NOWHERE WEAK DIFFERENTIABILITY OF THE PETTIS INTEGRAL

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Abstract. For an arbitrary infinite-dimensional Banach space $\mathfrak{X}$, we construct examples of strongly-measurable $\mathfrak{X}$-valued Pettis integrable functions whose indefinite Pettis integrals are nowhere weakly differentiable; thus, for these functions the Lebesgue Differentiation Theorem fails rather spectacularly. We also relate the degree of nondifferentiability of the indefinite Pettis integral to the cotype of $\mathfrak{X}$, from which it follows that our examples are reasonably sharp.

There are several generalizations of the space $L_1(\mathbf{R})$ of Lebesgue integrable functions taking values in the real numbers $\mathbf{R}$ (and defined on the usual Lebesgue measure space (Ω, Σ, μ) on $[0,1]$) to a space of strongly-measurable "integrable" (suitably formulated) functions taking values in a Banach space $\mathfrak{X}$.

The most common generalization is the space $L_1(\mathfrak{X})$ of Bochner-Lebesgue integrable functions. Using the fact [P1, Theorem 1.1] that a strongly-measurable function is essentially separably-valued, one can easily extend Lebesgue's Differentiation Theorem from $L_1(\mathbf{R})$ to $L_1(\mathfrak{X})$. Specifically [B; cf. DU, Theorem II.2.9], if $f \in L_1(\mathfrak{X})$, then

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} \|f(\omega) - f(t)\| \, d\mu(\omega) \;=\; 0$$

and so

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} f(\omega)\, d\mu(\omega) \;=\; f(t)$$

for almost all t in Ω.

Another generalization of $L_1(\mathbf{R})$ is the space $\mathcal{P}_1(\mathfrak{X})$ of strongly-measurable Pettis integrable functions. A function $f : \Omega \to \mathfrak{X}$ is Pettis integrable if for each $E \in \Sigma$ there is an element $x_E \in \mathfrak{X}$ satisfying

$$x^*(x_E) \;=\; \int_E x^* f \, d\mu$$

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for each x^* in the dual space $\mathfrak{X}^*$ of $\mathfrak{X}$. The element x_E is called the Pettis integral of f over E and we write

$$\mathcal{P} - \int_E f\, d\mu \;=\; x_E \;.$$

It is clear that $L_1(\mathfrak{X}) \subset \mathcal{P}_1(\mathfrak{X})$, while the reverse inclusion holds if and only if $\mathfrak{X}$ is finite dimensional (see e.g. [DG]).

If $f \in \mathcal{P}_1(\mathfrak{X})$, then for each $x^* \in \mathfrak{X}^*$ the function $x^*f \in L_1(\mathbf{R})$ and so there exists a set A (which depends on x^*) of full measure such that

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} x^* f(\omega)\, d\mu(\omega) \;=\; x^* f(t)$$

for each $t \in A$. In his paper [P1] introducing the Pettis integral, Pettis phrased this by saying that the Pettis integral of a function in $\mathcal{P}_1(\mathfrak{X})$ is *pseudo-differentiable*. He closed his paper by asking whether the Pettis integral of a function f in $\mathcal{P}_1(\mathfrak{X})$ enjoys the stronger property of being *a.e. weakly differentiable*; that is, does there necessarily exist a set A (*independent* of x^*) of full measure such that

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} x^* f(\omega)\, d\mu(\omega) \;=\; x^* f(t)$$

for each $t \in A$ and $x^* \in \mathfrak{X}^*$, or such that (which is the same thing of course)

$$\text{weak} - \lim_{h\to 0} \frac{1}{h}\, \mathcal{P} - \int_t^{t+h} f(\omega)\, d\mu(\omega) \;=\; f(t)$$

for each $t \in A$.

If $\mathfrak{X}$ is finite dimensional, then the Pettis integral of a function in $\mathcal{P}_1(\mathfrak{X})$ is a.e. weakly differentiable. R.S. Phillips [Ph] (for $\mathfrak{X} = \ell_2$) and M.E. Munroe [M] (for $\mathfrak{X} = C[0,1]$) each constructed an example of a function in $\mathcal{P}_1(\mathfrak{X})$ whose Pettis integral is not a.e. weakly differentiable. G.E.F. Thomas [T, p. 131] conjectured that such a function in $\mathcal{P}_1(\mathfrak{X})$ exists for every infinite-dimensional Banach space $\mathfrak{X}$.

At the recent May 1993 Kent State University Functional Analysis Conference, Joe Diestel requested a further investigation into Pettis's question. Independently, V. Kadets [K] recently constructed, for each infinite-dimensional Banach space $\mathfrak{X}$, a function in $\mathcal{P}_1(\mathfrak{X})$ whose Pettis integral fails to be a.e. weakly differentiable; specifically, it fails to be weakly differentiable on a set of positive, but not full, measure.

The main theorem of this paper constructs, for each infinite-dimensional Banach space $\mathfrak{X}$, a function in $\mathcal{P}_1(\mathfrak{X})$ whose Pettis integral is *nowhere* weakly differentiable. This theorem also addresses the degree of nondifferentiability of the Pettis integral. Our second theorem shows, for *arbitrary* Banach spaces, that the functions which we construct are close to being optimal with respect to their degree of nondifferentiability. From these two theorems it follows (Corollaries 3 and 4) that the cotype of a space is closely tied to the degree of nondifferentiability of the Pettis integral.

Theorem 2 was shown to us by Nigel Kalton in answer to a question posed in a preliminary version of this paper. We are grateful to him for permission to include this result here.

To state our main result we introduce the collection Ψ of all increasing functions $\psi\colon\ [0,\infty) \to [0,\infty)$ satisfying the growth condition

$$\sum_{n=1}^{\infty} \psi(2^{-p_{n-1}})\ \sqrt{2^{p_n}}\ <\ \infty\ , \tag{$\dagger$}$$

for some increasing sequence $\{p_n\}_{n=0}^{\infty}$ of integers. Examples of functions in Ψ are

$$\psi(s) = s^{\frac{1}{2}+\epsilon},$$

$$\psi(s) = s^{\frac{1}{2}} \left[\frac{1}{\log(1/s)}\right]^{1+\epsilon} \qquad \text{and} \qquad \psi(s) = s^{\frac{1}{2}} \left[\frac{1}{\log(1/s)}\right] \left[\frac{1}{\log\log(1/s)}\right]^{1+\epsilon}$$

for $p_n = n$ and any $\epsilon > 0$.

Theorem 1. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space. For each $\psi \in \Psi$, there exists $f \in \mathcal{P}_1(\mathfrak{X})$ such that*

$$\left\| \mathcal{P} - \int_I f\, d\mu \right\|_{\mathfrak{X}}\ \geqslant\ \psi\left(\mu\left(I\right)\right) \tag{$\ddagger$}$$

for each interval I contained in $[0,1]$.

Remark. Taking $\psi(t) = t^{\frac{3}{4}}$ gives a Pettis integrable function f such that for each $t \in \Omega$,

$$\lim_{h\to 0} \left\| \frac{1}{h}\ \mathcal{P} - \int_t^{t+h} f(\omega)\, d\mu(\omega) \right\|_{\mathfrak{X}}\ =\ \infty\ .$$

If the Pettis integral of this f were weakly differentiable at t, then the above limit would be finite.

Proof. Let $\{I_k^n : n = 0, 1, \ldots\ ; k = 1, \ldots, 2^n\}$ be the dyadic intervals on $[0,1]$, i.e.

$$I_k^n = \left[\frac{k-1}{2^n}, \frac{k}{2^n}\right)\ .$$

Define inductively a collection $\{A_k^n : n = 0, 1, \ldots\ ; k = 1, \ldots, 2^n\}$ of disjoint sets of strictly positive measure such that $A_k^n \subset I_k^n$ (e.g. appropriately chosen "fat Cantor" sets).

Fix $K > 1$. By a theorem of Mazur there is a basic sequence $\{x_n\}$ in $\mathfrak{X}$ with basis constant at most K. Take a blocking $\{F_n\}$ of the basis with each subspace F_n of large enough dimension to find (using the finite-dimensional version of Dvoretzky's Theorem [D]) a 2^n-dimensional subspace E_n of F_n such that the Banach-Mazur distance between E_n and $\ell_2^{2^n}$ is less than 2. Note that $\{E_n\}$ forms a finite-dimensional decomposition. Next find operators $T_n\colon \ell_2^{2^n} \to E_n$ such that $\|T_n\| \leqslant 2$ and $\left\|T_n^{-1}\right\| = 1$. Let $\{u_k^n\colon k = 1, \dots 2^n\}$ be the standard unit vectors of $\ell_2^{2^n}$ and let $e_k^n \equiv T_n u_k^n$.

By the growth condition (†) on ψ, there is an increasing sequence $\{p_n\}_{n=0}^{\infty}$ of integers, with $p_0 = 0$, satisfying

$$\sum_{n=1}^{\infty} \psi(4 \cdot 2^{-p_{n-1}}) \sqrt{2^{p_n}} < \infty .$$

Define $f\colon [0,1] \to \mathfrak{X}$ by

$$f(\omega) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} c_n \frac{1_{A_k^n}(\omega)}{\mu(A_k^n)} e_k^n ,$$

where

$$c_m = 2K \left[\psi\left(4 \cdot 2^{-p_{n-1}}\right)\right] \cdot \delta_{m,p_n} ,$$

(here $\delta_{j,k}$ is the usual Kronecker delta symbol). Clearly, f is strongly measurable.

The Pettis integral of f is easily computable; namely,

$$\mathcal{P} - \int_E f\, d\mu = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} c_n \left(\int_E \frac{1_{A_k^n}}{\mu(A_k^n)}\, d\mu \right) e_k^n . \qquad (*)$$

To see this, first note that the growth condition on ψ guarantees that the above series does indeed converge to an element of $\mathfrak{X}$, since

$$\begin{aligned}
\left\| \sum_{n=p}^{q} \sum_{k=1}^{2^n} c_n \left(\int_E \frac{1_{A_k^n}}{\mu(A_k^n)}\, d\mu \right) e_k^n \right\|_{\mathfrak{X}} &= \left\| \sum_{n=p}^{q} \sum_{k=1}^{2^n} c_n \left(\int_E \frac{1_{A_k^n}}{\mu(A_k^n)}\, d\mu \right) T_n u_k^n \right\|_{\mathfrak{X}} \\
&\leqslant 2 \sum_{n=p}^{q} c_n \left\| \sum_{k=1}^{2^n} \left(\int_E \frac{1_{A_k^n}}{\mu(A_k^n)}\, d\mu \right) u_k^n \right\|_{\ell_2^{2^n}} \\
&= 2 \sum_{n=p}^{q} c_n \left[\sum_{k=1}^{2^n} \left| \int_E \frac{1_{A_k^n}}{\mu(A_k^n)}\, d\mu \right|^2 \right]^{\frac{1}{2}} \\
&\leqslant 2 \sum_{n=p}^{q} c_n \sqrt{2^n} ,
\end{aligned}$$

which approaches zero as $p, q \to \infty$. Now fix $E \in \Sigma$ and $x^* \in \mathfrak{X}^*$ and let $\epsilon_k^n = \operatorname{sign}(x^* e_k^n)$. Then

$$\sum_{k=1}^{2^n} |x^* e_k^n| = \left| \sum_{k=1}^{2^n} \epsilon_k^n x^* T_n u_k^n \right| \leqslant \|T_n^*\| \; \|x^*\| \; \left\| \sum_{k=1}^{2^n} \epsilon_k^n u_k^n \right\|_{\ell_2^{2^n}} \leqslant 2 \; \|x^*\| \; \left(\sqrt{2^n}\right) ,$$

and so

$$\begin{aligned} \int_E \sum_{n=1}^{\infty} \left| \sum_{k=1}^{2^n} c_n \frac{1_{A_k^n}}{\mu(A_k^n)} x^*(e_k^n) \right| d\mu &= \sum_{n=1}^{\infty} \int_E \left| \sum_{k=1}^{2^n} c_n \frac{1_{A_k^n}}{\mu(A_k^n)} x^*(e_k^n) \right| d\mu \\ &\leqslant \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} c_n \left(\int_E \frac{1_{A_k^n}}{\mu(A_k^n)} \, d\mu \right) |x^* e_k^n| \\ &\leqslant \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} c_n \; |x^* e_k^n| \\ &\leqslant 2 \; \|x^*\| \sum_{n=1}^{\infty} c_n \left(\sqrt{2^n}\right) < \infty . \end{aligned}$$

Thus we may interchange the integral and summation below to see that

$$\begin{aligned} &\int_E x^* f \, d\mu = \int_E \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} c_n \frac{1_{A_k^n}}{\mu(A_k^n)} x^*(e_k^n) \, d\mu \\ &= \sum_{n=1}^{\infty} \int_E \sum_{k=1}^{2^n} c_n \frac{1_{A_k^n}}{\mu(A_k^n)} x^*(e_k^n) \, d\mu = x^* \left(\sum_{n=1}^{\infty} \sum_{k=1}^{2^n} c_n \left(\int_E \frac{1_{A_k^n}}{\mu(A_k^n)} \, d\mu \right) e_k^n \right) , \end{aligned}$$

as needed for $(*)$.

Fix an interval $I \in \Sigma$. Find a dyadic interval $I_j^m \subset I$ such that $4 \; \mu(I_j^m) \geqslant \mu(I)$ and then find n such that $p_{n-1} \leqslant m < p_n$. Let P be the natural projection from $\sum \oplus E_j$ onto E_{p_n}. Since $\|P\| \leqslant 2K$,

$$\begin{aligned} 2K \; \left\| \mathcal{P} - \int_I f \, d\mu \right\|_{\mathfrak{X}} &\geqslant \left\| P \left(\mathcal{P} - \int_I f \, d\mu \right) \right\|_{\mathfrak{X}} \\ &= c_{p_n} \left\| \sum_{k=1}^{2^{p_n}} \left(\int_I \frac{1_{A_k^{p_n}}}{\mu(A_k^{p_n})} \, d\mu \right) e_k^{p_n} \right\|_{\mathfrak{X}} \\ &\geqslant c_{p_n} \left\| \sum_{k=1}^{2^{p_n}} \left(\int_I \frac{1_{A_k^{p_n}}}{\mu(A_k^{p_n})} \, d\mu \right) u_k^{p_n} \right\|_{\ell_2^{2^{p_n}}} \\ &= c_{p_n} \left[\sum_{k=1}^{2^{p_n}} \left| \int_I \frac{1_{A_k^{p_n}}}{\mu(A_k^{p_n})} \, d\mu \right|^2 \right]^{\frac{1}{2}} , \end{aligned}$$

and so since $A_k^{p_n} \subset I_k^{p_n} \subset I_j^m \subset I$ for some k,

$$2K \left\| \mathcal{P} - \int_I f\, d\mu \right\|_{\mathfrak{X}} \geqslant c_{p_n} = 2K\, \psi\left(4 \cdot 2^{-p_{n-1}}\right) .$$

But ψ is increasing and $4 \cdot 2^{-p_{n-1}} \geqslant 4 \cdot 2^{-m} \geqslant \mu(I)$ and so

$$\left\| \mathcal{P} - \int_I f\, d\mu \right\|_{\mathfrak{X}} \geqslant \psi\left(\mu\left(I\right)\right) .$$

Thus f satisfies the statement of the theorem. ■

The functions in Ψ can be viewed as indicators of the degree of nondifferentiability (i.e. the poor "averaging behavior") of the indefinite Pettis integral. For instance, taking

$$\psi(s) = s^{\frac{1}{2}} \left[\frac{1}{\log\left(1/s\right)}\right]^{1+\epsilon},$$

we deduce from Theorem 1 that there exists $f \in \mathcal{P}_1(\mathfrak{X})$ such that, not only do we have

$$\lim_{h\to 0} \left\| \frac{1}{h}\, \mathcal{P} - \int_t^{t+h} f(\omega)\, d\mu(\omega) \right\|_{\mathfrak{X}} = \infty ,$$

but even worse,

$$\lim_{h\to 0} h^{\frac{1}{2}} \cdot \left[\log\left(\frac{1}{h}\right)\right]^{1+\epsilon} \left\| \frac{1}{h}\, \mathcal{P} - \int_t^{t+h} f(\omega)\, d\mu(\omega) \right\|_{\mathfrak{X}} = \infty$$

for all $t \in \Omega$.

The next theorem shows that Theorem 1 comes close to describing the *worst* type of averaging behavior of the Pettis integral that can occur in an *arbitrary* infinite-dimensional Banach space. In particular, it shows that, for spaces on which the identity operator is $(2,1)$-summing (i.e., spaces with the Orlicz property), Theorem 1 fails to hold for the function $\psi(s) = s^{\frac{1}{2}}$. Thus, the growth condition (†) on $\psi \in \Psi$ can *not* be replaced by $\psi(s) = O(s^{\frac{1}{2}})$ as $s \to 0$. We do not know, however, whether it can be replaced by $\psi(s) = o(s^{\frac{1}{2}})$ as $s \to 0$.

Theorem 2. *If the identity operator on an infinite-dimensional Banach space* $\mathfrak{X}$ *is* $(q,1)$*-summing for some* $2 \leqslant q < \infty$*, then, for every* $f \in \mathcal{P}_1(\mathfrak{X})$,

$$\left\| \mathcal{P} - \int_t^{t+h} f\, d\mu \right\|_{\mathfrak{X}} = o\left(h^{\frac{1}{q}}\right)$$

as $h \to 0^+$ *for* μ*-a.e.* t.

The proof below, which uses a factorization theorem of Pisier [P], was pointed out to us by Nigel Kalton.

Proof. Fix $f \in \mathcal{P}_1(\mathfrak{X})$ for an infinite-dimensional Banach space $\mathfrak{X}$. Consider the operator $K\colon L_\infty \to \mathfrak{X}$ given by

$$K(g) = \mathcal{P} - \int_\Omega g(\omega) f(\omega)\, d\mu(\omega) \ .$$

We need to show that

$$\left\| K\left(1_{[0,t+h]}\right) - K\left(1_{[0,t]}\right) \right\|_{\mathfrak{X}} \ = \ o\left(h^{\frac{1}{q}}\right)$$

as $h \to 0^+$ for μ-a.e. t. Fix $\epsilon > 0$.

Since K is compact and since the dual of L_∞ has the approximation property, there is [e.g. DU, Thm. VIII.3.6] a decomposition $K = K_1 + K_2$, with $K_i \in \mathcal{L}(L_\infty, \mathfrak{X})$, such that K_1 has finite rank and K_2 has norm at most ϵ^2. It is enough to show that there is some constant A, which depends only on $\mathfrak{X}$ and q, such that for each i,

$$\limsup_{h\to 0^+} \ h^{-\frac{1}{q}} \ \left\| K_i\left(1_{[0,t+h]}\right) - K_i\left(1_{[0,t]}\right) \right\|_{\mathfrak{X}} \ \leqslant \ A\, \epsilon \ , \qquad (\Diamond)$$

on a set of μ-measure at least $1 - \epsilon^q$.

Towards this, consider [see e.g. R] the natural surjective isometry $\tau\colon L_\infty \to C(\Delta)$ for the appropriate extremally disconnected compact Hausdorff space Δ. Recall that τ takes an indicator function of a Borel set in $[0,1]$ to an indicator function of a clopen set in Δ, say $\tau(1_A) = 1_{\widehat{A}}$ in such a way that if $A \subset B \subset \Omega$, then $\widehat{A} \subset \widehat{B} \subset \Delta$ and $\widehat{B \setminus A} = \widehat{B} \setminus \widehat{A}$. Let $\widehat{K}_i$ be the composite map:

$$\widehat{K}_i \ : \ C(\Delta) \ \xrightarrow{\tau^{-1}} \ L_\infty \ \xrightarrow{K_i} \ \mathfrak{X} \ .$$

First we deal with K_1. We assume, without loss of generality, that K_1 is of rank one. So the mapping $\widehat{K}_1$ is of the form

$$\widehat{K}_1(\varphi) \ = \ \left[\int_\Delta \varphi\, d\lambda\right] \ x$$

for some norm one element x in $\mathfrak{X}$ and a finite regular signed Borel measure λ on Δ. Thus

$$\begin{aligned} \left\| K_1\left(1_{[0,t+h]}\right) - K_1\left(1_{[0,t]}\right) \right\|_{\mathfrak{X}} \ &= \ \left\| \widehat{K}_1\left(1_{\widehat{[0,t+h]}}\right) - \widehat{K}_1\left(1_{\widehat{[0,t]}}\right) \right\|_{\mathfrak{X}} \\ &= \ \left| \lambda\left(\widehat{[0,t+h]}\right) - \lambda\left(\widehat{[0,t]}\right) \right| \\ &= \ \left| \alpha(t+h) - \alpha(t) \right| \ , \end{aligned}$$

where $\alpha\colon [0,1] \to \mathbb{R}$ is given by $\alpha(t) = \lambda\left(\widehat{[0,t]}\right)$. Since $\widehat{[0,t]} \subset \widehat{[0,t+h]}$ for positive h, the function α is of bounded variation and so is differentiable μ-almost everywhere. Thus, $\left\| K_1\left(1_{[0,t+h]}\right) - K_1\left(1_{[0,t]}\right) \right\|_{\mathfrak{X}} \ = \ O(h)$ μ-a.e. and so $(\Diamond)$ holds for any $q > 1$.

Now we deal with K_2. Fix $2 \leqslant q < \infty$. If the identity operator on $\mathfrak{X}$ is $(q,1)$-summing, then [P, Cor. 2.7] there is a probability measure ν on the Borel sets of Δ such that the operator $\widehat{K}_2$ admits a factorization of the form

$$\begin{array}{ccc} C(\Delta) & \xrightarrow{\widehat{K}_2} & \mathfrak{X} \\ & {}_J\searrow \quad \nearrow_T & \\ & L_{q,1}(\nu) & \end{array}$$

where J is the natural inclusion map and T is a bounded linear operator with operator norm at most $C\|\widehat{K}_2\| \leq C\epsilon^2$, where C depends only on $\mathfrak{X}$ and q. Here, $L_{q,1}(\nu)$ is the usual Lorentz space of all real-valued ν-measurable functions f on Δ for which the norm $\|f\|_{q,1}$ is finite, where

$$\|f\|_{q,1} = \int_0^\infty t^{\frac{1}{q}-1} f^*(t)\, dt$$

and f^* is the non-increasing rearrangement of $|f|$. As above

$$\begin{aligned} \left\|K_2\left(1_{[0,t+h]}\right) - K_2\left(1_{[0,t]}\right)\right\|_{\mathfrak{X}} &= \left\|K_2\left(1_{(t,t+h]}\right)\right\|_{\mathfrak{X}} \\ &= \left\|\widehat{K}_2\left(1_{\widehat{(t,t+h]}}\right)\right\|_{\mathfrak{X}} \\ &\leqslant C\epsilon^2 \left\|J\left(1_{\widehat{(t,t+h]}}\right)\right\|_{L_{q,1}(\nu)} . \end{aligned}$$

Since the non-increasing rearrangement of $J\left(1_{\widehat{(t,t+h]}}\right)$ is just the indicator function of the set $\left[0, \nu\left(\widehat{(t,t+h]}\right)\right)$, we have

$$\left\|J\left(1_{\widehat{(t,t+h]}}\right)\right\|_{L_{q,1}(\nu)} = q \left[\nu\left(\widehat{(t,t+h]}\right)\right]^{\frac{1}{q}} ,$$

and so

$$h^{-\frac{1}{q}} \left\|K_2\left(1_{[0,t+h]}\right) - K_2\left(1_{[0,t]}\right)\right\|_{\mathfrak{X}} \leqslant Cq\epsilon^2 \left[\frac{|\beta(t+h) - \beta(t)|}{h}\right]^{\frac{1}{q}} ,$$

where $\beta\colon [0,1] \to \mathbb{R}$ is given by $\beta(t) = \nu\left(\widehat{[0,t]}\right)$. The function β is increasing and hence differentiable μ-almost everywhere. Thus

$$\limsup_{h\to 0^+} h^{-\frac{1}{q}} \left\|K_2\left(1_{[0,t+h]}\right) - K_2\left(1_{[0,t]}\right)\right\|_{\mathfrak{X}} \leqslant C\, q\, \epsilon^2 \, [\beta'(t)]^{\frac{1}{q}}$$

for μ-a.e. t. From $\int_0^1 \beta'(t)\, dt \leqslant \beta(1) - \beta(0) \leqslant 1$, it follows that $\mu\left[\beta'(t) \geqslant \epsilon^{-q}\right] \leqslant \epsilon^q$. Thus, on a set of measure at least $1 - \epsilon^q$,

$$\limsup_{h\to 0^+} h^{-\frac{1}{q}} \left\|K_2\left(1_{[0,t+h]}\right) - K_2\left(1_{[0,t]}\right)\right\|_{\mathfrak{X}} \leqslant C\, q\, \epsilon ,$$

which implies ($\Diamond$) for K_2. ■

Recall that the identity operator on a space with finite cotype q is $(q,1)$-summing. Indeed, cotype plays a major rôle in the unfolding drama. To see this, consider a space $\mathfrak{X}$ which contains a finite-dimensional decomposition $\sum \oplus E_n$ where the Banach-Mazur distance between E_n and $\ell_p^{2^n}$ is less than M for each n for some fixed $1 \leqslant p \leqslant \infty$ and $M > 1$. By modifying Mazur's construction [see e. g. LT] of a basic sequence and using the fact (a simple compactness argument suffices) that finite representability of ℓ_p is inherited by subspaces of finite codimension, it is possible to construct such a finite-dimensional decomposition in $\mathfrak{X}$ whenever ℓ_p is finitely representable in $\mathfrak{X}$. By the Maurey-Pisier Theorem [MP], ℓ_{q_0} is finitely representable in $\mathfrak{X}$ where $2 \leqslant q_0 \leqslant \infty$ and

$$q_0 \;=\; \inf\left\{q\colon \mathfrak{X} \text{ has cotype } q\right\} .$$

In the same spirit as in the proof of Theorem 1 (and with similar notation), for $1 \leqslant p \leqslant \infty$ let Ψ_p be the collection of all increasing functions $\psi\colon\ [0,\infty) \to [0,\infty)$ satisfying the growth condition

$$\sum_{n=1}^{\infty} \psi\left(2^{-p_{n-1}}\right)\ \left[2^{p_n}\right]^{\frac{1}{p}} \;<\; \infty \tag{$\dagger_p$}$$

for some increasing sequence $\{p_n\}_{n=0}^{\infty}$ of integers (following the convention that $1/\infty$ is 0). For $1 \leqslant p < \infty$, a typical function in Ψ_p is

$$\psi(s) = s^{\frac{1}{p}+\epsilon}$$

with $p_n = n$ and for any $\epsilon > 0$. For $p = \infty$, ($\dagger_p$) reduces to the condition

$$\lim_{s\to 0^+} \psi(s) = 0 \ .$$

Fix $\psi \in \Psi_p$ and find an increasing sequence $\{p_n\}_{n=0}^{\infty}$ of integers, with $p_0 = 0$, satisfying

$$\sum_{n=1}^{\infty} \psi(4 \cdot 2^{-p_{n-1}})\ \left[2^{p_n}\right]^{\frac{1}{p}} \;<\; \infty$$

(again, $1/\infty$ is 0). Define $f\colon\ [0,1] \to \mathfrak{X}$ by

$$f(\omega) \;=\; \sum_{n=1}^{\infty}\sum_{k=1}^{2^n} c_n\ \frac{1_{A_k^n}(\omega)}{\mu(A_k^n)}\ e_k^n\ ,$$

where

$$c_m \;=\; 2K\ \left[\psi\left(4 \cdot 2^{-p_{n-1}}\right)\right] \cdot\ \delta_{m,p_n}\ ,$$

where K is the finite-dimensional decomposition constant. Minor variations of the proof of Theorem 1 show that this function f satisfies

$$\left\| \mathcal{P} - \int_I f \, d\mu \right\|_{\mathfrak{X}} \geqslant \psi\left(\mu\left(I\right)\right)$$

for each interval I contained in $[0,1]$.

Theorems 1 and 2, along with the above observations, give the following corollaries.

Corollary 3. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space with finite cotype and let $q_0 = \inf\{q \colon \mathfrak{X}_0$ has cotype $q\}$. Then the following hold.*

(1) *If $p > q_0$, then for each $f \in \mathcal{P}_1(\mathfrak{X})$, we have*

$$\left\| \mathcal{P} - \int_t^{t+h} f \, d\mu \right\|_{\mathfrak{X}} = o\left(h^{\frac{1}{p}}\right)$$

as $h \to 0^+$ for μ-a.e. t.

(2) *If $p < q_0$, then there is an $f \in \mathcal{P}_1(\mathfrak{X})$ such that*

$$\left\| \mathcal{P} - \int_t^{t+h} f \, d\mu \right\|_{\mathfrak{X}} \geqslant h^{\frac{1}{p}}$$

for all $t \in [0,1]$.

Corollary 4. *For an infinite-dimensional Banach space $\mathfrak{X}$, the following are equivalent.*

(1) *$\mathfrak{X}$ fails cotype.*

(2) *For each $\psi \in \Psi_\infty$, there exists $f \in \mathcal{P}_1(\mathfrak{X})$ such that*

$$\left\| \mathcal{P} - \int_I f \, d\mu \right\|_{\mathfrak{X}} \geqslant \psi\left(\mu\left(I\right)\right)$$

for each interval I contained in $[0,1]$.

Remark. Note that Corollary 4 proves the existence of a *reflexive* Banach space for which the Pettis integral has essentially no kind of differentiability property whatsoever.

Theorem 1 can be reformulated by considering the indefinite Pettis integral

$$g(t) = \mathcal{P} - \int_0^t f(\omega) \, d\mu(\omega) \ ,$$

and then expressing (‡) as

$$\|g(s) - g(t)\| \geqslant \psi(|s-t|) \ . \qquad (‡')$$

Corollary 4 shows that if g is the indefinite integral of a Pettis-integrable function taking values in a space failing cotype, then there are (essentially) no restrictions on ψ in ($\ddagger'$). Since $g(t)$ is always *continuous* [P1, Thm. 2.5], it is not unreasonable to inquire, in the case of an arbitrary infinite-dimensional Banach space, whether there are any restrictions on ψ which are attributable merely to the continuity of g as opposed to the additional fact that g is an indefinite Pettis integral. Our final result answers this question with a resounding no.

Theorem 5. *Let $\mathfrak{X}$ be an infinite-dimensional Banach space and let $\psi \in \Psi_\infty$. Then there exists a continuous function $f\colon \Omega \to \mathfrak{X}$ such that*

$$\|f(s) - f(t)\|_{\mathfrak{X}} \;\geqslant\; \psi\left(|s-t|\right)$$

for each s and t in Ω.

Remark. As Ralph Howard pointed out, Theorem 5 does not hold if $\mathfrak{X}$ is finite-dimensional. In fact, if f is a continuous function taking values in $\mathbb{R}^n$ and satisfying the lower estimate given above, then an easy Hausdorff dimension argument (see e.g. [Kah]) shows that the function ψ must satisfy $\liminf_{t\to 0} \psi(t) t^{\epsilon - 1/n} < \infty$ for every $\epsilon > 0$.

Proof. Find an increasing sequence $\{p_n\}_{n=0}^{\infty}$ of integers with $p_0 = 0$ such that $\sum_n \psi\left(2^{-p_n}\right)$ is finite and fix $K > 1$. Keeping with the notations and ideas of Theorem 1, find a finite-dimensional decomposition $\{E_n\}$ in $\mathfrak{X}$ and, to avoid excessive superscripts, let $J_k^n = I_k^{p_n}$ and likewise $\tilde{e}_k^n = e_k^{p_n}$ and $\tilde{u}_k^n = u_k^{p_n}$ for each admissible n and k.

Consider the continuous piecewise-linear function

$$f_n(\omega) \;=\; \sum_{k=1}^{2^{p_n}} 2^{p_n} \left[\left(\frac{k}{2^{p_n}} - \omega\right) \tilde{e}_k^n + \left(\omega - \frac{k-1}{2^{p_n}}\right) \tilde{e}_{k+1}^n\right] 1_{J_k^n}(\omega)\,.$$

If $\omega \in J_k^n$, then $f_n(\omega)$ is of the form $\alpha\, \tilde{e}_k^n \;+\; (1-\alpha)\, \tilde{e}_{k+1}^n$ for some $0 \leqslant \alpha \leqslant 1$. Thus the norm of $f_n(\omega)$ is at most 2 for each $\omega \in \Omega$. Define $f\colon \Omega \to \mathfrak{X}$ by

$$f(\omega) \;=\; \sum_{n=2}^{\infty} c_n\, f_n(\omega)\;,$$

where

$$c_{n+2} \;=\; 2\, K\, \psi\left(2^{-p_n}\right)\;.$$

Since each f_n is uniformly continuous and

$$\left\|\sum_{n=p}^{q} c_n f_n(\omega)\right\| \;\leqslant 2 \sum_{n=p}^{q} c_n\;,$$

the choice of $\{p_n\}$ guarantees not only that $f(\omega)$ is indeed in $\mathfrak{X}$ for each $\omega \in \Omega$ but also that f is uniformly continuous.

Fix $s, t \in \Omega$. Find p_n such that $2^{-p_n} < |s-t| \leqslant 2^{-p_{n-1}}$. Since s and t are in neither the same nor adjacent intervals of the partition $\{J_k^{n+1}\}_k$ of Ω, for appropriate *distinct* integers $k-1$, k, j, and $j+1$,

$$\begin{aligned} f_{n+1}(s) &= \alpha\ \tilde{e}_{k-1}^{n+1} + (1-\alpha)\ \tilde{e}_k^{n+1} \\ f_{n+1}(t) &= \beta\ \tilde{e}_j^{n+1} + (1-\beta)\ \tilde{e}_{j+1}^{n+1} \end{aligned}$$

for some $0 \leqslant \alpha, \beta \leqslant 1$ and so

$$\begin{aligned} \|f_{n+1}(s) - f_{n+1}(t)\|_{\mathfrak{X}} &\geqslant \left\|\alpha\ \tilde{u}_{k-1}^{n+1} + (1-\alpha)\ \tilde{u}_k^{n+1} - \beta\ \tilde{u}_j^{n+1} - (1-\beta)\ \tilde{u}_{j+1}^{n+1}\right\|_{\ell_2} \\ &= \left[(\alpha)^2 + (1-\alpha)^2 + (\beta)^2 + (1-\beta)^2\right]^{\frac{1}{2}} \\ &\geqslant 1\ . \end{aligned}$$

Let P be the natural projection from $\sum \oplus E_j$ onto $E_{p_{n+1}}$. Since ψ is increasing, we see that

$$\begin{aligned} 2\ K\ \|f(s) - f(t)\|_{\mathfrak{X}} &\geqslant \|P(f(s) - f(t))\|_{\mathfrak{X}} \\ &= c_{n+1}\ \|(f_{n+1}(s) - f_{n+1}(t))\|_{\mathfrak{X}} \\ &\geqslant c_{n+1} \\ &= 2\ K\ \psi\left(2^{-p_{n-1}}\right) \\ &\geqslant 2\ K\ \psi(|s-t|)\ . \end{aligned}$$

Thus f satisfies the statement of the theorem. ■

Remark. Theorem 5 really only uses the existence of a basic sequence inside $\mathfrak{X}$, while Theorem 1 makes full use of Dvoretzky's Theorem.

We close with a few observations. [DG, Ex. 3] constructs, for each fixed infinite-dimensional Banach space $\mathfrak{X}$, a strongly-measurable $\mathfrak{X}$-valued function that is Pettis integrable but not Bochner-Lebesgue integrable; however, that function *is* Bochner-Lebesgue integrable over *any* interval not containing 0. Theorem 1 pushes this construction a bit further to give a Pettis integrable function that *is not* Bochner-Lebesgue integrable over *any* interval.

Consider the collection $K(\mu, \mathfrak{X})$ of the μ-continuous countably additive $\mathfrak{X}$-valued vector measure with relatively compact range. If f is in $\mathcal{P}_1(\mathfrak{X})$, then the corresponding measure $\nu_f(E) = \mathcal{P} - \int_E f\, d\mu$ is in $K(\mu, \mathfrak{X})$ [cf. DU, Thm. VIII.1.5]. The measure $\nu_f(E)$ is of bounded semi-variation; furthermore, $\nu_f(E)$ is of bounded variation if and only if f is in $L_1(\mathfrak{X})$ [cf. DU, Thm. II.2.4, Cor. 2.5]. Theorem 1 (consider the measure ν_f corresponding to f as above) and [JK, Theorem 2] both construct, for each fixed infinite-dimensional Banach space $\mathfrak{X}$, a vector measure in $K(\mu, \mathfrak{X})$ that is of bounded semi-variation but of infinite variation on every interval. The measure in [JK, Theorem 2] cannot arise, however, as an indefinite Pettis integral, while the measure from Theorem 1 is (of course) precisely an indefinite Pettis integral.

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