a.e. whenever $\sum \|f_n\|_{L_1} < \infty$.

If Y is a subset of $\mathfrak{X}$, then $\operatorname{sp} Y$ denotes the linear span of Y and $[Y]$ denotes the closed linear span of Y. All notation and terminology, not otherwise explained, are as in [3] or [14].

2. CONVERGENCE PROPERTIES

Proposition 1.2 suggests that it would be of interest to study the following properties that a Banach space $\mathfrak{X}$ might enjoy.

- (A'_p) Each $L_p(\mathfrak{X})$-bounded sequence of functions that converges scalarly a.e. also converges weakly a.e.
- (A_p) Each $L_p(\mathfrak{X})$-bounded sequence of functions that converges scalarly a.e. has a subsequence that converges weakly a.e.
- (B_p) Each $L_p(\mathfrak{X})$-bounded sequence of functions that converges scalarly in measure has a subsequence that converges weakly a.e.
- (C_p) Each $L_p(\mathfrak{X})$-bounded sequence of functions that converges scalarly in L_p has a subsequence that converges weakly a.e.
- (D_p) Each $L_p(\mathfrak{X})$-bounded sequence of functions that converges scalarly in measure has a subsequence that converges scalarly a.e.
- (E_p) Each $L_p(\mathfrak{X})$-bounded sequence of functions that converges scalarly in L_p has a subsequence that converges scalarly a.e.

For convenience, a schematic summary of the properties is given below, in which a double arrow indicates an implication that is always valid.

$$\begin{array}{cccccc}
\text{weakly a.e.} & \Rightarrow & \xleftarrow{A} & \text{scalarly a.e.} & & \\
\text{C}\uparrow & \text{E}\nearrow & \nwarrow\text{B} & \uparrow\text{D} & \Downarrow & \\
\text{scalarly in } L_p & \Rightarrow & & \text{scalarly in measure} & & .
\end{array}$$

Remarks. 1. Clearly, subsequential convergence is the most one can expect in passing from scalar in measure or scalar in L_p to scalar a.e. convergence.

2. Note that if $\mathfrak{X}$ has (Property$_p$) and $p < q$, then $\mathfrak{X}$ also has (Property$_q$).

3. Note the following obvious implications.

$$\begin{array}{ccccc}
(A_p) & \longleftarrow & (B_p) & \longrightarrow & (C_p) \\
 & & \downarrow & & \downarrow \\
 & & (D_p) & \longrightarrow & (E_p)
\end{array}$$

4. For a fixed $1 \leq p < \infty$, a sequence $\{f_n\}$ in $L_0(\mathfrak{X})$ converges scalarly in L_p if and only if (i) $\{f_n\}$ converges scalarly in measure and (ii) for each $x^* \in \mathfrak{X}^*$, the set $\{|x^* f|^p\}$ is uniformly integrable. Note that (ii) holds when $\{f_n\}$ is $L_r(\mathfrak{X})$-bounded for some $r > p$.

3. $L_\infty(\mathfrak{X})$-BOUNDED SEQUENCES

[3.i] From scalar convergence to weak a.e. convergence. In this subsection, we characterize those spaces having (A_∞), (B_∞), and (C_∞).

Theorem 3.1. *Let $\mathfrak{X}^*$ have the Radon-Nikodým property. Then $\mathfrak{X}$ enjoys the following properties:*

(1) *Let $\{f_n\}$ be a sequence in $L_0(\mathfrak{X})$ which converges scalarly a.e. to some $f \in L_0(\mathfrak{X})$. Then $\{f_n\}$ converges weakly a.e. to f if and only if $\{f_n\}$ is pointwise-bounded a.e.*

(2) *Let $\{f_n\}$ be a sequence in $L_0(\mathfrak{X})$ which converges scalarly a.e. to some $f \in L_0(\mathfrak{X})$. Then $\{f_n\}$ has a subsequence which converges weakly a.e. to f if and only if $\{f_n\}$ has a subsequence which is pointwise-bounded a.e.*

(3) *Let $\{f_n\}$ be a sequence in $L_0(\mathfrak{X})$ which converges scalarly in measure. Then $\{f_n\}$ has a subsequence which converges weakly a.e. if and only if $\{f_n\}$ has a subsequence which is pointwise-bounded a.e.*

Proof. By Fact 1.4 each a.e. weakly convergent sequence is bounded a.e., and so necessity in (1)-(3) (for an *arbitrary* Banach space) is clear. To prove sufficiency observe that we may take $f = 0$ without loss of generality. We may also assume that $\mathfrak{X}^*$ is separable. (Indeed, by the Pettis measurability theorem [15], there is a separable subspace $\mathfrak{X}_0$ of $\mathfrak{X}$ such that the f_n's are essentially valued in $\mathfrak{X}_0$. Because $\mathfrak{X}^*$ has the RNP, $\mathfrak{X}_0^*$ must be separable [18, Theorem 2].) Let $\{x_i^*\}$ be dense in $\mathfrak{X}^*$. Now we prove sufficiency in (1). Since $\{f_n\}$ is scalarly null a.e., it follows that for each i there is a set A_i of full measure such that if $\omega \in A_i$, then $\lim_n x_i^* f_n(\omega) = 0$. Put $A = \cap_i A_i$. Since the f_n's are pointwise-bounded on some set B of full measure, and since $\{x_i^*\}$ is dense in $\mathfrak{X}^*$, it follows that $\lim_n x^* f_n(\omega) = 0$ for each $x^* \in \mathfrak{X}^*$ and for each $\omega \in A \cap B$. Thus, $\{f_n\}$ is weakly null a.e. Sufficiency in (2) follows at once. Finally, we prove sufficiency in (3). By first passing to a subsequence we may assume that $\{f_n\}$ is pointwise bounded almost everywhere. Since $\{f_n\}$ converges to zero scalarly in measure, for each i the sequence $\{x_i^* f_n\}_n$ converges in measure to the null function. So by a Cantor diagonalization argument there exists a subsequence $\{f_{n_k}\}$ such that for almost all ω

$$\lim_k x_i^* \left(f_{n_k}(\omega)\right) = 0 \ .$$

for all i. Now, arguing as before, the pointwise boundedness implies that $\{f_{n_k}\}$ is weakly null a.e. ■

For the Banach spaces ℓ_1, $C(\Delta)$, and the James tree space, Davis and Johnson [2] constructed examples of $L_\infty(\mathfrak{X})$-bounded sequences that converge scalarly a.e. but not weakly a.e. They conjectured that such a sequence exists for any space $\mathfrak{X}$ whose dual fails the Radon-Nikodým property (RNP). Combined with work of Uhl [20] and Stegall [19], a result of Edgar [7] shows that their conjecture was correct. In fact, rather more can be said as the following theorem (whose proof was inspired by [7]) shows.

Theorem 3.2. *For a Banach space $\mathfrak{X}$, the following are equivalent:*

(1) *$\mathfrak{X}^*$ has the Radon-Nikodým property (i.e. $\mathfrak{X}$ is an Asplund space);*
(2) *$\mathfrak{X}$ has (A'_∞);*
(3) *$\mathfrak{X}$ has (A_∞);*
(4) *$\mathfrak{X}$ has (B_∞).*

Proof. Several implications follow from Theorem 3.1. To prove the other implications, suppose that $\mathfrak{X}^*$ fails the RNP. Then [20] there is a separable subspace $\mathfrak{X}_0$ of $\mathfrak{X}$ such that $\mathfrak{X}_0^*$ is not separable. We shall construct an $L_\infty(\mathfrak{X})$-bounded sequence $\{g_n\}$ of $\mathfrak{X}_0$-valued functions such that $g_n \to 0$ scalarly a.e. and scalarly in $L_r(\mathfrak{X})$ for $1 \le r < \infty$, but such that no subsequence of $\{g_n\}$ converges weakly a.e. (This particular construction has been fruitful in several similar characterizations of $\mathfrak{X}^*$ having the RNP [5,7,8].) This will show that $\mathfrak{X}$ fails (A_∞), thus completing the proof of the theorem.

Let $\Delta = \{-1,1\}^{\mathbb{N}}$ be the Cantor group with Haar measure ν. Let $\{\Delta_k^n \colon k = 1,\dots,2^n\}$ be the standard n-th partition of Δ. Thus $\Delta_1^0 = \Delta$ and $\Delta_k^n = \Delta_{2k-1}^{n+1} \cup \Delta_{2k}^{n+1}$ and $\nu(\Delta_k^n) = 2^{-n}$. Instead of our usual Lebesgue measure space on $[0,1]$, we shall now take our underlying measure space to be the completion of ν for the completion of the Borel σ-algebra of Δ. Thus $L_p(\mathfrak{X})$ will denote $L_p(\Delta,\nu;\mathfrak{X})$.

We consider the space $C(\Delta)$ of real-valued continuous functions on Δ as a subspace of $L_\infty(\Delta,\nu)$. Let $\{1_\Delta\} \cup \{h_k^n \colon n = 0,1,2,\dots$ and $k = 1,\dots,2^n\}$ be the usual Haar basis of $C(\Delta)$, where $h_k^n \colon \Delta \to \mathbb{R}$ is given by

$$h_k^n \ = \ 1_{\Delta_{2k-1}^{n+1}} \ - \ 1_{\Delta_{2k}^{n+1}} \ .$$

Let $\{e_k^n : n = 0, 1, 2, \ldots$ and $k = 1, \ldots, 2^n\}$ be an enumeration (lexicographically) of the usual ℓ_1 basis and let $H : \ell_1 \to L_\infty$ be the Haar operator that takes e_k^n to h_k^n.

By Stegall's Factorization Theorem [S, Theorem 4], H factors through $\mathfrak{X}_0$, i.e. there are bounded linear operators $R : \ell_1 \to \mathfrak{X}_0$ and $S : \mathfrak{X}_0 \to L_\infty$ such that $H = SR$.

$$\begin{tikzcd} \ell_1 \arrow[rr, "H"] \arrow[dr, "R"'] & & L_\infty(\Delta) \supset C(\Delta) \\ & \mathfrak{X}_0 \arrow[ur, "S"'] & \end{tikzcd}$$

Let $\tilde{R}$ be the natural extension of R to a bounded linear operator from $L_1(\ell_1)$ to $L_1(\mathfrak{X}_0)$.

Consider the sequence $\{f_m\}$ of $L_1(\ell_1)$ functions given by

$$f_m(\cdot) \ = \ \frac{1}{m} \sum_{n=1}^{m} \sum_{k=1}^{2^n} h_k^n(\cdot) e_k^n \ .$$

Let $g_m = \tilde{R}(f_m)$. Clearly, $\{g_m\}$ is $L_\infty(\mathfrak{X})$-bounded since $||f_m(\omega)||_{\ell_1} = 1$ for ν-a.e. ω and each m.

To examine the scalar behavior of $\{g_m\}$, note that if $y^* \in \mathfrak{X}_0^*$, then

$$y^* g_m(\cdot) = (R^* y^*) f_m(\cdot) \ ,$$

where $R^* y^* \in \ell_1^*$. So to show that $\{g_m\}$ converges to the null function scalarly a.e. and scalarly in $L_r(\mathfrak{X})$ we need only show the same for $\{f_m\}$. So fix a functional x^* in ℓ_1^*; let x^* have the form $(\alpha_k^n) \in \ell_\infty$, lexicographically ordered. Then

$$x^* f_m(\omega) = \frac{1}{m} \sum_{n=1}^{m} X_n(\cdot) \qquad \text{where} \qquad X_n(\cdot) = \sum_{k=1}^{2^n} h_k^n(\cdot) \alpha_k^n \ .$$

Note that $\|X_n\|_\infty \le \|x^*\|$, that each X_n has zero mean, and that $\int X_n X_m \, d\nu = 0$ when $n \ne m$. The Strong Law of Large Numbers for uncorrelated random variables with uniformly bounded second moments [1, Theorem 5.1.2] gives that $\{x^* f_m\}$ converges to the null function a.e. Since $\|x^* f_m\|_\infty \le \|x^*\|$ it also follows that $\{x^* f_m\}$ converges to the null function in L_p for $0 \le p < \infty$.

We shall now show that no subsequence of $\{g_m\}$ converges weakly a.e. Since $\{g_m\}$ is scalarly null a.e., it suffices to show that no subsequence is weakly null a.e. For $\omega \in \Omega$, let $\ell_\omega \in [C(\Delta)]^*$ be the point evaluation at ω functional and let $\widetilde{\ell}_\omega \in [L_\infty(\Delta)]^*$ be any Hahn-Banach extension. Then

$$\left(S^*\widetilde{\ell}_\omega\right)(g_n(\omega)) = \ell_\omega(Hf_n(\omega)) = \left(\frac{1}{m}\sum_{n=1}^{m}\sum_{k=1}^{2^n} h_k^n(\omega)h_k^n(\cdot)\right)(\omega) = 1 \ ,$$

and thus no subsequence of $\{g_n\}$ converges weakly a.e. (to the null function) in $L_0(\mathfrak{X}_0)$. ∎

Property C_∞, on the other hand, is a much weaker property according to the following mildly surprising result.

Proposition 3.3. *Let $\{f_n\}$ be a sequence in $L_0(\mathfrak{X})$ that converges to the null function scalarly in L_∞. Then $\{f_n\}$ is weakly null a.e. In particular, every Banach space enjoys (C_∞) (and a fortiori (E_∞)).*

Proof. For each $n \geq 1$, we may write $f_n = g_n + h_n$, where h_n has $L_\infty(\mathfrak{X})$-norm at most $1/n$ and g_n is countably-valued (see e.g. [3, II.1.3]). By choosing a suitable representative of g_n in $L_\infty(\mathfrak{X})$, we may express g_n as

$$g_n = \sum_{k=1}^{\infty} x_k^n \ 1_{E_k^n},$$

where each E_k^n has strictly positive measure and, for each n, Ω is the disjoint union of $\{E_k^n\}_k$. Note that $\{g_n\}$ also converges to the null function scalarly in $L_\infty(\mathfrak{X})$. Hence, for each $\omega \in \Omega$ and each $x^* \in X^*$, we have

$$|x^*(g_n(\omega))| \leq \sup_k |x^*(x_k^n)| = \|x^*(g_n)\|_\infty \to 0$$

as $n \to \infty$. Hence $\{g_n(\omega)\}$ is weakly null for all $\omega \in \Omega$. Clearly, $\{h_n(\omega)\}$ is norm-null a.e., whence $\{f_n\}$ is weakly null a.e. ∎

Perhaps the following theorem is the most useful analogue for scalar convergence in general Banach spaces of the familiar fact from real analysis that each sequence that converges in measure has a subsequence that converges a.e.

Theorem 3.4. *Let K be a weakly compact subset of a Banach space $\mathfrak{X}$ and let $\{f_n\}$ be a sequence in $L_0(\mathfrak{X})$ such that each f_n is essentially K-valued. If $\{f_n\}$ converges scalarly in measure to $f \in L_0(\mathfrak{X})$, then some subsequence $\{f_{n_k}\}$ converges weakly a.e. to f. (In particular, f is essentially K-valued.)*

Proof. Clearly, we may assume that $\mathfrak{X}$ is separable. First, we show that $f(\omega) \in K' = \overline{\mathrm{conv}(K)}$ a.e. Suppose this is not the case. Then there exists a closed ball

$B \subset \mathfrak{X} \setminus K'$ such that $\mu(A) > 0$, where $A = \{\omega : f(\omega) \in B\} > 0$. By the Hahn-Banach Theorem there exists $x^* \in \mathfrak{X}^*$ and $\alpha, \beta \in \mathbb{R}$ such that

$$\sup_{k \in K'} x^*(k) < \alpha < \beta < \inf_{b \in B} x^*(b) \ .$$

Since each f_n is (without loss of generality) K-valued, it follows that

$$x^*(f_n(\omega)) < \alpha < \beta < x^*(f(\omega))$$

for all $n \geq 1$ and all $\omega \in A$. This contradicts the fact that $\{f_n\}$ converges scalarly in measure to f. Hence, by replacing f_n by $f_n - f$ and K by $K' - K'$, we may assume without loss of generality that $\{f_n\}$ converges scalarly in measure to the null function and that K is a separable weakly compact set containing zero. It is easily seen that the weak topology on K is generated by a sequence $\{x_n^*\}$ in $\mathfrak{X}^*$. By a Cantor diagonal argument there exists a subsequence $\{f_{n_k}\}$ and a set $\Omega' \subset \Omega$ of full measure such that $x_n^*(f_{n_k}(\omega)) \to 0$ as $k \to \infty$ for all $n \geq 1$ and for all $\omega \in \Omega'$ Now let $x^* \in \mathfrak{X}^*$ and let $\varepsilon > 0$. There exists $\delta > 0$ and $N \geq 1$ such that

$$\{k \in K : |x^*(k)| < \varepsilon\} \supset \cap_{i=1}^{N} \{k \in K : |x_i^*(k)| < \delta\} \ .$$

It follows that

$$\{\omega \in \Omega : |x^*(f_{n_k}(\omega))| < \varepsilon \quad \text{for all sufficiently large k}\} \supset \Omega' \ ,$$

and so $x^*(f_{n_k}(\omega)) \to 0$ as $n \to \infty$ for all $\omega \in \Omega'$. Hence $\{f_{n_k}\}$ is weakly null a.e. ■

Minor variations in the above proof gives the following result.

Theorem 3.5. *Let K be a weakly compact subset of a Banach space $\mathfrak{X}$ and let $\{f_n\}$ be a sequence in $L_0(\mathfrak{X})$ such that each f_n is essentially K-valued. If $\{f_n\}$ converges scalarly a.e. to $f \in L_0(\mathfrak{X})$, then $\{f_n\}$ converges weakly a.e. to f.*

[3.ii] From scalar convergence in measure to scalar a.e. convergence. In the previous subsection, Proposition 3.3 shows that each Banach space enjoys (E_∞). In this subsection, we explore (D_∞).

For each $1 \leq p \leq \infty$, it follows directly from the definitions that (B_p) implies (D_p); however, they are not equivalent. Indeed, Theorem 3.2 implies that ℓ_1 fails (B_p) for each $1 \leq p \leq \infty$. However, ℓ_1 has (D_p) for each $1 \leq p \leq \infty$, as the next theorem, which was pointed out to us by W.B. Johnson [12], shows. We are grateful to him for permission to include this result here.

Theorem 3.6. *A scalarly null in measure sequence of $L_0(\ell_1)$ functions contains a scalarly null a.e. subsequence. In particular, ℓ_1 satisfies properties (D_p) for each $1 \le p \le \infty$.*

The following lemma will be used in the proofs of several results, including Theorem 3.6.

Lemma 3.7. *Let $\mathfrak{X}$ be a Banach space with a basis and $\{f_n\}$ be a scalarly in measure null sequence of $L_0(\mathfrak{X})$ functions. There exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$, a blocking $\{\mathfrak{X}_k\}$ of the basis, and sequences $\{g_k\}$ and $\{h_k\}$ of $L_0(\mathfrak{X})$ functions so that:*

(i) *$\{h_k\}$ converges a.e (in $\mathfrak{X}$-norm) to the null function;*

and for each k:

(ii) *$f_{n_k} = g_k + h_k$;*

(iii) *$g_k = P_k f_{n_k}$, where P_k is the natural projection of $\mathfrak{X}$ onto $\mathfrak{X}_k$.*

In particular, $\{g_k\}$ is also scalarly null in measure.

Proof of Lemma 3.7. Let $\{x_n\}_{n\ge1}$ be a normalized basis for $\mathfrak{X}$ and let $\{x_n^*\}$ be the corresponding biorthogonal functionals. Consider a sequence $\{f_n\}_{n\ge1}$ of $L_0(\mathfrak{X})$ functions that is scalarly null in measure. It suffices to construct inductively two increasing sequences $\{n_k\}_{k\ge1}$ and $\{m_k\}_{k\ge0}$ of integers and a sequence $\{g_k\}$ of functions such that, for $\mathfrak{X}_k \equiv \mathrm{sp}\,\{x_i : m_{k-1} < i \le m_k\}$, each g_k satisfies (iii) and

$$\mu(\{\omega : \|g_k(\omega) - f_{n_k}(\omega)\| \ge 2^{-k}\}) \le 2^{-k}. \tag{1}$$

To start the induction set $m_0 = 0$ and $n_0 = 0$. Suppose that $k \ge 1$ and that n_i and m_i have been chosen for $i \le k-1$. Since $\{f_n\}$ is assumed to be scalarly null in measure, it follows that, for each fixed i, the sequence $\{x_i^*(f_n)\}_n$ converges to zero in measure. So there exists $n_k > n_{k-1}$ such that

$$\mu\left(\left\{\omega : \sum_{i=1}^{m_{k-1}} |x_i^*(f_{n_k})| \ge 2^{-k-1}\right\}\right) \le 2^{-k-1} .$$

Hence there exists $m_k > m_{k-1}$ such that, for $\mathfrak{X}_k \equiv \mathrm{sp}\,\{x_i : m_{k-1} < i \le m_k\}$, the function g_k as given in (iii) satisfies (1), which completes the induction. ■

Proof of Theorem 3.6. Consider a sequence $\{f_n\}$ of $L_0(\ell_1)$ functions that is scalarly null in measure. Let $\{e_n\}$ be the standard basis of ℓ_1. Find a blocking $\{\mathfrak{X}_k\}$ of $\{e_n\}$ and sequences $\{g_k\}$ and $\{h_k\}$ as given by Lemma 3.7.

In view of (i), it is enough to show that $\{g_k\}$ has a subsequence that is scalarly null a.e. With that in mind, we establish the following claim.

Claim. Given $\varepsilon > 0$,

$$\sup_{\|x^*\|\le 1} \mu\left(\{|x^*(g_k)| > \varepsilon\}\right) \to 0$$

as $k \to \infty$.

Proof of Claim. Suppose not. Then there exist $\varepsilon > 0$ and $x_k^* \in \ell_1^*$, with $\|x_k^*\| \le 1$, such that

$$\limsup_k \mu(\{|x_k^*(g_k)| > \varepsilon\}) > \varepsilon.$$

Note that $\ell_1 = (\sum \oplus E_k)_1$ and so $\ell_1^* = (\sum \oplus E_k^*)_\infty$. Thus there exists $x^* \in \ell_1^*$ such that, for each k, the functionals x_k^* and x^* have identical restrictions to $\mathfrak{X}_k^*$. Hence,

$$\limsup_k \mu(\{|x^*(g_k)| > \varepsilon\}) > \varepsilon,$$

which contradicts the fact that $\{g_k\}$ is scalarly null in measure.

It follows from the claim that there exists a subsequence $\{g_{n_k}\}$ such that

$$\sup_{\|x^*\|\le 1} \mu(\{|x^*(g_{n_k})| > 2^{-k}\}) < 2^{-k}.$$

Clearly $\{g_{n_k}\}$ is scalarly null a.e. ■

However, we know of at least one space that fails (D_∞).

Theorem 3.8. *$C[0,1]$ fails property (D_∞).*

Proof. Consider a sequence $\{f_n\}$ in the unit ball of $L_\infty\left(C\left[0,1\right]\right)$. With $\Omega_i = [0,1]$, write $f_n(\cdot) \equiv f_n\left(\cdot, t\right) : \Omega_1 \times \Omega_2 \to \mathbb{R}$ so that

(i) $f_n(s,\cdot)\colon \Omega_2 \to \mathbb{R}$ is in $C\left[0,1\right]$ for almost all $s \in \Omega_1$

(ii) $|f_n(s,t)| \leqslant 1$ for each $t \in \Omega_2$ for almost all $s \in \Omega_1$.

Such a sequence $\{f_n\}$ is scalarly null in measure if and only if

(iii) $f_n(\cdot,t)\colon \Omega_1 \to \mathbb{R}$ converges in measure to the null function for each $t \in \Omega_2$.

To see this, note that if $t \in \Omega_2$ is fixed, then $\left(x_t^* f_n\right)(\cdot) = f_n(\cdot,t)$ where $x_t^* \in \left(C\left[0,1\right]\right)^*$ is the point evaluation at t functional. As for the reverse implication, assume that (iii) holds and let $x^* \equiv \nu \in \left(C\left[0,1\right]\right)^*$ be a finite regular positive Borel measure on $\mathcal{B}\left(\Omega_2\right)$. It suffices to show that $x^* f_n(\cdot) \equiv \int_{\Omega_2} f_n(\cdot,t)\, d\nu(t)$ converges to the null function in μ-measure. Towards this, let $\lambda = \mu \times \nu$ be the corresponding product measure on the completion $\mathcal{A}$ of $\mathcal{B}\left(\Omega_1 \times \Omega_2\right)$. Then (iii) implies that

$f_n(\cdot,\cdot)\colon \Omega_1 \times \Omega_2 \to \mathbb{R}$ converges to the null function in λ-measure and hence (by (ii)) also in $L_1(\Omega_1 \times \Omega_2, \mathcal{A}, \lambda)$. Since

$$\int_{\Omega_1} \left(\int_{\Omega_2} |f_n(s,t)| \, d\nu(t) \right) d\mu(s) \;=\; \iint_{\Omega_1 \times \Omega_2} |f_n(s,t)| \, d\lambda \to 0$$

as $n \to \infty$, it follows that the sequence $\{l_n\}$ of L_1 functions given by

$$l_n(\cdot) \;\equiv\; \int_{\Omega_2} |f_n(\cdot,t)| \, d\nu(t)$$

converges to the null function in μ-measure, which gives the result.

For each positive integer n, let $\widetilde{n}$ be its binary representation as a finite sequence of 0 and 1's. For $t \in \Omega$, let t_3 be its unique (nonterminating) ternary expansion into 0, 1, and 2's. For $1 \leqslant k \leqslant n$, let $\Gamma(k,n)$ be the collection of all k-tuples $(n_1, n_2, \ldots, n_k)$ of positive integers that satisfy $1 \leqslant n_1 < n_2 < \ldots < n_{k-1} < n_k = n$. For $\gamma = (n_1, n_2, \ldots, n_k)$ in $\Gamma(k,n)$, let A_γ be the set of $t \in \Omega$ for which t_3 is of the form

$$0.\, \widetilde{n}_1 \ 2\ \widetilde{n}_2 \ 2\ \widetilde{n}_3 \ 2\ \ldots \widetilde{n}_{k-1} \ 2\ \widetilde{n}_k \ 2\ \ldots ,$$

i.e.

$$A_\gamma \;=\; (\ 0.\, \widetilde{n}_1 \ 2\ \widetilde{n}_2 \ 2\ \ldots \widetilde{n}_{k-1} \ 2\ \widetilde{n}_k \ 1\ \overline{2} \quad , \quad 0.\, \widetilde{n}_1 \ 2\ \widetilde{n}_2 \ 2\ \ldots \widetilde{n}_{k-1} \ 2\ \widetilde{n}_k \ \overline{2}\ ] \ .$$

For technical reasons, consider the subset

$$\widetilde{A}_\gamma \;=\; (\ 0.\, \widetilde{n}_1 \ 2\ \widetilde{n}_2 \ 2\ \ldots \widetilde{n}_{k-1} \ 2\ \widetilde{n}_k \ 20\ \overline{2} \quad , \quad 0.\, \widetilde{n}_1 \ 2\ \widetilde{n}_2 \ 2\ \ldots \widetilde{n}_{k-1} \ 2\ \widetilde{n}_k \ 220\overline{2}\ ]$$

of A_γ along with the corresponding unions

$$A_k^n \;=\; \bigcup_{\gamma \in \Gamma(k,n)} A_\gamma \qquad \text{and} \qquad \widetilde{A}_k^n \;=\; \bigcup_{\gamma \in \Gamma(k,n)} \widetilde{A}_\gamma \ .$$

The following properties of these sets will be used:

(1) $A_{k_1}^n \cap A_{k_2}^n \;=\; \emptyset$ if $k_1 \neq k_2$;
(2) if $t \in \cap_j A_{k_j}^{n_j}$ for an increasing sequence $\{n_j\}$, then $\{k_j\}$ is also (strictly) increasing;
(3) if $\{n_k\}_k$ is an increasing sequence of positive integers and $t_3 = 0.\, \widetilde{n}_1 \ 2\ \widetilde{n}_2 \ 2\ \widetilde{n}_3 \ 2\ \ldots$, then $t \in \widetilde{A}_k^{n_k}$ for each k.

For each admissible n and k, find a continuous function $g_k^n\colon \Omega_2 \to [0,1]$ that is supported on A_k^n and takes the value 1 on $\widetilde{A}_k^n$. Lexicographically order the dyadic interval $\{I_k\}_{k\geqslant 1}$ of $[0,1]$ and let $h_k\colon \Omega_1 \to [0,1]$ be the indicator function of I_k. Define $f_n\colon \Omega_1 \times \Omega_2 \to \mathbb{R}$ by

$$f_n(s,t) = \sum_{k=1}^{n} h_k(s) g_k^n(t) \ .$$

Clearly, the corresponding sequence $\{f_n\}$ is in the unit ball of $L_\infty\left(C\left[0,1\right]\right)$ and it satisfies conditions (i), (ii), and, by (1) and (2), also (iii). Thus $\{f_n\}$ converges scalarly in measure to the null function.

However, for any subsequence $\{n_k\}_k$ of the positive integers, for the corresponding point $t_3 = 0.\ \widetilde{n}_1\ 2\ \widetilde{n}_2\ 2\ \widetilde{n}_3\ 2\ \ldots$, it follows from condition (3) that $f_{n_k}(s,t) = h_k(s)$, which does not go pointwise a.e. to the null function. ∎

We shall prove below (Corollary 4.11) that L_1 fails (D_p) for $1 \le p < \infty$, but we do not know what happens when $p = \infty$.

Question 3.9. *Does L_1 enjoy (D_∞)?*

This question can be reformulated as a question about functions of two variables as follows. For $n \ge 1$, let $f_n(s,t)$ be real-valued functions on the unit square which satisfy the following:

(i) $f_n(s,t) = \sum_{k=1}^{N(n)} 1_{E_{n,k}}(s) g_{n,k}(t)$, where $\{E_{n,k}\}_k$ is a partition of $[0,1]$ into sets of strictly positive measure;

(ii) $\int |g_{n,k}(t)|\, d\mu(t) \le 1$ for all $1 \le k \le N(n)$;

(iii) $F_n(s) = \int_A f_n(s,t) d\mu(t) \to 0$ in measure as $n \to \infty$ for every measurable set $A \subset [0,1]$.

Question 3.9 paraphrased. *For $\{f_n(s,t)\}$ as above, does there always exist a subsequence $\{f_{n_k}(s,t)\}$ such that $\int_A f_{n_k}(s,t) d\mu(t) \to 0$ a.e. for each $A \in \Sigma$?*

4. $L_p(\mathfrak{X})$-BOUNDED SEQUENCES

We now investigate what happens when $L_\infty(\mathfrak{X})$-boundedness is weakened to $L_p(\mathfrak{X})$-boundedness.

[4.i] From scalar convergence to weak a.e. convergence. In this subsection, we shall use Dvoretzky's theorem on the existence of almost spherical sections [6] to prove that for $1 \le p < \infty$ none of the properties (A_p), (B_p) nor (C_p) can hold in any infinite-dimensional Banach space.

Let us first recall the q-Pettis norm of an $L_0(\mathfrak{X})$ function (which might be infinite):

$$\|f\|_{\mathcal{P}_q(\mathfrak{X})} = \sup_{x^* \in B(\mathfrak{X}^*)} \left(\int_\Omega |x^* f(\omega)|^q \, d\mu \right)^{1/q} ,$$

for $1 \leq q < \infty$. The building block used in our construction is the basic example of [5] which we now recall (and refine slightly) for the reader's convenience.

Proposition 4.1. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space and let E be a measurable subset of Ω. Given $\varepsilon > 0$, there exists $f \in L_\infty(\mathfrak{X})$ such that $\|f(\cdot)\| = 1_E(\cdot)$ and $\|f\|_{\mathcal{P}_q(\mathfrak{X})} < 2\varepsilon^{1/q}$ for each $1 \leq q < \infty$.*

Proof. Since $q \to \|f\|_{\mathcal{P}_q(\mathfrak{X})}$ is an increasing function for a fixed $f \in L_0(\mathfrak{X})$, it suffices to consider only $2 \leq q < \infty$. First we prove the result for $E = \Omega$.

Let

$$\{I_k^n = [(k-1)/2^n, k/2^n) : n \geq 0 \text{ and } 1 \leq k \leq 2^n\}$$

be the collection of dyadic subintervals of Ω. By Dvoretzky's Theorem there exist unit vectors $\{e_k^n\}_{k=1}^{2^n}$ in $\mathfrak{X}$ such that

$$\frac{1}{2}\left(\sum_{k=1}^{2^n} |a_k^n|^2\right)^{1/2} \leq \left\|\sum_{k=1}^{2^n} a_k^n e_k^n\right\| \leq 2\left(\sum_{k=1}^{2^n} |a_k^n|^2\right)^{1/2} \tag{1}$$

for all real numbers a_k^n. Define $f_n : \Omega \to \mathfrak{X}$ by

$$f_n(\omega) = \sum_{k=1}^{2^n} 1_{I_k^n}(\omega) e_k^n.$$

Note that $\|f_n(\omega)\| = 1$ for all $\omega \in \Omega$. Fix $x^* \in B(\mathfrak{X}^*)$. Then (1) implies that $\left(\sum_{k=1}^{2^n} |x^*(e_k^n)|^2\right)^{1/2} \leq 2$. Thus, for $q \geq 2$, we have

$$\begin{aligned}
\int_\Omega |x^* f_n(\omega)|^q \, d\mu &= \int_\Omega \left|\sum_{k=1}^{2^n} x^*(e_k^n) 1_{I_k^n}(\omega)\right|^q d\mu \\
&= 2^{-n} \sum_{k=1}^{2^n} |x^*(e_k^n)|^q \\
&\leq 2^{-n} \left(\sum_{k=1}^{2^n} |x^*(e_k^n)|^2\right)^{q/2} \leq 2^{q-n} ,
\end{aligned}$$

and so $\|f_n\|_{\mathcal{P}_q(\mathfrak{X})} \leq 2 \cdot (2^{-n})^{\frac{1}{q}}$, which gives the result.

An analogous construction can be carried out in any set E of positive measure, and the result is trivial anyhow for a set E of measure zero. ■

Now we beef up the previous result.

Proposition 4.2. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space and let h be a non-negative countably-valued measurable function defined on Ω. Given $\varepsilon > 0$ and $1 \le q_0 < \infty$ there exists $f \in L_0(\mathfrak{X})$ with the following properties:*

(1) $\|f(\cdot)\| = h(\cdot)$;
(2) $\|f\|_{\mathcal{P}_q(\mathfrak{X})} < \infty$ *for each* $1 \le q < \infty$;
(3) $\|f\|_{\mathcal{P}_{q_0}(\mathfrak{X})} < \varepsilon$.

Proof. Write $h = \sum_{k=1}^{\infty} a_k 1_{E_k}$, where the a_k's are positive numbers and the E_k's are disjoint measurable sets. Select positive numbers $\{\varepsilon_k\}$ such that $\sum_{k=1}^{\infty} a_k^q \varepsilon_k$ is finite for each $1 \le q < \infty$ and $\sum_{k=1}^{\infty} a_k^{q_0} \varepsilon_k < (\varepsilon/2)^{q_0}$. By Proposition 4.1, for each k there exists $f_k \in L_\infty(\mathfrak{X})$ such that $\|f_k(\cdot)\| = 1_{E_k}(\cdot)$ and $\|f_k\|_{\mathcal{P}_q(\mathfrak{X})} < 2\varepsilon_k^{1/q}$. Clearly, $f = \sum_{k=1}^{\infty} a_k f_k$ has the required properties. ■

Theorem 4.3. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space and let g be any non-negative measurable function that is not essentially-bounded. There exists a sequence $\{g_n\}$ in $L_0(\mathfrak{X})$ such that the following hold:*

(1)
$$\mu(\{\omega : \|g_n(\omega)\| > t\}) \le \mu(\{\omega : g(\omega) > t\})$$
for all $n \ge 1$ and for all $t > 0$;

(2)
$$\sum_{n=1}^{\infty} \|g_n\|_{\mathcal{P}_q(\mathfrak{X})} < \infty$$
for each $1 \le q < \infty$;

(3) *$\{g_n\}$ converges scalarly a.e. to the null function;*

(4) *for each subsequence $\{g_{n_j}\}$ there exists a set $A \subset \Omega$ of full measure such that*
$$\limsup_j \|g_{n_j}(\omega)\| = \infty$$
for each $\omega \in A$.

In particular, no subsequence of $\{g_n\}$ converges weakly on any set of strictly positive measure.

Proof. Let h be a non-negative countably-valued measurable function on Ω which is *not* essentially bounded and which satisfies $h(\omega) \le g(\omega)$ for $\omega \in \Omega$. Use Proposition 4.2 to construct a sequence $\{g_n\}$ of *independent* $\mathfrak{X}$-valued random variables such that

(i) each $\|g_n\|$ has the same distribution as h,
(ii) $\|g_n\|_{\mathcal{P}_q(\mathfrak{X})}$ is finite for each $1 \le q < \infty$, and
(iii) $\|g_n\|_{\mathcal{P}_n(\mathfrak{X})} \le 2^{-n}$.

Clearly (1) is satisfied. Condition (2) follows from the observation that, if $N \in \mathbb{N}$ and $1 \le q \le N$, then by (ii) and (iii)

$$\begin{aligned}\sum_{n=1}^{\infty} \|g_n\|_{\mathcal{P}_q(\mathfrak{X})} &\le \sum_{n=1}^{N} \|g_n\|_{\mathcal{P}_q(\mathfrak{X})} + \sum_{n=N+1}^{\infty} \|g_n\|_{\mathcal{P}_n(\mathfrak{X})} \\ &\le \sum_{n=1}^{N} \|g_n\|_{\mathcal{P}_q(\mathfrak{X})} + \sum_{n=N+1}^{\infty} 2^{-n} \quad < \quad \infty .\end{aligned}$$

Clearly, (3) follows from (2) using Fact 1.5. To prove (4), fix a subsequence $\{g_{n_j}\}$. Then, for each $M > 0$,

$$\sum_{j=1}^{\infty} \mu(\{\omega : \|g_{n_j}(\omega)\| > M\}) = \sum_{j=1}^{\infty} \mu(\{\omega : h(\omega) > M\}) = \infty$$

since h does not belong to L_∞. So by the Borel-Cantelli lemma $\|g_{n_j}(\omega)\| > M$ infinitely often a.e. ■

An appropriate choice of the measurable function g (e.g. $g(\omega) = |\log \omega|$) in Theorem 4.3 yields the following corollary.

Corollary 4.4. *Let $\mathfrak{X}$ be a Banach space and let $1 \le p < \infty$. The following are equivalent:*

(1) *$\mathfrak{X}$ is finite-dimensional;*
(2) *$\mathfrak{X}$ satisfies (A'_p);*
(3) *$\mathfrak{X}$ satisfies (A_p);*
(4) *$\mathfrak{X}$ satisfies (B_p);*
(5) *$\mathfrak{X}$ satisfies (C_p).*

Remarks. 1. Theorem 4.3 shows that there is no analogue for scalar convergence of the *uniform boundedness principle*: if $\mathfrak{X}$ is infinite-dimensional then 'scalar boundedness a.e.' does *not* imply 'norm-boundedness a.e.'
2. If $\|f_n\|_{L_q(\mathfrak{X})} \to 0$ then clearly some subsequence is $\mathfrak{X}$-norm-null a.e. However, condition (2) of Theorem 4.3 suggests that searching for a non-trivial scalar integrability condition which implies weak a.e. convergence is probably futile.

[4.ii] From scalar convergence in measure or in L_p to scalar a.e. convergence. In this subsection we examine the properties (D_p) and (E_p) more closely for $1 \le p < \infty$.

First we recall some notation from [16]. Let $\{x_n\}$ be a basic sequence in a Banach space $\mathfrak{X}$ with coefficient functional sequence $\{x_n^*\}$ in $\mathfrak{X}^*$. A family $\{\mathfrak{X}_n\}$

of finite-dimensional subspaces of $[x_n]$ is a *blocking* of $\{x_n\}$ provided there exists an increasing sequence of integers $\{n_k\}$ with $n_1 = 1$ such that $\mathfrak{X}_k = [x_i]_{i=n_k}^{n_{k+1}-1}$ for each k. For $1 \le p \le \infty$, if there is a positive constant c so that, for each collection of vectors $\{y_i\}_{i=1}^n$, where $y_i \in \mathfrak{X}_i$,

$$\left\| \sum_{i=1}^n y_i \right\| \ \le\ c \, \| \, (\|y_i\|) \, \|_{\ell_p} \ \ ,$$

resp.

$$c \, \| \, (\|y_i\|) \, \|_{\ell_q} \ \le\ \left\| \sum_{i=1}^n y_i \right\| \ \ ,$$

then we say that the blocking $\{\mathfrak{X}_k\}$ satisfies an upper (resp. lower) p-estimate.

Theorem 4.5. *Fix $1 < q \le \infty$ and let q' be its conjugate exponent. Suppose that $\mathfrak{X}$ has a basis $\{x_n\}$ with the property that each blocking of this basis satisfies an upper q-estimate. Then $\mathfrak{X}$ enjoys (D_p) for each $q' \le p \le \infty$.*

Proof. Wlog, $p = q'$. Fix a sequence $\{f_n\}$ in $B(L_p(\mathfrak{X}))$ that is scalarly null in measure. We need to extract an scalarly null a.e. subsequence. To this end, let $\{\mathfrak{X}_k\}$, $\{P_k\}$, $\{g_k\}$, and $\{h_k\}$ be as provided from Lemma 3.7. It suffices to show that $\{g_k\}$ converges to the null function scalarly a.e.

Fix $x^* \in \mathfrak{X}^*$ and let $x_k^* = x^* \circ P_k \in X^*$. Note that

$$|x^* g_k(\omega)| \le \|x_k^*\| \ \ \|g_k(\omega)\|_{\mathfrak{X}} \ \ .$$

Thus, for $\varepsilon > 0$ fixed

$$\mu(\{|x^* g_k| \ge \varepsilon\}) \ \le\ \left[\frac{\|x_k^*\|}{\varepsilon} \ \ \|g_k\|_{L_p} \right]^p \ \ .$$

Note that each $\|g_k\|_{L_p}$ is bounded above by $2K$ where K is the basis constant of $\{x_n\}$. Thus

$$\sum_{k=1}^\infty \mu(\{|x^* g_k| \ge \varepsilon\}) \ \le\ \left[\frac{2K}{\varepsilon} \right]^p \sum_{k=1}^\infty \|x_k^*\|^p \ \ .$$

Since the blocking $\{\mathfrak{X}_k\}$ satisfies an upper q-estimate (say with constant C),

$$\| \, (\|x_k^*\|) \, \|_{\ell_p} \ \le\ 2CK \liminf_n \left\| \sum_{k=1}^n x_k^* \right\| \ \le\ 2CK^2 \, \|x^*\| \ \ .$$

Thus $\sum_k \mu(\{|x^* g_k| \ge \varepsilon\}) < \infty$. So by Borel-Cantelli, $\{x^* g_k\}$ converges to the null function, as needed. ∎

Minor variations in the above proof give that Theorem 4.5 remains true if the word *basis* is replaced by *finite-dimensional decomposition*.

However, there are many spaces that fail (E_p) (and hence fail (D_p)).

Theorem 4.6. *Fix $1 < q \leq \infty$ and let q' be its conjugate exponent. Suppose that $\mathfrak{X}$ contains a weakly null semi-normalized basic sequence $\{x_n\}$ which satisfies a lower q-estimate. Then $\mathfrak{X}$ fails (E_p) for each $1 \leq p < q'$.*

Proof. We may assume without loss of generality that $\mathfrak{X} = [x_n]$. Fix $p \in [1, q')$ and choose $q_0 \in \left(\frac{1}{q'}, \frac{1}{p}\right)$. Let $\{g_n\}$ be a sequence of i.i.d. random variables defined on Ω with the same distribution as $g_0(t) = t^{-q_0}$. Define $f_n \colon \Omega \to \mathfrak{X}$ by

$$f_n(\cdot) \ = \ g_n(\cdot)\, x_n \ .$$

Since $g_0 \in L_p$ and since $\{x_n\}$ is semi-normalized, $\{f_n\}$ is an $L_p(\mathfrak{X})$-bounded sequence. For each $x^* \in \mathfrak{X}^*$,

$$\|x^* f_n\|_{L_p} \ = \ |x^*(x_n)| \ \|g_n\|_{L_p} \ .$$

Thus $\{f_n\}$ converges scalarly in L_p to the null function.

Fix a subsequence $\{f_{n_j}\}$ of $\{f_n\}$. It suffices to show that $\{f_{n_j}\}$ is not scalarly null a.e. To this end, let $\{x_n^*\}$ be the sequence of biorthogonal functionals satisfying $x_n^*(x_m) = \delta_{nm}$. Since $\{x_n\}$ is semi-normalized and satisfies a lower q-estimate, it follows that $\{x_n^*\}$ satisfies an upper q'-estimate. Consider the element $x^* \in \mathfrak{X}^*$ given by $x^* = \sum_j j^{-q_0}\ x_{n_j}^*$, which converges in $\mathfrak{X}^*$ since $\{x_n^*\}$ satisfies an upper q'-estimate. Fix $M > 0$. Since

$$\mu\{x^* f_{n_j} > M\} \ = \ \mu\{j^{-q_0} g_{n_j} > M\} \ = \ [M j^{q_0}]^{-\frac{1}{q_0}} \ = \ M^{-\frac{1}{q_0}}\ j^{-1} \ ,$$

we see that $\sum_j \mu\{x^* f_{n_j} > M\} = \infty$, and so by the Borel-Cantelli lemma there is a set A of full measure such that if $\omega \in A$ then $|x^* f_{n_j}(\omega)| > M$ infinitely often. Thus this subsequence does not converge scalarly a.e. ∎

By a theorem of Prus [Pr] every *nearly uniformly convex* space (see [H] for the definition of this property) satisfies the hypothesis of Theorem 4.6 for some $1 < q < \infty$ and so we obtain the following corollary.

Corollary 4.7. *Suppose that $\mathfrak{X}$ contains a super-reflexive or (more generally) a nearly uniformly convexifiable infinite-dimensional subspace. Then $\mathfrak{X}$ fails (E_p) for some $1 < p < \infty$.*

The following two corollaries follow from the previous two theorems by considering the standard unit vector basis of ℓ_p and c_0.

Corollary 4.8. *Fix $1 < q < \infty$ and let q' be its conjugate exponent. Then ℓ_q has (D_p) (resp. (E_p)) if and only if $1 < q' \leq p \leq \infty$.*

Corollary 4.9. *c_0 has (D_p) (and thus (E_p)) for each $1 \le p \le \infty$.*

Note that in Corollary 4.8, if $q \downarrow 1$ then $p \uparrow \infty$, which suggests that ℓ_1 should fail D_p for each $1 \le p < \infty$. Strangely, however, the truth is the complete opposite as was proved in Theorem 3.6 above: ℓ_1 has D_p for all $1 \le p \le \infty$.

Finally, we determine the range of values of p for which L_q satisfies (D_p).

Corollary 4.10. *Fix $1 < q < \infty$ and let q' be its conjugate exponent. Then L_q has (D_p) (resp. (E_p)) if and only if $\max(2, q') \le p \le \infty$.*

Proof. Since ℓ_2 and ℓ_q each embed into L_q, it follows from Corollary 4.8 that L_q fails (E_p) for $1 \le p < \max(2, q')$. Since L_q has type $\min(2, q)$ and the Haar system $\mathcal{H}$ forms an unconditional basis for L_q, each blocking of $\mathcal{H}$ satisfies an upper $\min(2, q)$-estimate. So by Theorem 4.5, if $p = \max(2, q')$ then L_q enjoys (D_p). ■

Let H_1 denote the Hardy space of analytic functions on the unit disk in the complex plane with the usual L_1 norm (see e.g. [18]). It is known that H_1 contains subspaces that are isomorphic to ℓ_q for $1 \le q \le 2$ (see e.g. [4]). Hence Corollary 4.8 implies the next result (cf. Question 3.9).

Corollary 4.11. *H_1 (and therefore also L_1) fails (E_p) for $1 \le p < \infty$.*

Remark. We do not know of a reflexive space that satisfies (D_1).

5. Completeness

In this section we prove some completeness results for the topologies of scalar convergence considered in this paper. First we recall the appropriate definitions. Let E be a topological vector space. A sequence $\{x_n\}$ in E is a Cauchy sequence if for every zero-neighborhood U there exists $N \ge 1$ such that $x_n - x_m \in U$ for all $n, m \ge N$. We shall say that E is complete if every Cauchy sequence converges.

Theorem 5.1. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space. The topologies of scalar convergence in measure and scalar convergence in L_p $(1 \le p < \infty)$ are incomplete.*

Proof. This result is very similar in spirit to the fact that the Pettis norm is incomplete whenever $\mathfrak{X}$ is infinite-dimensional [11]. We refer the reader to the proof of the incompleteness of the Pettis norm that is given in [5]. The construction there, which utilizes Dvoretzky's Theorem on almost spherical sections [6], can easily be modified, using the estimates of Proposition 4.1, to construct a sequence of functions that is Cauchy but not convergent in the topology of scalar convergence in measure and L_p for $1 \le p < \infty$. ■

Theorem 5.2. *The topology of scalar convergence in L_∞ is complete if and only if $\mathfrak{X}$ is weakly sequentially complete.*

Proof. First suppose that $\mathfrak{X}$ is not weakly sequentially complete. Let $\{x_n\}$ be a weak Cauchy sequence that does not converge weakly and let $f_n(\omega) = x_n$ $(n \geq 1)$. Clearly, $\{f_n\}$ is a Cauchy sequence in the topology of scalar convergence in L_∞. By Proposition 3.3, a limit of this sequence, say f, would have to satisfy $f(\omega) = \text{weak-lim } f_n(\omega)$ almost everywhere. Hence $\{f_n\}$ does not converge.

For the converse, suppose that $\mathfrak{X}$ is weakly sequentially complete. Let $\{f_n\}$ be a Cauchy sequence in the topology of scalar convergence in L_∞. By adapting the proof of Proposition 3.3, we see that the weak sequential completeness of $\mathfrak{X}$ guarantees that there exists a function $f : \Omega \to \mathfrak{X}$ such that $f(\omega) = \text{weak-lim } f_n(\omega)$ a.e. By the Pettis measurability theorem [P] and Proposition 1.1, f is strongly-measurable, i.e., $f \in L_0(\mathfrak{X})$. It now follows easily from the fact that $\{f_n\}$ is a Cauchy sequence that $\{f_n\}$ converges to f scalarly in L_∞. ∎

Of more relevance to this paper is the convergence of a pointwise-bounded or an $L_p(\mathfrak{X})$-bounded Cauchy sequence. We investigate this question next for the topology of scalar convergence in measure. For brevity's sake we shall say that a sequence is "scalarly Cauchy in measure" if it is a Cauchy sequence for the topology of scalar convergence in measure.

Theorem 5.3. *Let $\mathfrak{X}$ be a reflexive Banach space. Then each pointwise-bounded sequence $\{f_n\}$ in $L_0(\mathfrak{X})$ that is scalarly Cauchy in measure converges scalarly in measure.*

Proof. We may assume that $\mathfrak{X}$ is separable and hence that $\mathfrak{X}^*$ is separable. Arguing now as in Proposition 3.1 there exists a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(\omega)\}$ is a weakly Cauchy sequence in $\mathfrak{X}$ almost everywhere. Since $\mathfrak{X}$ is reflexive it is weakly sequentially complete and so (by the Pettis Measurability Theorem and Proposition 1.1) there exists f in $L_0(\mathfrak{X})$ such that f_{n_k} converges to f weakly a.e., thus also scalarly in measure, which is enough. ∎

Theorem 5.4. *Let $\mathfrak{X}$ be a reflexive Banach space. Then each $L_1(\mathfrak{X})$-bounded sequence $\{f_n\}$ that is scalarly Cauchy in measure converges scalarly in measure.*

Proof. First, we may assume that $\mathfrak{X}$ is separable. By a deep result of Zippin [21] every separable reflexive Banach space is isomorphic to a closed subspace of a reflexive Banach space with a basis. So we may assume that $\mathfrak{X}$ is isomorphically embedded into a reflexive Banach space $\mathfrak{Y}$ with a basis. Clearly, $\{f_n\}$ is scalarly Cauchy in measure when viewed as a sequence in $L_0(\mathfrak{Y})$. By Proposition 1.1, it suffices to show that $\{f_n\}$ converges to some f in $L_0(\mathfrak{Y})$.

Since $\mathfrak{Y}$ is reflexive, a normalized basis $\{e_k\}$ for $\mathfrak{Y}$ is both boundedly complete and shrinking [14]. Let C be the basis constant of $\{e_k\}$. For each n, we can expand f_n with respect to the basis $\{e_k\}$ thus:

$$f_n(\omega) = \sum_k f_{n,k}(\omega) e_k .$$

For each k, the sequence $\{f_{n,k}\}_n$ is Cauchy in measure, and hence converges in measure to some $g_k \in L_0$. By Fatou's Lemma, we have

$$\begin{aligned}
\int_\Omega \sup_N \left\| \sum_{k=1}^N g_k(\omega) e_k \right\| \, d\mu &\le C \int_\Omega \liminf_{N\to\infty} \left\| \sum_{k=1}^N g_k(\omega) e_k \right\| \, d\mu \\
&\le C \liminf_{N\to\infty} \int_\Omega \left\| \sum_{k=1}^N g_k(\omega) e_k \right\| \, d\mu \\
&\le C \liminf_{N\to\infty} \left(\liminf_{n\to\infty} \int_\Omega \left\| \sum_{k=1}^N f_{n,k}(\omega) e_k \right\| \, d\mu \right) \\
&\le C \liminf_{N\to\infty} \left(C \liminf_{n\to\infty} \int_\Omega \left\| \sum_{k=1}^\infty f_{n,k}(\omega) e_k \right\| \, d\mu \right) \\
&\le C^2 \sup_n \|f_n\|_{L_1(\mathfrak{Y})} < \infty . \qquad (1)
\end{aligned}$$

Hence

$$\sup_N \left\| \sum_{k=1}^N g_k(\omega) e_k \right\| < \infty \quad a.e.$$

Since $\{e_n\}$ is boundedly complete it follows that $f(\cdot) \equiv \sum_{k=1}^\infty g_k(\cdot) e_k$ is in $L_0(\mathfrak{X})$; moreover, it follows from (1) that $f \in L_1(\mathfrak{Y})$.

Fix $y^* \in \mathfrak{Y}^*$ and $N \ge 1$. Clearly,

$$y^*\left(\sum_{k=1}^N f_{n,k} e_k \right) \quad \to \quad y^*\left(\sum_{k=1}^N g_k e_k \right) \qquad (2)$$

in measure as $n \to \infty$. Let α_n denote the norm of the restriction of y^* to $[e_k]_{k \ge n}$. Since $\{e_n\}$ is a shrinking basis, $\alpha_n \to 0$ as $n \to \infty$. Now

$$\begin{aligned}
\int \left| y^*\left(\sum_{k=N+1}^\infty f_{n,k} e_k \right) \right| \, d\mu &\le \alpha_{N+1} \int \left\| \sum_{k=N+1}^\infty f_{n,k} e_k \right\| \, d\mu \\
&\le \alpha_{N+1} (1 + C) \sup_n \|f_n\|_{L_1(\mathfrak{Y})} \quad \to \quad 0 \qquad (3)
\end{aligned}$$

as $N \to \infty$. Combining (2) and (3) we see that $\{f_n\}$ converges to f scalarly in measure. ■

The proof of Theorem 5.3 apparently uses only the weak sequential completeness of $\mathfrak{X}$ and the fact that $\mathfrak{X}^*$ has the RNP. However, by Rosenthal's ℓ_1 theorem [17], these two properties are *equivalent* to $\mathfrak{X}$ being reflexive. Clearly, a necessary condition for the conclusion of Theorems 5.3 and 5.4 to hold is that $\mathfrak{X}$ is weakly sequentially complete, and when $\mathfrak{X}$ has an unconditional basis this condition is also sufficient, as our next two results show. However, we have not been able to determine general necessary and sufficient conditions on $\mathfrak{X}$ so that the conclusions of Theorems 5.3 and 5.4 hold. In view of the next two theorems, which establish the desired conclusions for ℓ_1, it is clear that the method of proof of Theorem 3.2 will not be of use in this situation.

Theorem 5.5. *Let $\mathfrak{X}$ be a weakly sequentially complete Banach space with an unconditional basis. Then each pointwise-bounded sequence $\{f_n\}$ in $L_0(\mathfrak{X})$ that is scalarly Cauchy in measure converges scalarly in measure.*

Proof. Let $\{e_n\}$ be a normalized unconditional basis for $\mathfrak{X}$. We may assume, without loss of generality, that

$$\left\|\sum_n a_n e_n\right\| \le \left\|\sum_n b_n e_n\right\| \tag{1}$$

whenever $|a_n| \le |b_n|$ for all n. The fact that $\mathfrak{X}$ is weakly sequentially complete implies that $\{e_n\}$ is boundedly complete [14].

By assumption,

$$\sup_n \|f_n(\omega)\| = M(\omega) < \infty \quad \text{a.e. .} \tag{2}$$

Also, for each n, we can expand f_n with respect to the basis $\{e_k\}$ thus:

$$f_n(\omega) = \sum_k f_{n,k}(\omega) e_k \ .$$

For each k, the sequence $\{f_{n,k}\}_n$ is a Cauchy sequence in the topology of convergence in measure, and hence converges in measure to some g_k. Now (1) and (2) imply that

$$\sup_N \left\|\sum_{k=1}^N g_k(\omega)\right\| \le M(\omega) \quad \text{a.e. .}$$

Since $\{e_n\}$ is boundedly complete it follows that $f(\cdot) = \sum_{n=1}^\infty g_n(\cdot) e_n$ is in $L_0(\mathfrak{X})$.

Let $h_n = f - f_n$, and so

$$h_n(\cdot) = \sum_k (g_k - f_{n,k})(\cdot)\ e_k \ .$$

To complete the proof of the theorem, it suffices to show that $\{h_n\}$ is scalarly null in measure.

So suppose, to derive a contradiction, that $\{h_n\}$ is not scalarly null in measure. Then there exists $x^* \in S(\mathfrak{X}^*)$ and $\varepsilon > 0$ such that

$$\mu\{\omega : |x^* h_n(\omega)| > \varepsilon\} > \varepsilon \tag{3}$$

for infinitely many n.

The gliding hump argument of Lemma 3.7 yields a subsequence $\{h_{n_k}\}_k$ and a blocking $\{\mathfrak{X}_k\}$ of the basis such that each h_{n_k} satisfies (3) and

$$\mu\{\omega : \|h_{n_k}(\omega) - P_k h_{n_k}(\omega)\| > \varepsilon/4\} < \varepsilon/4, \tag{4}$$

where P_k is the natural projection of $\mathfrak{X}$ onto $\mathfrak{X}_k$. We may define $y^* \in \mathfrak{X}^*$ by defining its action on each $x_k \in \mathfrak{X}_k$:

$$y^*(x_k) = 0 \quad (k \text{ odd}); \quad y^*(x_k) = x^*(x_k) \quad (k \text{ even}) \; . \tag{5}$$

Then by (1), we have $\|y^*\| \le \|x^*\| \le 1$, and so from (3), (4) and (5) we deduce that

$$\mu\{\omega : |y^* h_{n_k}(\omega)| > \varepsilon/4\} < \varepsilon/4 \quad (k \text{ odd}) \tag{6}$$

while

$$\mu\{\omega : |y^* h_{n_k}(\omega)| > \varepsilon/2\} > 3\varepsilon/4 \quad (k \text{ even}). \tag{7}$$

Clearly, (6) and (7) contradict the fact that $\{h_n\}$ is scalarly Cauchy in measure ■

Theorem 5.6. *Let $\mathfrak{X}$ be a weakly sequentially complete Banach space with an unconditional basis. Then each $L_1(\mathfrak{X})$-bounded sequence $\{f_n\}$ that is scalarly Cauchy in measure converges scalarly in measure.*

Proof. Let $\{e_k\}$ be a normalized unconditional basis for $\mathfrak{X}$. Let

$$f_n(\omega) = \sum_k f_{n,k}(\omega) e_k$$

be the expansion of f_n. Now, arguing as in the first half of Theorem 5.4, it can be shown that, for each k, $f_{n,k} \to g_k$ in measure as $n \to \infty$ and that $f(\cdot) \equiv \sum_k g_k(\cdot) e_k$ belongs to $L_1(\mathfrak{X})$. Now, arguing as in second half of Theorem 5.5, one uses the unconditionality of $\{e_k\}$ to prove that $\{f_n\}$ converges to f scalarly in measure. ■

Remark. Note that Theorems 5.5 and 5.6 apply to both ℓ_1 and H_1.

Finally, straightforward modifications to the proofs of Theorems 5.4 and 5.6 yield the following.

Theorem 5.7. *Fix $1 \le p < \infty$. Let $\mathfrak{X}$ be a weakly sequentially complete Banach space that is either reflexive or has an unconditional basis. Then each $L_p(\mathfrak{X})$-bounded sequence that is scalarly Cauchy in L_p converges scalarly in L_p.*

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