

UNIVERSAL NON-COMPLETELY-CONTINUOUS OPERATORS

MARIA GIRARDI*[‡] AND WILLIAM B. JOHNSON*

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ABSTRACT. A bounded linear operator between Banach spaces is called **completely continuous** if it carries weakly convergent sequences into norm convergent sequences. Isolated is a universal operator for the class of non-completely-continuous operators from L_1 into an arbitrary Banach space, namely, the operator from L_1 into ℓ_∞ defined by

$$T_0(f) = \left(\int r_n f d\mu \right)_{n \geq 0},$$

where r_n is the n^{th} Rademacher function. It is also shown that there does not exist a universal operator for the class of non-completely-continuous operators between two arbitrary Banach space. The proof uses the factorization theorem for weakly compact operators and a Tsirelson-like space.

Suppose that \mathfrak{C} is a class of (always bounded, linear, between Banach spaces) operators so that an operator S is in \mathfrak{C} whenever the domain of S is the domain of some operator in \mathfrak{C} and there exist operators A, B so that BSA is in \mathfrak{C} ; the natural examples of such classes are all the operators that do not belong to a given operator ideal. A subset \mathfrak{S} of such a class \mathfrak{C} is said to be **universal** for \mathfrak{C} provided for each U in \mathfrak{C} , some member of \mathfrak{S} factors through U ; that is, there exist operators A and B so that BUA is in \mathfrak{S} . In case \mathfrak{S} is singleton; say, $\mathfrak{S} = \{S\}$; we say that S is universal for \mathfrak{C} .

In order to study a class \mathfrak{C} of operators, it is natural to try to find a universal subclass of \mathfrak{C} consisting of specific, simple operators. For certain classes, such a subclass is known to exist. For example, Lindenstrauss and Pełczyński, who introduced the concept of universal operator, proved [LP] that the “summing operator” from ℓ_1 to ℓ_∞ , defined by $\{a_n\}_{n=1}^\infty \mapsto \{\sum_{k=1}^n a_k\}_{n=1}^\infty$, is universal for the class of non-weakly-compact operators; while in [J] it was pointed out that the formal identity from ℓ_1 to ℓ_∞ is universal for the class of non-compact operators.

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[‡]Participant, NSF Workshop in Linear Analysis & Probability, Texas A&M University.

An operator between Banach spaces is called **completely continuous** if it carries weakly convergent sequences into norm convergent sequences. The operator from L_1 into ℓ_∞ given by

$$T_0(f) = \left\{ \int r_n f d\mu \right\}_{n=0}^{\infty},$$

where r_n is the n^{th} Rademacher function, is not completely continuous. We prove in Corollary 4 that T_0 is universal for the class of non-completely-continuous operators from an L_1 -space; however, in Theorem 5 we show that there does not exist a universal non-completely-continuous operator.

Throughout this paper, \mathfrak{X} denotes an arbitrary Banach space, \mathfrak{X}^* the dual space of \mathfrak{X} , and $S(\mathfrak{X})$ the unit sphere of \mathfrak{X} . The triple (Ω, Σ, μ) refers to the Lebesgue measure space on $[0, 1]$, Σ^+ to the sets in Σ with positive measure, and L_1 to $L_1(\Omega, \Sigma, \mu)$. All notation and terminology, not otherwise explained, are as in [DU] or [LT].

To crystalize the ideas in Theorem 1, we introduce some terminology. A system $\mathcal{A} = \{A_k^n \in \Sigma : n = 0, 1, 2, \dots \text{ and } k = 1, \dots, 2^n\}$ is a **dyadic splitting** of $A_1^0 \in \Sigma^+$ if each A_k^n is partitioned into the two sets A_{2k-1}^{n+1} and A_{2k}^{n+1} of equal measure for each admissible n and k . Thus the collection $\pi_n = \{A_k^n : k = 1, \dots, 2^n\}$ of sets along the n^{th} -level partition A_1^0 with π_{n+1} refining π_n and $\mu(A_k^n) = 2^{-n} \mu(A_1^0)$. To a dyadic splitting corresponds a (normalized) Haar system $\{h_j\}_{j \geq 1}$ along with its natural blocking $\{H_n\}_{n \geq 0}$ where

$$h_1 = \frac{1}{\mu(A_1^0)} 1_{A_1^0} \quad \text{and} \quad h_{2^n+k} = \frac{2^n}{\mu(A_1^0)} (1_{A_{2k-1}^{n+1}} - 1_{A_{2k}^{n+1}})$$

for $n = 0, 1, 2, \dots, k = 1, \dots, 2^n$, and $H_n = \{h_j : 2^{n-1} < j \leq 2^n\}$. The usual Haar system $\{\tilde{h}_j\}$ corresponds to the usual dyadic splitting $\{[\frac{k-1}{2^n}, \frac{k}{2^n})\}_{n,k}$. Let $L_1(\mathcal{A})$ be the closed subspace of L_1 with basis $\{h_j\}_{j \geq 1}$.

A set N in the unit sphere of the dual of a Banach space \mathfrak{X} is said to norm a subspace \mathfrak{X}_0 within $\tau > 1$ if for each $x \in \mathfrak{X}_0$ there is $x^* \in N$ such that $\|x\| \leq \tau x^*(x)$. It is well known and easy to see that a sequence $\{\mathfrak{X}_j\}_{j \geq 1}$ of subspaces of \mathfrak{X} forms a finite dimensional decomposition with constant at most τ provided that for each $n \in \mathbb{N}$ the space generated by $\{\mathfrak{X}_1, \dots, \mathfrak{X}_n\}$ can be normed by a set from $S(\mathfrak{X}_{n+1}^\perp)$ within $\tau_n > 1$ where $\prod \tau_n \leq \tau$.

To help demystify Theorem 1, we examine more closely the operator $T_0 : L_1 \rightarrow \ell_\infty$ given above. This operator does more than just map the Rademacher functions $\{r_n\}$ to the standard unit vectors $\{e_n\}$ in ℓ_∞ (which suffices to guarantee that it is not completely continuous). Let x_n^* be the n^{th} unit vector of ℓ_1 , viewed as an element in

the dual of ℓ_∞ . For the usual dyadic splitting of the unit interval, r_n is just the sum of the Haar functions in H_n , properly normalized. Thus $1 = \|T_0 r_n\| = x_n^*(T_0 r_n)$ follows from the stronger condition that

$$x_n^*(T_0 h) = \delta_{n,m} \quad \text{for each } h \in H_m .$$

Note that $T_0^* x_n^*$ is just r_n , which as a sequence in L_1^* is weak*-null and equivalent to the unit vector basis of ℓ_1 . Since T_0 maps each element in H_n to e_n , the collection $\{sp T_0 H_n\}$ forms a finite dimensional decomposition. Theorem 1 states that each non-completely-continuous operator T on L_1 behaves like the operator T_0 in the sense that there is some dyadic splitting of some subset of $[0, 1]$ so that the corresponding Haar system with T enjoys the above properties of the usual Haar system with T_0 .

Theorem 1. *Let Y be a subset of $S(\mathfrak{X}^*)$ that norms \mathfrak{X} within some fixed constant greater than one and let \mathcal{Y} be a subspace of \mathfrak{X}^* that contains Y . If the operator $T: L_1 \rightarrow \mathfrak{X}$ is not completely continuous and $\{\tau_n\}_{n \geq 0}$ is a sequence of numbers larger than 1, then there exist*

- (A) a dyadic splitting $\mathcal{A} = \{A_k^n\}$
- (B) a sequence $\{x_n^*\}_{n \geq 0}$ in $S(\mathfrak{X}^*) \cap \mathcal{Y}$
- (C) a finite set $\{z_{n,i}^*\}_{i=1}^{p_n}$ in $S(\mathfrak{X}^*)$ for each $n \geq 0$

such that for the Haar system $\{h_j\}_{j \geq 1}$ and the blocking $\{H_n\}_{n \geq 0}$ corresponding to \mathcal{A} , for some $\delta > 0$, and each $n, m \geq 0$,

- (1) $x_n^*(Th) = \delta \cdot \delta_{n,m}$ for each $h \in H_m$
- (2) $\{T^* x_n^*\}$ is weak*-null in L_∞
- (3) $\{T^* x_n^*\}$ is equivalent to the unit vector basis of ℓ_1
- (4) $\{z_{n,i}^*\}_{i=1}^{p_n}$ norms $sp(\cup_{j=0}^n TH_j)$ within τ_n
- (5) $TH_{n+1} \subset {}^\perp \{z_{n,i}^*\}_{i=1}^{p_n}$.

Note that condition (3) implies that $\{x_n^*\}$ is also equivalent to the standard unit vector basis of ℓ_1 . If $\Pi\tau_n$ is finite, then the last two conditions guarantee that $\{sp TH_n\}_{n \geq 0}$ forms a finite dimensional decomposition with constant at most $\Pi\tau_n$.

The proof uses the following two standard lemmas.

Lemma 2. *Let $E = sp \{x_i\}_{i=0}^m$ be a finite dimensional subspace of a Banach space \mathfrak{X} and let \mathcal{Y} be a total subspace of \mathfrak{X}^* . For each $\epsilon > 0$ there exists $\eta > 0$ such that if $y^* \in \mathfrak{X}^*$ satisfies $|y^*(x_i)| < \eta$ for each $1 \leq i \leq m$, then there exists $x^* \in E^\perp$ of norm 0 or $\|y^*\|$ such that $\|x^* - y^*\| < \epsilon$. Furthermore, if y^* is in \mathcal{Y} then x^* can be taken to be in \mathcal{Y} .*

Proof of Lemma 2. Assume, without loss of generality, $\{x_i\}_{i=0}^m$ is linearly independent. Consider the isomorphism $l: E \rightarrow \ell_1^m$ that takes x_i to the i^{th} unit basis vector of ℓ_1^m and let P be a projection from \mathfrak{X} onto E that is $w(\mathcal{Y})$ -continuous, so that P^*E^* is a subspace of \mathcal{Y} . Such a projection exists because \mathcal{Y} is total. Then $\tilde{x}^* \equiv y^* \cdot (I_{\mathfrak{X}} - P)$ is in E^\perp . It is easy to check that for $\eta = \frac{\epsilon}{3\|l\|\|P\|}$, if $|y^*(x_i)| < \eta$ for each i , then $\|\tilde{x}^* - y^*\| \leq \frac{\epsilon}{3}$. If $\|\tilde{x}^*\| = 0$, then let $x^* = \tilde{x}^*$. Otherwise, let $x^* = (\|y^*\| / \|\tilde{x}^*\|) \tilde{x}^*$. Then $\|x^* - y^*\| \leq 2\|\tilde{x}^* - y^*\|$. Thus x^* does what it should do. ■

Recall that the extreme points of $B(L_\infty)$ are just the ± 1 -valued measurable functions.

Lemma 3. *If $\{f_i\}_{i=0}^n$ is a finite subset of L_1 , $\{\alpha_i\}_{i=0}^n$ are scalars, and*

$$S = \left\{ g \in B(L_\infty) : \int f_i g d\mu = \alpha_i \text{ for each } 0 \leq i \leq n \right\},$$

then $\text{ext } S = S \cap \text{ext } B(L_\infty)$, where ext denotes the extreme points of a set. Also, if S is non-empty then so is $\text{ext } S$.

Specifically, we use the following version of this extreme point argument lemma.

Lemma 3'. *If $F = \{f_1, \dots, f_n\}$ and there exists g in $B(L_\infty) \cap F^\perp$ such that $\int f_0 g d\mu \geq \alpha_0 > 0$, then there exists a ± 1 -valued function u in $B(L_\infty) \cap F^\perp$ such that $\int f_0 u d\mu = \alpha_0$.*

Proof of Lemma 3. Consider, if there is one, a function g in S for which there exists a subset A of positive measure and $\epsilon > 0$ such that $-1 + \epsilon < g1_A < 1 - \epsilon$. Since the set $\{f \in L_\infty : |f| \leq 1_A\} \cap \{f_i\}_{i=0}^n^\perp$ is infinite dimensional, it contains a non-zero element h of norm less than ϵ . But then $g \pm h \in S$ and so g is not an extreme point of S . Thus $\text{ext } S = S \cap \text{ext } B(L_\infty)$.

Since S convex and weak*-compact in L_∞ , if S is non-empty then so is $\text{ext } S$.

As for the last claim of the lemma, just note that if $g \in B(L_\infty) \cap F^\perp$ satisfies $\int f_0 g d\mu \equiv \beta \geq \alpha_0 > 0$, then $\frac{\alpha_0}{\beta} g$ is in the set S where $\alpha_i = 0$ for $i > 0$. By the first part of the lemma, any extreme point u of S will do. ■

Although the proof of Theorem 1 is somewhat technical, the overall idea is simple. Since T is not completely continuous, we start by finding a weakly convergent sequence $\{g_n\}$ in L_1 and norm one functionals y_n^* such that $\delta_0 \leq y_n^*(Tg_n)$. Each x_n^* will be a small perturbation of some $y_{j_n}^*$. Conditions (2) and (3) can be arranged by standard arguments.

Now the proof gets technical. We begin by finding a subset A_1^0 where the L_∞ function $(T^*y_n^*)g_n$, which in the motivating example of T_0 is the function r_n , is

large in some sense. We then proceed by induction on the level n . Given a finite dyadic splitting up to n^{th} -level provides the subsets $\{H_m\}_{m=0}^n$ of corresponding Haar functions. We need to split each A_k^n into 2 sets A_{2k-1}^{n+1} and A_{2k}^{n+1} (thus finding h_{2^n+k}) and find the desired functionals so that all works. It is easy to find the functionals to satisfy condition (4). In the search for x_{n+1}^* , apply Lemma 2 to the set E given in (†) so that we need only to almost (within some η) satisfy (1-i') for some y_j^* ; for then we can perturb y_j^* to find x_{n+1}^* that satisfies (1-i') exactly. Next, for each A_k^n , apply Lemma 3' with F as given in (‡) and $f_0 = T^*y_j^*1_{A_k^n}$ and g being a small perturbation of $g_j1_{A_k^n}$. All is set up so that such a perturbation exists for a j (dependent on n but independent of k) sufficiently large enough. Now Lemma 3' gives that desired ± 1 -valued Haar-like function that yields the desired splitting of the $(n+1)^{\text{th}}$ -level. The sets F_k^n are chosen exactly so that conditions (1-ii'), (1-iii'), and (5') hold.

Proof of Theorem 1. Let $T: L_1 \rightarrow \mathfrak{X}$ be a norm one operator that is not completely continuous. Then there is a sequence $\{g_n\}$ in L_1 and a sequence $\{y_n^*\}$ in $S(\mathfrak{X}^*) \cap \mathcal{Y}$ satisfying:

- (a) $\|g_n\|_{L_\infty} \leq 1$
- (b) g_n is weakly null in L_1
- (c) $\delta_0 \leq y_n^*(Tg_n)$ for some $\delta_0 > 0$.

Using (a), (b), and (c) along with Rosenthal's ℓ_1 theorem [cf. LT, Prop. 2.e.5], by passing to a further subsequence, we also have that

- (d) $\{T^*y_n^*\}$ is equivalent to the standard unit vectors basis of ℓ_1 .

Since $B(L_\infty)$ is weak* sequentially-compact in L_∞ , by passing to a subsequence and considering differences we may assume that

- (e) $T^*y_n^*$ is weak*-null in L_∞ ,

where (d) allows normalization of the new y_n^* 's so as to keep them in $S(\mathfrak{X}^*)$ and, used with care, (b) ensures that (c) still holds for some (new) positive δ_0 . But $\{(T^*y_n^*) \cdot g_n\}$ is also in $B(L_\infty)$ and so, by passing to yet another subsequence, we have that

- (f) $\{(T^*y_n^*) \cdot g_n\} \rightarrow h$ weak* in L_∞

for some $h \in L_\infty$.

Since $\int h d\mu \geq \delta_0$, the set $A \equiv [h \geq \delta_0]$ has positive measure. We may assume, by replacing y_n^* by $-y_n^*$ and g_n by $-g_n$ when needed, that $\|T^*y_n^*|_A\|_{L_\infty} = \text{ess sup } T^*y_n^*|_A$. So from (a) and (f) it follows that $\delta_0 \leq \liminf \text{ess sup } T^*y_n^*|_A$ while from (e) it follows that $\limsup \mu[T^*y_n^*|_A \geq \delta_0 - \eta] < \mu(A)$ for each $0 < \eta < \delta_0$.

Thus, since the closure of the set

$$\left\{ \frac{\int_E f d\mu}{\mu(E)} : E \subset A, E \in \Sigma^+ \right\}$$

is the interval $[\text{ess inf } f, \text{ess sup } f]$, there is a subset A_1^0 of A with positive measure and j_0 such that $y_{j_0}^* T(1_{A_1^0}) = \delta \mu(A_1^0)$ for some positive δ less than δ_0 , say $\delta \equiv \delta_0 - 3\epsilon$. Put $x_0^* = y_{j_0}^*$ and $H_0 \equiv \{h_1\} = \{1_{A_1^0}/\mu(A_1^0)\}$.

We shall construct, by induction on the level n , a dyadic splitting of A_1^0 along with the desired functionals. Towards this, take a decreasing sequence $\{\epsilon_n\}_{n \geq 0}$ of positive numbers such that $\epsilon_0 < \epsilon$ and $\sum \epsilon_n < \frac{\delta_0}{2K}$ where K is the basis constant of $\{T^* y_n^*\}$. The sequence $\{x_n^*\}$ will be chosen such that $\|x_n^* - y_{j_n}^*\| \leq \epsilon_n$ for some increasing sequence $\{j_n\}_n$ of integers, which will ensure conditions (2) and (3). Note that condition (1) is equivalent to the following 3 conditions holding

- (1-i) $x_n^*(Th) = 0$ for $h \in H_m$ and $0 \leq m < n$
- (1-ii) $x_m^*(Th) = 0$ for $h \in H_n$ and $0 \leq m < n$
- (1-iii) $x_n^*(Th) = \delta$ for $h \in H_n$

for each n . Clearly these three conditions hold for $n = 0$. Fix $n \geq 0$.

Suppose that we are given a finite dyadic splitting $\{A_k^m : m = 0, \dots, n \text{ and } k = 1, \dots, 2^m\}$ of A_1^0 up to n^{th} -level, which gives the subsets $\{H_m\}_{m=0}^n$ of corresponding Haar functions. Thus we can find a finite set $\{z_{n,i}^*\}_{i=1}^{p_n}$ in $S(\mathfrak{X}^*)$ such that $\{z_{n,i}^*\}_{i=1}^{p_n}$ norms $\text{sp}(\cup_{j=0}^n TH_j)$ within τ_n . Suppose that we are also given $\{x_m^*\}_{m=0}^n$ in $\mathcal{Y} \cap S(\mathfrak{X}^*)$ such that the three subconditions of (1) hold and if $k = 1, 2, \dots, n$ then $\|x_k^* - y_{j_k}^*\| \leq \epsilon_k$ for some j_k .

We shall find x_{n+1}^* along with $j_{n+1} > j_n$ such that $\|x_{n+1}^* - y_{j_{n+1}}^*\| \leq \epsilon_{n+1}$ and we shall partition, for each $1 \leq k \leq 2^n$, the set A_k^n into 2 sets A_{2k-1}^{n+1} and A_{2k}^{n+1} of equal measure (thus finding h_{2^n+k} and so finding the corresponding set $\{H_{n+1}\}$) such that

- (1-i') $x_{n+1}^*(Th) = 0$ for $h \in H_m$ and $0 \leq m < n+1$
- (1-ii') $x_m^*(Th) = 0$ for $h \in H_{n+1}$ and $0 \leq m < n+1$
- (1-iii') $x_{n+1}^*(Th) = \delta$ for $h \in H_{n+1}$
- (5') $TH_{n+1} \subset \perp \{z_{n,i}^*\}_{i=1}^{p_n}$.

Towards this, apply Lemma 2 to

$$(\dagger) \quad E \equiv \{Th : h \in H_m, 0 \leq m \leq n\}$$

and ϵ_{n+1} to find the corresponding η_{n+1} . Let

$$(\ddagger) \quad F_k^n = \{1_{A_k^n}\} \cup \{T^* x_m^* 1_{A_k^n}\}_{m=0}^n \cup \{T^* z_{n,i}^* 1_{A_k^n}\}_{i=1}^{p_n} \subset L_1$$

and $F_n = \text{sp} \left[\cup_{k=1}^{2^n} F_k^n \right]$.

Pick $j \equiv j_{n+1} > j_n$ so large that for $k = 1, \dots, 2^n$

$$(g) \quad |(T^* y_j^*) h| < \eta_{n+1} \quad \text{for all } h \in \cup_{m=0}^n H_m$$

$$(h) \quad \left| \int_{\Omega} g_j f d\mu \right| \leq \frac{\epsilon}{3} \|f\| \quad \text{for all } f \text{ in } F_n$$

$$(i) \quad \int_{A_k^n} T^* y_j^* \cdot g_j d\mu \geq (\delta_0 - \epsilon) \mu(A_k^n).$$

Condition (g) follows from (e), condition (h) follows from (b) and the fact that F_n is finite dimensional, condition (i) follows from (f) and the definition of A .

By Lemma 2 and (g), there is $x_{n+1}^* \in S(\mathfrak{X}^*) \cap \mathcal{Y}$ such that $\|x_{n+1}^* - y_{j_{n+1}}^*\|$ is at most ϵ_{n+1} and $x_{n+1}^* T h = 0$ for each $h \in \cup_{m=0}^n H_m$. Thus (1-i') holds.

Condition (h) gives that the L_{∞} -distance from g_j to

$$F_n^{\perp} \equiv \left\{ g \in L_{\infty} : \int_{\Omega} f g d\mu = 0 \text{ for each } f \in F_n \right\}$$

is at most $\frac{\epsilon}{3}$. So there is $\tilde{g}_j \in F_n^{\perp} \cap B(L_{\infty})$ such that $\|\tilde{g}_j - g_j\|_{L_{\infty}}$ is less than ϵ . Clearly $\tilde{g}_j 1_{A_k^n} \in F_k^n \perp \cap B(L_{\infty})$ for each admissible k . By condition (i), for each admissible k ,

$$\int_{\Omega} (T^* x_{n+1}^*) \cdot (\tilde{g}_j 1_{A_k^n}) d\mu \geq \delta \mu(A_k^n)$$

and so, by Lemma 3, there exists a function $u_k^n \in B(L_{\infty}) \cap F_k^n \perp$ such that

$$(*) \quad \int_{\Omega} (T^* x_{n+1}^*) \cdot (u_k^n) d\mu = \delta \mu(A_k^n)$$

and u_k^n is of the form $1_{A_{2k-1}^{n+1}} - 1_{A_{2k}^{n+1}}$ for 2 disjoint sets A_{2k-1}^{n+1} and A_{2k}^{n+1} whose union is A_k^n . Furthermore, A_{2k-1}^{n+1} and A_{2k}^{n+1} are of equal measure since $1_{A_k^n} \in F_k^n$. Since $u_k^n \in F_k^n \perp$, conditions (1-ii') and (5') hold. Condition (1-iii') is just (*). \blacksquare

Theorem 1 contains much information. For example, the next corollary crystallizes the role of the previously mentioned operator T_0 .

Corollary 4. *If the operator $T: L_1 \rightarrow \mathfrak{X}$ is not completely continuous, then there exist an isometry A and an operator B such that the following diagram commutes.*

$$\begin{array}{ccc} L_1 & \xrightarrow{T} & \mathfrak{X} \\ A \uparrow & & \downarrow B \\ L_1 & \xrightarrow{T_0} & \ell_{\infty} \end{array}$$

Furthermore, if \mathfrak{X} is separable, then T_0 and B may be viewed as operators into c_0 .

Proof of Corollary 4. Let j_1 be the natural injection of $L_1(\mathcal{A})$ into L_1 , let \mathfrak{X}_0 be the norm closure of $T(j_1 L_1(\mathcal{A}))$, and let \tilde{x}_n^* be the restriction of x_n^* to \mathfrak{X}_0 .

Since $\{T^*x_n^*\}$ is weak*-null in L_∞ , \tilde{x}_n^* is weak*-null in \mathfrak{X}_0^* . Thus the mapping $U: \ell_1 \rightarrow \mathfrak{X}_0^*$ that take the n^{th} unit basis vector of ℓ_1 to \tilde{x}_n^* is weak* to weak* continuous and so U is the adjoint of the operator $S: \mathfrak{X}_0 \rightarrow c_0$ where $S(x) = (\tilde{x}_n^*(x))_{n \geq 0}$.

Consider the (commutative) diagram:

$$\begin{array}{ccccc}
 L_1 & & \xrightarrow{\tau} & & \mathfrak{X} \\
 & & & & \\
 j_1 \uparrow & & & & \uparrow j_2 \\
 L_1(\mathcal{A}) & & \xrightarrow{\tau_{\mathcal{A}}} & & \mathfrak{X}_0 \\
 & & & & \\
 R \uparrow & & & & \downarrow S \\
 L_1 & & \longrightarrow & & c_0 \xrightarrow{j_3} \ell_\infty
 \end{array}$$

where $R: L_1 \rightarrow L_1(\mathcal{A})$ is the natural isometry that takes a usual Haar function \tilde{h}_j in L_1 to the corresponding associated Haar function h_j in $L_1(\mathcal{A})$, the maps j_i are the natural injections, and $T_{\mathcal{A}}$ is such that the upper square commutes.

For an arbitrary space \mathfrak{X} , since ℓ_∞ is injective, the operator $j_3 S$ extends to an operator $\tilde{S}: \mathfrak{X} \rightarrow \ell_\infty$. For a separable space \mathfrak{X} , since c_0 is separably injective, this extension \tilde{S} may be view as taking values in c_0 .

Let $A = j_1 R$ and $B = \frac{1}{\delta} \tilde{S}$. Then $BTA(\tilde{h}_j) = \frac{1}{\delta} (\tilde{x}_n^*(Th_j))_{n \geq 0}$. Property 1 of Theorem 1 gives that $BTA = T_0$. \blacksquare

Corollary 4 says that, viewed as an operator into ℓ_∞ (respectively, into c_0), T_0 is universal for the class of non-completely-continuous operators from L_1 into an arbitrary (respectively, separable) Banach space.

Theorem 5. *There does not exist a universal operator for the class of non-completely-continuous operator.*

The proof of the nonexistence of such an operator uses the existence of a factorization through a reflexive space for a weakly compact operator.

Proof. Suppose that there did exist a universal non-completely-continuous operator, say $T_1: \mathfrak{X} \rightarrow \mathcal{Z}$ where \mathfrak{X} and \mathcal{Z} are Banach spaces. Then there is a sequence $\{x_n\}$ in \mathfrak{X} of norm one elements that converge weakly to zero but whose images $\{T_1 x_n\}$ are uniformly bounded away from zero. Furthermore, by passing to a subsequence, we also have that $\{T_1 x_n\}$ is a basic sequence in \mathcal{Z} .

The first step of the proof uses T_1 to construct a “nice” universal non-completely continuous operator. By Corollary 7 in [DFJP], there exists a reflexive space \mathcal{Y} with a normalized unconditional basis $\{y_n\}$ such that the map $S: \mathcal{Y} \rightarrow \mathfrak{X}$ that sends y_n to x_n is continuous. Consider the map $U: \mathcal{Z} \rightarrow \ell_\infty$ that sends z to $(z_n^*(z))$ where $\{z_n^*\}$ is a bounded sequence in \mathcal{Z}^* such that $\{T_1 x_n, z_n^*\}$ is a biorthogonal system. The map $I_{\mathcal{Y}} \equiv UT_1 S$ sends y_n to the n^{th} unit vector of ℓ_∞ . The reflexivity of \mathcal{Y} guarantees that $I_{\mathcal{Y}}$ is not completely continuous. Since $I_{\mathcal{Y}}$ factors through the universal operator T_1 , the operator $I_{\mathcal{Y}}$ must also be universal. We now work with this “nice” operator $I_{\mathcal{Y}}$.

For any linearly independent finite set $\{x_k\}_{k=1}^n$, let $\mathcal{D}\{x_k\}_{k=1}^n$ be the norm of the operator from the span of $\{x_k\}_{k=1}^n$ to ℓ_1^n that sends x_k to the k^{th} unit vector of ℓ_1^n . Set $d_n = \mathcal{D}\{y_k\}_{k=1}^n$. Reflexivity of \mathcal{Y} gives that d_n tends to infinity. Let T be a (reflexive) Tsirelson-like space with normalized unconditional basis $\{t_n\}$ such that for all finite subsets F of natural numbers,

$$\mathcal{D}\{t_n\}_{n \in F} \leq \max \left\{ 2, \sqrt{d_{|F|}} \right\} ,$$

where $|F|$ is the cardinality of F . For example, $\{t_n\}$ can just be an appropriately chosen subsequence of the usual basis of the usual Tsirelson space [cf. CS, Chapter I]. Consider the non-completely-continuous map $I_T: T \rightarrow \ell_\infty$ that sends t_n to the n^{th} unit vector of ℓ_∞ . By the universality of $I_{\mathcal{Y}}$, there exists maps A and B such that the following diagram commutes.

$$\begin{array}{ccccc} T & & \xrightarrow{I_T} & & \ell_\infty \\ & & & & \\ A \uparrow & & & & \downarrow B \\ \mathcal{Y} & & \xrightarrow{I_{\mathcal{Y}}} & & \ell_\infty \end{array}$$

Since each $I_{\mathcal{Y}}(y_n)$ is of norm one, there exists $\delta > 0$ such that $\delta < \|I_T A y_n\|$ for each n . Each $A y_n$ is of the form

$$A y_n = \sum_{m=1}^{\infty} \alpha_{n,m} t_m$$

and so there is a sequence $\{m(n)\}_n$ of natural numbers such that $\delta < |\alpha_{n,m(n)}|$. Since $\{y_n\}$ tends weakly to zero, for each m the set of all n for which $m(n) = m$ is finite. Thus by replacing \mathcal{Y} with the closed span of a suitable subsequence of $\{y_n\}$, we may assume that the $m(n)$'s are distinct.

Let T_* be the subspace of T spanned by $\{t_{m(n)}\}_n$. Since $\{y_n\}$ and $\{t_{m(n)}\}$ are both unconditional bases, by the diagonalization principle [cf. LT, Prop. 1.c.8],

the correspondence $y_n \mapsto \alpha_{n,m(n)} t_{m(n)}$ extends to an operator $D: \mathcal{Y} \rightarrow T_*$. Since $\{t_{m(n)}\}$ is an unconditional basis and $\delta < |\alpha_{n,m(n)}|$, the correspondence $\alpha_{n,m(n)} t_{m(n)} \mapsto t_{m(n)}$ extends to an operator $M: T_* \rightarrow T_*$.

By the definition of d_n , there exists a sequence $\{\beta_i^n\}_{i=1}^n$ such that $\sum_{i=1}^n |\beta_i^n| = 1$ and

$$\left\| \sum_{i=1}^n \beta_i^n y_i \right\|_{\mathcal{Y}} = \frac{1}{d_n}.$$

By the choice of T , for large n ,

$$\frac{1}{\sqrt{d_n}} \leq \left\| \sum_{i=1}^n \beta_i^n t_{m(i)} \right\|_{T_*}.$$

Since $MD: \mathcal{Y} \rightarrow T_*$ maps y_n to $t_{m(n)}$,

$$\left\| \sum_{i=1}^n \beta_i^n t_{m(i)} \right\|_{T_*} \leq \|MD\| \left\| \sum_{i=1}^n \beta_i^n y_i \right\|_{\mathcal{Y}}.$$

This gives that

$$\frac{1}{\sqrt{d_n}} \leq \frac{\|MD\|}{d_n},$$

which cannot be since d_n tends to infinity. ■

The first two paragraphs of the proof of Theorem 5 yield part (a) of the next proposition. Part (b) follows from similar considerations and the Gurarii-James theorem [Ja, Thm. 2].

Proposition 6.

- (a) *Let \mathfrak{S} be the collection of all formal identity operators into ℓ_∞ from reflexive sequence spaces for which the unit vectors form a normalized unconditional basis. Then \mathfrak{S} is universal for the class of all non-completely-continuous operators.*
- (b) *The collection $\{I: \ell_p \rightarrow \ell_\infty; 1 < p < \infty\}$ of formal identity operators is universal for the class of all non-completely-continuous operators whose domain is superreflexive.*

Recall that a Banach space \mathfrak{X} has the Radon-Nikodým Property (RNP) [respectively, is strongly regular, has the Complete Continuity Property (CCP)] if each bounded linear operator from L_1 into \mathfrak{X} is representable [respectively, strongly regular, completely continuous]. The books [DU], [GGMS], and [T] contain splendid surveys of these properties. Here we only recall that a representable operator is strongly regular and a strongly regular operator is completely continuous. The first

paragraph of the proof of Theorem 1 uses elementary methods to construct, from an operator $T: L_1 \rightarrow \mathfrak{X}$ that is not completely continuous, a copy of ℓ_1 in the closed span of a norming set of \mathfrak{X} . On a much deeper level, the following fact is well-known.

Fact. *The following are equivalent.*

- (1) ℓ_1 embeds into \mathfrak{X} .
- (2) L_1 embeds into \mathfrak{X}^* .
- (3) \mathfrak{X}^* fails the CCP.
- (4) \mathfrak{X}^* is not strongly regular.

The well-known equivalence of (1) and (2) was shown by Pełczyński [P, for separable \mathfrak{X}] and Hagler [H, for non-separable \mathfrak{X}]. The other downward implications follow from the definitions. Bourgain [B] used a non-strongly-regular operator into a dual space to construct a copy of ℓ_1 in the pre-dual. Here the authors wish to formalize the following essentially known fact which, to the best of our knowledge, has not appeared in print as such.

Fact. *The following are equivalent.*

- (1) \mathfrak{X} has trivial type.
- (2) \mathfrak{X} fails super CCP.
- (3) \mathfrak{X} is not super strongly regular.

Proof. To see that (1) implies (2), recall that \mathfrak{X} has trivial type if and only if ℓ_1 is finitely representable in \mathfrak{X} and that L_1 is finitely representable in ℓ_1 . Thus, if \mathfrak{X} has trivial type, then L_1 is finitely representable in \mathfrak{X} and so \mathfrak{X} cannot have the super CCP. Property (3) formally follows from (2). Towards seeing that (3) implies (1), consider a space \mathfrak{X} that is not strongly regular. From the above fact it follows that ℓ_1 embeds into \mathfrak{X}^* . Thus \mathfrak{X}^* has trivial type, which implies the same for \mathfrak{X} . ■

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MARIA GIRARDI

UNIVERSITY OF SOUTH CAROLINA, DEPARTMENT OF MATHEMATICS, COLUMBIA, SC 29208

E-mail address: girardi@math.sc.edu

W. B. JOHNSON

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843

E-mail address: johnson@math.tamu.edu