DUAL BANACH SPACES WHICH CONTAIN AN ISOMETRIC COPY OF $L_1$

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Abstract. A Banach space contains asymptotically isometric copies of $\ell_1$ if and only if its dual space contains an isometric copy of $L_1$.

1. Introduction

The duality between a Banach space containing a ‘nice’ copy of $\ell_1$ and its dual space containing a ‘nice’ copy of $L_1$ is summarized in the diagram below. Each upward implication follows straight from the definitions and the absence of a downward arrow indicates that the corresponding implication does not hold in general.

\[
\begin{array}{ccc}
\ell_1 & \overset{\text{isomorphic}}{\longrightarrow} & X & \overset{\text{isomorphic}}{\longrightarrow} & L_1 & \overset{\text{isomorphic}}{\longrightarrow} & X^* \\
\downarrow & & & & & & \\
\ell_1 & \overset{\text{almost isometric}}{\longrightarrow} & X & \overset{\text{almost isometric}}{\longrightarrow} & L_1 & \overset{\text{almost isometric}}{\longrightarrow} & X^* \\
\downarrow & & & & & & \\
\ell_1 & \overset{\text{asympt. isometric}}{\longrightarrow} & X & & & & \\
\downarrow & & & & & & \\
\ell_1 & \overset{\text{isometric}}{\longrightarrow} & X & & & & \\
\end{array}
\]

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The investigation of this duality began when Pełczyński [P] proved that if \( X \) contains an isomorphic copy of \( \ell_1 \) then \( X^* \) contains an isomorphic copy of \( L_1 \). He also proved the converse result under a technical assumption which was later removed by Hagler [H2]. Earlier, James [J] had shown that if \( X \) contains \( \ell_1 \) isomorphically then \( X \) contains \( \ell_1 \) almost isometrically. Recently, Dowling, N. Randrianantoanina and Turett [DRT] proved that a dual Banach space contains almost isometric copies of \( L_1 \) whenever it contains isomorphic copies of \( L_1 \) (see also [H1, Corollary 2.32] for this result). The main result of this paper, Theorem 2, shows that \( X \) contains asymptotically isometric copies of \( \ell_1 \) if and only if \( X^* \) contains \( L_1 \) isometrically. In the real case, this is a hitherto unpublished result of Hagler [H1, Theorem 2.2].

2. Notation and Terminology

Henceforth, all Banach spaces are either real or complex. \( X, Y, \) and \( Z \) will denote arbitrary (infinite-dimensional) Banach spaces. Let \( C(K) \) be the space of continuous functions on some compact Hausdorff space \( K \), let \( L_1 \) be the space of Lebesgue-integrable functions on \([0,1]\), and let \( L_p(\Gamma) \) be the space of scalar-valued functions on the set \( \Gamma \) with finite \( \| \cdot \|_p \)-norm where \( 1 \leq p \leq \infty \), all with their usual norms. Let \( \Delta \) be the Cantor set, \( \ell_p \) be \( \ell_p(\mathbb{N}) \), and \( C \) be \( C([0,1]) \).

The concept of asymptotically isometric copies of \( \ell_1 \) was introduced by Hagler [H1, pg. 14]. It was revitalized recently by Dowling and Lennard in fixed point theory [DL]. A Banach space contains asymptotically isometric copies of \( \ell_1 \) provided it satisfies one (hence all) of the conditions in the lemma below.

**Lemma 1.** For a Banach space \( X \), the following are equivalent.

(A1) There exist a null sequence \( (\varepsilon_n) \) of positive numbers less than one and a sequence \( (x_n) \) in \( X \) such that

\[
\sum_{n=1}^{m} (1 - \varepsilon_n)|a_n| \leq \left\| \sum_{n=1}^{m} a_n x_n \right\| \leq \sum_{n=1}^{m} |a_n|
\]

for each finite sequence \( (a_n)_{n=1}^{m} \) of scalars.

(A2) There exist a null sequence \( (\varepsilon_n) \) of positive numbers and a sequence \( (x_n) \) in \( X \) such that

\[
\sum_{n=1}^{m} |a_n| \leq \left\| \sum_{n=1}^{m} a_n x_n \right\| \leq \sum_{n=1}^{m} (1 + \varepsilon_n)|a_n|
\]

for each finite sequence \( (a_n)_{n=1}^{m} \) of scalars.

(A3) There exist a null sequence \( (\varepsilon_n) \) of positive numbers and a sequence \( (x_n) \) in \( X \) such that

\[
\sum_{n=k}^{m} |a_n| \leq \left\| \sum_{n=k}^{m} a_n x_n \right\| \leq (1 + \varepsilon_k) \sum_{n=k}^{m} |a_n|
\]

for each finite sequence \( (a_n)_{n=k}^{m} \) of scalars and \( k \in \mathbb{N} \).
The proof of this lemma is elementary (cf. [DLT, Theorem 1.7] for further equivalent formulations). Note that each condition is equivalent to the variant obtained by replacing ‘There exist a’ by ‘For each’ and ‘and’ by ‘there exists.’ A sequence \((x_n)\) satisfying one of the conditions in the lemma is called an asymptotically isometric copy of \(\ell_1\). See [DLT] for a splendid survey of this topic and its applications to fixed point theory.

The proof of James’s theorem [J] for \(\ell_1\) shows that if \(X\) contains \(\ell_1\) almost isometrically, then for each null sequence \((\varepsilon_n)\) of positive numbers there exists a sequence \((x_n)\) in \(X\) such that

\[
(1 - \varepsilon_k) \sum_{n=k}^{m} |a_n| \leq \left\| \sum_{n=k}^{m} a_n x_n \right\| \leq \sum_{n=k}^{m} |a_n|
\]

for each finite sequence \((a_n)_{n=k}^{m}\) of scalars and \(k \in \mathbb{N}\). Indeed, the line between containing \(\ell_1\) almost isometrically and asymptotically isometrically is very fine.

A sequence \((x_n)\) in a Banach space \(X\) is a \((1 + \varepsilon)\)-perturbation of an isometric copy of \(\ell_1\) (for short, a \((1 + \varepsilon)\)-p.i. \(\ell_1\)-sequence) provided that there exist a Banach space \(Y\), a linear isometric embedding \(T: X \rightarrow Y\), and a sequence \((y_n)\) in \(Y\) such that \((y_n)\) is isometrically equivalent to the unit vector basis of \(\ell_1\) and \(\|y_n - Tx_n\| \leq \varepsilon\) for each \(n \in \mathbb{N}\). If furthermore

\[
\|y_n - Tx_n\| \overset{def}{=} \varepsilon_n \quad n \rightarrow \infty, 0,
\]

then \((x_n)\) is a perturbation of an isometric copy of \(\ell_1\) (for short, a p.i. \(\ell_1\)-sequence) with respect to \((\varepsilon_n)\). Note that if \(X\) is separable then \(Y\) may be taken to be separable.

If \(X\) is a Banach space, then \(X^*\) is its dual space, \(B(X)\) is its closed unit ball, and \(S(X)\) is its unit sphere. The closed linear span of a subset \(A\) of \(X\) is \([A]\). If \(Y\) is a subspace of \(X\) then \(\pi: X \rightarrow X/Y\) is the natural quotient mapping.

For a surjective bounded linear operator \(T: X \rightarrow Z\), the corresponding bounded linear operator \(T_q\) is defined by the following (commutative) diagram.

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Z \\
\phantom{X} \downarrow \pi & & \phantom{X} \downarrow T_q \\
X/\ker T & & \\
\end{array}
\]

The operator \(T\) is called an isometric quotient mapping provided \(T_q\) is an isometry, which is the case if and only if \(T^*\) is an isometric embedding. If \(S: X \rightarrow Z\) is an isomorphic embedding, then \(S^*\) is an isometric quotient mapping if and only if \(S\) is an isometric embedding.

All notation and terminology, not otherwise explained, are as in [LT].
3. Main Result

Theorem 2, the main result of this paper, may be viewed as the isometric version of the theorems of Pełczyński and Hagler.

**Theorem 2.** For a Banach space $X$, the following are equivalent.

(a) $X$ contains asymptotically isometric copies of $\ell_1$.
(b) $X$ contains a perturbation of an isometric copy of $\ell_1$.
(c) $\ell_1$ is linearly isometric to a quotient space of a subspace $X$.
(d) $L_1$ is linearly isometric to a subspace of $X^*$.
(e) $C^*$ is linearly isometric to a subspace of $X^*$.
(f) $X^*$ contains an infinite set $\Gamma$ which is isometrically equivalent to the usual basis of $\ell_1(\Gamma)$ and which is dense-in-itself in the weak-star topology on $X^*$.

And if $X$ is separable, then the following is equivalent to each of the above conditions.

(g) $C(\Delta)$ is isometric to a quotient space of $X$.

Recall that a subset $K$ of a topological space is dense-in-itself if $K$ has no isolated points in the relative topology. Our proof of Theorem 2 uses the following results.

**Lemma 3.** If $(x_n)$ is a p.i. $\ell_1$-sequence, then $(\lambda_n x_n)$ is an asymptotically isometric copies of $\ell_1$ satisfying (A2) for some suitable choice of scalars $(\lambda_n)$. Conversely, an asymptotically isometric copies of $\ell_1$ satisfying (A2) is a p.i. $\ell_1$-sequence.

**Proof.** Let $(\tilde{x}_n)$ be a p.i. $\ell_1$-sequence with respect to $(\tilde{\varepsilon}_n)$. Then

$$\sum_{n=1}^m (1 - \tilde{\varepsilon}_n)|a_n| \leq \left\| \sum_{n=1}^m a_n \tilde{x}_n \right\| \leq \sum_{n=1}^m (1 + \tilde{\varepsilon}_n)|a_n| .$$

for each finite sequence $(a_n)_{n=1}^m$ of scalars. Define

$$\varepsilon_n \overset{\text{def}}{=} \frac{1 + \tilde{\varepsilon}_n}{1 - \tilde{\varepsilon}_n} - 1 \quad \text{and} \quad x_n \overset{\text{def}}{=} \frac{\tilde{x}_n}{1 - \tilde{\varepsilon}_n} .$$

Then $(\varepsilon_n)$ and $(x_n)$ satisfy (A2); thus, $(x_n)$ is an asymptotically isometric copy of $\ell_1$.

Conversely, let $(\varepsilon_n)$ and $(x_n)$ satisfy (A2). Then $(x_n)$ is a p.i. sequence. To see this, let $X_0 = [x_n]$ and $W = \{ (w_n)_{n=1}^\infty : w_n \in \mathbb{C} \text{ and } |w_n| = 1 \text{ for each } n \in \mathbb{N} \}$.

For each $\omega = (w_n) \in W$, define $f_\omega \in B(X_0^*)$ by $f_\omega(x_n) = w_n$; for indeed,

$$\left| f_\omega \left( \sum_{n=1}^m a_n x_n \right) \right| = \left| \sum_{n=1}^m a_n w_n \right| \leq \sum_{n=1}^m |a_n| \leq \left\| \sum_{n=1}^m a_n x_n \right\|_{X_0}$$

for each finite sequence $(a_n)_{n=1}^m$ of scalars. For each $\omega \in W$, let $\tilde{f}_\omega \in B(X^*)$ be a norm-preserving Hahn-Banach extension of $f_\omega$. 


Let
\[ Y \overset{\text{def}}{=} C(B(X^*), \sigma(X^*, X)) , \]
endowed with the usual sup norm, and consider the isometric embedding
\[ T : X \to Y \]
given by
\[ (Tx)(x^*) \overset{\text{def}}{=} x^*(x) . \]
Let \( y_n \in B(Y) \) be the ‘truncation’ of \( Tx_n \); specifically,
\[ y_n(x^*) = \begin{cases} (Tx_n)(x^*) & \text{if } |(Tx_n)(x^*)| \leq 1 \\ \frac{(Tx_n)(x^*)}{|(Tx_n)(x^*)|} & \text{if } |(Tx_n)(x^*)| > 1 . \end{cases} \]

For each \( n \in \mathbb{N} \), condition (A2) gives that \( kx_n k \leq 1 + \varepsilon_n \), and so by (1)
\[ \|y_n - Tx_n\|_{Y} \leq \varepsilon_n . \]

Since for each \( n \in \mathbb{N} \) and \( \omega = (w_j) \in W \)
\[ (Tx_n)(\tilde{f}_\omega) = \tilde{f}_\omega (x_n) = f_\omega (x_n) = w_n = y_n(\tilde{f}_\omega) , \]
it follows that
\[ \left\| \sum_{n=1}^{m} a_n y_n \right\|_{Y} \geq \sup_{\omega \in W} \left| \sum_{n=1}^{m} a_n y_n(\tilde{f}_\omega) \right| = \sup_{(w_n) \in W} \left| \sum_{n=1}^{m} a_n w_n \right| = \sum_{n=1}^{m} |a_n| . \]
for each finite sequence \((a_n)_{n=1}^{m}\) of scalars. Also, \( \|y_n\| \leq 1 \) for each \( n \in \mathbb{N} \). Thus \((y_n)\) is isometrically equivalent to the unit vector basis of \( \ell_1 \).

**Remark 4.** Minor modifications to the above proof give an isomorphic version of Lemma 3. Indeed, if \((\bar{x}_n)\) be a \((1 + \varepsilon)\)-p.i. \( \ell_1 \)-sequence with \( \varepsilon < 1 \) and
\[ \varepsilon \overset{\text{def}}{=} \frac{1 + \varepsilon}{1 - \varepsilon} - 1 \quad \text{and} \quad x_n \overset{\text{def}}{=} \frac{\bar{x}_n}{1 - \varepsilon} , \]
then
\[ \sum_{n=1}^{m} |a_n| \leq \left\| \sum_{n=1}^{m} a_n x_n \right\| \leq (1 + \varepsilon) \sum_{n=1}^{m} |a_n| \] (2)
for each finite sequence \((a_n)_{n=1}^{m}\) of scalars. Conversely, if \((x_n)\) satisfies (2) for each finite sequence \((a_n)_{n=1}^{m}\) of scalars, then \((x_n)\) is a \((1 + \varepsilon)\)-p.i. \( \ell_1 \)-sequence.

**Lemma 5.** If \( X \) satisfies \((f)\) of Theorem 2, then there exists a separable subspace \( X_0 \) of \( X \) and a countable subset \( \Gamma' \) of \( X_0^* \) which satisfies \((f)\) of Theorem 2.

**Proof.** Let \( X \) be a Banach space satisfying \((f)\) of Theorem 2. We shall inductively construct a sequence \((\Lambda_i)\) of countably infinite subsets of \( \Gamma \) and a sequence \((Z_i)\) of separable subspaces of \( X \) which satisfy, for each \( n \in \mathbb{N} \),
1. \( Z_n \subset Z_{n+1} , \)
(2) $Z_n$ norms $[\bigcup_{i=1}^{n} \Lambda_i]$, i.e., if $x^* \in [\bigcup_{i=1}^{n} \Lambda_i]$ then
$$\|x^*\| = \sup_{z \in B(Z_n)} |x^*(z)|,$$

(3) $\bigcup_{i=1}^{n} \Lambda_i$ is contained in the $Z_n$-cluster points of $\Lambda_{n+1}$,
i.e., if $x^* \in \bigcup_{i=1}^{n} \Lambda_i$ and $(w_i)_{i=1}^{k}$ are from $Z_n$ and $\varepsilon > 0$ then
$$\{y^* \in \Lambda_{n+1} : |(y^* - x^*)(w_i)| < \varepsilon \text{ for } 1 \leq i \leq k\} \setminus \{x^*\} \neq \emptyset.$$

For the first step of the induction choose a countably infinite subset $\Lambda_1$ of $\Gamma$ and find a separable subspace $Z_1$ of $X$ which satisfies (2). Suppose that we have chosen $(\Lambda_i)_{i=1}^{n}$ and $(Z_i)_{i=1}^{n}$ satisfying the three conditions. Since $Z_n$ is separable and elements of $\Gamma$ are of norm one, there is a countable subset $\Lambda_{n+1}$ of $\Gamma$ satisfying (3). Next we find a separable subspace $Z_{n+1}$ of $X$ which satisfies (1) and (2). This completes the inductive step.

Now let $X_0 = [\bigcup_{n=1}^{\infty} Z_n]$ and
$$\Gamma' \overset{\text{def}}{=} \{x^*|_{X_0} : x^* \in \bigcup_{n=1}^{\infty} \Lambda_n\}.$$

Condition (2) gives that $\Gamma'$ is isometrically equivalent to the usual basis of $\ell_1(\Gamma')$. Conditions (1) and (3), along with the fact that $\Gamma \subset S(X^*)$, give that $\Gamma'$ is dense-in-itself in the weak-star topology on $X_0^*$.

Fact 6. (cf. [HS, Lemma 4]) Let $N$ and $M$ be compact Hausdorff spaces with $M$ perfect and suppose that $\phi: N \to M$ is continuous and onto. Then there exists a subset $Q$ of $N$ such that $Q$ is dense-in-itself and $\phi|_{Q}: Q \to M$ is a bijection.

Fact 7. (Haskell P. Rosenthal [R, Proposition 3 and its Remark 2]) Let $X_0$ be a separable Banach space satisfying (f) of Theorem 2. Then there exists $K \subset B(X_0^*)$, which is homeomorphic to $\Delta$, such that the restriction operator
$$R: X_0 \to C(K)$$
given by $(Rx_0)(x_n^*) = x_n^*(x_0)$ is an isometric quotient mapping.

Proof of Theorem 2. We shall assume that $X$ is a complex Banach space as the proof in the real case is easier.

The equivalence of (a) and (b) follows directly from Lemma 3.

To see that (a) implies (c), let $(\varepsilon_n)$ and $(x_n)$ be sequences satisfying condition (A3). Partition $\mathbb{N}$ into infinite sets $\{J_n\}_{n \in \mathbb{N}}$ and let $T: [x_n] \to \ell_1$ be the bounded linear operator that maps $x_j$ to the $n^{\text{th}}$ unit vector of $\ell_1$ when $j \in J_n$. Then $T$ is an isometric quotient mapping.

To see that (c) implies (a), let
$$T: X_0/X_1 \to \ell_1$$
be an isometry from a quotient space of a subspace $X_0$ of $X$ onto $\ell_1$. Fix a null sequence $(\varepsilon_n)$ of positive numbers. Find a sequence $(x_n)$ in $X_0$ such that $T(x_n + X_1)$ is the $n^{\text{th}}$ unit vector of $\ell_1$ and
$$1 \leq \|x_n\| \leq 1 + \varepsilon_n.$$
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Then $(\varepsilon_n)$ and $(x_n)$ satisfy (A2).

To see that (a) implies (e), let $(\varepsilon_n)$ and $(x_n)$ satisfy (A2). We shall define the bounded linear operators in the (commutative) diagram below

$$X \xrightarrow{j} Y \xrightarrow{T} C \xrightarrow{\tilde{\gamma}} C^{**}$$

as follows. Let $(z_n)$ be dense in the unit sphere of $C$. Define $T$ by $Tx_n = z_n$. Condition (A2) gives that $T$ is a surjective norm-one bounded linear operator. Furthermore, $T$ is an isometric quotient mapping for if $f \in S(C)$ then there is a subsequence $(z_{k_n})$ converging in norm to $f$ and so

$$1 \leq \|T^{-1}f\|_{Y/\ker T} \leq \lim_{n \to \infty} \|x_{k_n}\|_X = 1.$$  

Let $j$ be the natural embedding and let $\tilde{\gamma}$ be the canonical isometric embedding given by point evaluation. Since $C^{**}$ has the Hahn-Banach Extension Property, $T$ admits a norm-preserving extension $\tilde{T}$. Dualizing gives the commutative diagram

$$C^* \xrightarrow{h} C^{**} \xrightarrow{\tilde{\gamma}^*} C^* \xrightarrow{T^*} Y^*$$

$$\xrightarrow{\tilde{T}^*} X^* \xrightarrow{j^*}$$

where $h$ is the canonical isometric embedding given by point evaluation. To see that $R \overset{\text{def}}{=} \tilde{T}^* h$ is the desired isometric embedding, let $\mu \in C^*$. Then, since $T^*$ is an isometric embedding and $\tilde{T}^* h$ is the identity mapping,

$$\|\mu\|_{C^*} = \|T^* \mu\|_{Y^*} = \|j^* R \mu\|_{Y^*} \leq \|R \mu\|_{X^*} \leq \|\mu\|_{C^*}.$$  

Clearly, (e) implies (d).

To see that (d) implies (a), let $T: L_1 \to X^*$ be an isometric embedding and let $(\varepsilon_n)$ be a null sequence of positive numbers. Then $T^*: X^{**} \to L_\infty$ is a weak-star continuous isometric quotient mapping. By Goldstine’s Theorem,

$$W \overset{\text{def}}{=} T^*(B(X))$$

is weak-star dense in $B(L_\infty)$. For each $n \in \mathbb{N}$, let

$$F_n \overset{\text{def}}{=} \{z_j^n : 1 \leq j \leq M(n)\}$$

be an $(\varepsilon_n/2)$-net for $\{z \in \mathbb{C} : |z| = 1\}$.

Let $T$ be the tree

$$T = \bigcup_{n \in \mathbb{N}} T_n$$
where $T_n$, the $n$th-level of $T$, is
\[
T_n \overset{\text{def}}{=} \{(m_0, m_1, m_2, \ldots m_{n-1}) \in \mathbb{N}^n : m_0 = 1 \text{ and } 1 \leq m_j \leq M(j) \text{ for each } j \in \mathbb{N}\}.
\]
If $\alpha = (m_0, m_1, m_2, \ldots m_{n-1}) \in T$ then
\[
(\alpha, j) \overset{\text{def}}{=} (m_0, m_1, m_2, \ldots m_{n-1}, j);
\]
thus, for each $n \in \mathbb{N}$
\[
T_{n+1} = \{(a, j) : \alpha \in T_n \text{ and } 1 \leq j \leq M(n)\}.
\]
We will inductively, for each $n \in \mathbb{N}$, a collection $\{A_\alpha\}_{\alpha \in T_n}$ of disjoint sets of positive (Lebesgue) measure and a function $f_n \in W$ such that, for each $n \in \mathbb{N}$ and $\alpha \in T_n$,
\[
\bigcup_{j=1}^{M(n)} A_{(\alpha,j)} \subset A_\alpha \subset [0,1] \tag{4}
\]
and, for each $1 \leq j \leq M(n)$,
\[
|f_n - z_j^n| < \frac{\varepsilon_n}{2} \text{ on } A_{(\alpha,j)} \tag{5}
\]
To start the induction, let
\[
A_{(m_0)} = [0,1].
\]
For the inductive step, let $n \in \mathbb{N}$ and suppose that we have constructed disjoint sets
\[
\{A_\alpha : \alpha \in T_n\}
\]
of positive measure. For each $\alpha \in T_n$, partition $A_\alpha$ into sets $\{D_{(\alpha,j)}\}_{j=1}^{M(n)}$ of positive measure. Consider the function $g_n \in B(L_\infty)$ defined by
\[
g_n(t) = \begin{cases} z_j^n & \text{if } t \in D_{(\alpha,j)} \text{ and } \alpha \in T_n \\ 0 & \text{otherwise} \end{cases}
\]
Since $W$ is weak-star dense in $B(L_\infty)$ there exists $f_n \in W$ approximating $g_n$ closely enough to ensure that the sets
\[
A_{(\alpha,j)} \overset{\text{def}}{=} \{|f_n - z_j^n| < \varepsilon_n/2\} \cap D_{(\alpha,j)}
\]
all have positive measure. This completes the proof of the inductive step.

For each $n \in \mathbb{N}$, select $x_n \in B(X)$ such that $T^n(x_n) = f_n$. To see that $(x_n)$ is an asymptotically isometric copy of $\ell_1$, let $(a_n)_{n=1}^m$ be a finite complex sequence. Define $(\tilde{a}_n)_{n=1}^m$ from $\{z \in \mathbb{C} : |z| = 1\}$ by
\[
\tilde{a}_n = \begin{cases} \frac{a_n}{|a_n|} & \text{if } a_n \neq 0 \\ 1 & \text{if } a_n = 0 \end{cases}.
\]
thus, $a_n \tilde{a}_n = |a_n|$. For each $1 \leq n \leq m$, find $1 \leq j_n \leq M(n)$ so that

$$|\tilde{a}_n - z_{j_n}^n| < \frac{\varepsilon_n}{2}.$$  

Then $\alpha \overset{\text{def}}{=} (1, j_1, \ldots, j_m) \in T_{m+1}$ and so by (4) and (5)

$$|f_n - z_{j_n}^n| < \frac{\varepsilon_n}{2} \text{ on } A_\alpha$$

for each $1 \leq n \leq m$. Thus

$$\left\| \sum_{n=1}^{m} a_n x_n \right\| \geq \left\| T^* \left( \sum_{n=1}^{m} a_n x_n \right) \right\|_{L_\infty} = \left\| \sum_{n=1}^{m} a_n f_n \right\|_{L_\infty} \geq \sum_{n=1}^{m} |a_n| - \varepsilon_n |a_n| = \sum_{n=1}^{m} (1 - \varepsilon_n)|a_n| .$$

So $(\varepsilon_n)$ and $(x_n)$ do indeed satisfy (A1).

To show that (a) implies (f), let (a) hold. Then we have the situation depicted in (3). For $t \in [0, 1]$, let $\delta_t \in C^*$ denote the point mass measure at $t$. Since $T^*$ is a weak-star continuous isometric embedding

$$M \overset{\text{def}}{=} \{ T^*(\delta_t) : t \in [0, 1] \} \subset B(Y^*) ,$$

equipped with the weak-star topology of $Y^*$, is homeomorphic to $[0, 1]$ and is isometrically equivalent to the usual basis of $\ell_1([0, 1])$. By the Hahn-Banach Theorem $j^*(B(X^*)) = B(Y^*)$ and so

$$N \overset{\text{def}}{=} j^*-1(M) \cap B(X^*)$$

is weak-star compact and satisfies $j^*(N) = M$. By Fact 6 there exists

$$\Gamma = \{ n_t : t \in [0, 1] \} \subset N$$

such that $j^*(n_t) = T^*(\delta_t)$ and such that $\Gamma$ is dense-in-itself in the weak-star topology on $X^*$. Since for any finite set $\{ a_t \}_{t \in F}$ of scalars

$$\sum_{t \in F} |a_t| = \left\| \sum_{t \in F} a_t T^* \delta_t \right\|_{Y^*} = \left\| \sum_{t \in F} a_t j^* n_t \right\|_{Y^*} \leq \left\| \sum_{t \in F} a_t n_t \right\|_{X^*} \leq \sum_{t \in F} |a_t| ,$$

the set $\Gamma$ is isometrically equivalent to the usual basis of $\ell_1([0, 1])$.

To see that (f) implies (e), let $X$ satisfy (f). Then by Lemma 5, there is a separable subspace $X_0$ of $X$ which satisfies (f). From Fact 7 and the fact that $C^*(\Delta)$ is linearly isometric to $C^*$, it follows that $X_0$ satisfies (e). The equivalence of (a) and (e) gives that $X$ also satisfies (e).
Thus (a) through (f) are equivalent. The fact that $C^*(\Delta)$ is linearly isometric to $C^*$ gives that (g) implies (e). That (f) implies (g) when $X$ is separable is due to Rosenthal: Fact 7.

**Remark 8.** Without the added assumption of separability, (g) is not equivalent to the other conditions. Clearly, $\ell_\infty$ satisfies conditions (a) through (f). But by a result of Grothendieck [G], a separable quotient of $\ell_\infty$ is reflexive and so $\ell_\infty$ does not satisfy (g).

**Remark 9.** A complemented isomorphic version of Theorem 2 is due to Hagler and Stegall [HS, Theorem 1]. A $K$-complemented isometric version of Theorem 2 is due to Hagler ([H1, Theorem 2.13] or [H3]).

**Remark 10.** Many Banach spaces (and their subspaces) which arise naturally in analysis contain an abundance of asymptotically isometric copies of $\ell_1$: for example, Carothers, Dilworth and Lennard [CDL] proved that every non-reflexive subspace of the Lorentz space $L_{w,1}(0,\infty)$ contains asymptotically isometric copies of $\ell_1$ whenever the weight $w$ satisfies very mild regularity conditions. On the other hand, $L_{w,1}$ does not contain an isometric copy of the 2-dimensional space $\ell_1^2$ whenever $w$ is strictly decreasing [CDT].

**Remark 11.** Theorem 2 improves a recent result of Shuatao Chen and Bor-Luh Lin [CL] who proved that $X$ contains an asymptotically isometric copy of $\ell_1$ whenever $X^*$ contains an isometric copy of $\ell_\infty$.

Dowling, Johnson, Lennard and Turett [DJLT] gave some concrete examples of equivalent norms on $\ell_1$ such that the corresponding Banach spaces do not contain asymptotically isometric copies of $\ell_1$. Theorem 2 easily yields other equivalent norms on $\ell_1$ with this property.

**Corollary 12.** Let $(\gamma_n)$ be a sequence of non-zero scalars with $\|\gamma_n\|_2 < \varepsilon$. Then the Banach space $(\ell_1, \| \cdot \|'_1)$, where

$$
\| (a_n) \|'_1 \overset{\text{def}}{=} \inf \left\{ \| (a_n + b_n) \|_1^2 + \| (\gamma_n^{-1}b_n) \|_2^2 : (\gamma_n^{-1}b_n) \in \ell_2 \right\},
$$

does not contain asymptotically isometric copies of $\ell_1$ and

$$
(1 + \varepsilon^2)^{-\frac{1}{2}} \| (a_n) \|_1 \leq \| (a_n) \|'_1 \leq \| (a_n) \|_1
$$

for each $(a_n) \in \ell_1$.

**Proof.** We exhibit $(\ell_1, \| \cdot \|'_1)$ as a quotient space of $X = \ell_1 \oplus \ell_2$ with its usual norm

$$
\| ((a_n), (b_n)) \|_X = \left[ \| (a_n) \|_1^2 + \| (b_n) \|_2^2 \right]^{1/2}.
$$

Let

$$
Y \overset{\text{def}}{=} \{ ((a_n), (b_n)) \in X : b_n = -\gamma_n^{-1}a_n \} \overset{\text{note}}{=} \{ ((a_n), (\gamma_n^{-1}a_n)) : (\gamma_n^{-1}a_n) \in \ell_2 \}.
$$
Since each element of $X/Y$ has a representative of the form $((a_n), 0)$,
\[
\|((a_n), 0) + Y\|_{X/Y} = \inf \{ \|(a_n), 0\) + ((b_n), (-\gamma_n^{-1}b_n))\|_X : (\gamma_n^{-1}b_n) \in \ell_2 \} = \|(a_n)\|_1.
\]
Thus $X/Y$ is isometrically isomorphic to $(\ell_1, \| \cdot \|_1')$.

Observe that $X^* = \ell_\infty \oplus_2 \ell_2$ with its usual norm
\[
\|((c_n), (d_n))\|_{X^*} = \left[ \|c_n\|_{\ell_\infty}^2 + \|d_n\|_2^2 \right]^{1/2}
\]
and
\[
Y^\perp = \{(c_n), (d_n)\} : d_n = \gamma_n c_n \} = \{(c_n), (\gamma_n c_n) : (c_n) \in \ell_\infty \}.
\]
It follows that $Y^\perp$ is isometrically isomorphic to $(\ell_\infty, \| \cdot \|_\infty')$ where
\[
\|(c_n)\|_{\ell_\infty'} = \left[ \|c_n\|_{\ell_\infty}^2 + \|\gamma_n c_n\|_2^2 \right]^{1/2}
\]
and the mapping
\[
i : (\ell_\infty, \| \cdot \|_\infty') \to (\ell_1, \| \cdot \|_1')^*
\]
given by $(i(c_n))(a_n) = \sum_n a_n c_n$ is an isometry. Since the norm $\| \cdot \|_\infty'$ is strictly convex, the space $(\ell_\infty, \| \cdot \|_\infty')$ does not contain an isometric copy of $L_1$. Thus by Theorem 2 the space $(\ell_1, \| \cdot \|_1')$ does not contain asymptotically isometric copies of $\ell_1$. From (8) it follows that
\[
\|(c_n)\|_{\ell_\infty} \leq \|(c_n)\|_{\ell_\infty'} \leq \sqrt{1 + \varepsilon^2} \|(c_n)\|_{\ell_\infty}
\]
for each $(c_n) \in \ell_\infty$ and so (7) holds by duality.

Finally, Alspach’s [A] example of an isometry $T : K \to K$ without a fixed point, where $K$ is a certain weakly compact convex subset of $L_1$, yields an obvious corollary.

**Corollary 13.** If $X$ contains asymptotically isometric copies of $\ell_1$, then there exists a (nonempty) weakly compact convex subset $K$ of $X^*$ and an isometry $T : K \to K$ without a fixed point.

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**References**


Shutao Chen and Bor-Luh Lin, Dual action of asymptotically isometric copies of $\ell_p$ ($1 \leq p < \infty$) and $c_0$, *Collect. Math.* **48** (1997), 449–458.


J. Hagler, Some more Banach spaces which contain $\ell^1$, *Studia Math.* **46** (1973), 35–42.

J. Hagler, in preparation.


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