NOWHERE WEAK DIFFERENTIABILITY
OF THE PETTIS INTEGRAL

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Abstract. For an arbitrary infinite-dimensional Banach space $\mathcal{X}$, we construct examples of strongly-measurable $\mathcal{X}$-valued Pettis integrable functions whose indefinite Pettis integrals are nowhere weakly differentiable; thus, for these functions the Lebesgue Differentiation Theorem fails rather spectacularly. We also relate the degree of nondifferentiability of the indefinite Pettis integral to the cotype of $\mathcal{X}$, from which it follows that our examples are reasonably sharp.

There are several generalizations of the space $L_1(\mathbb{R})$ of Lebesgue integrable functions taking values in the real numbers $\mathbb{R}$ (and defined on the usual Lebesgue measure space $(\Omega, \Sigma, \mu)$ on $[0,1]$) to a space of strongly-measurable “integrable” (suitably formulated) functions taking values in a Banach space $\mathcal{X}$.

The most common generalization is the space $L_1(\mathcal{X})$ of Bochner-Lebesgue integrable functions. Using the fact [P1, Theorem 1.1] that a strongly-measurable function is essentially separably-valued, one can easily extend Lebesgue’s Differentiation Theorem from $L_1(\mathbb{R})$ to $L_1(\mathcal{X})$. Specifically [B; cf. DU, Theorem II.2.9], if $f \in L_1(\mathcal{X})$, then

$$\lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} \| f(\omega) - f(t) \| \, d\mu(\omega) = 0$$

and so

$$\lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} f(\omega) \, d\mu(\omega) = f(t)$$

for almost all $t$ in $\Omega$.

Another generalization of $L_1(\mathbb{R})$ is the space $\mathcal{P}_1(\mathcal{X})$ of strongly-measurable Pettis integrable functions. A function $f : \Omega \to \mathcal{X}$ is Pettis integrable if for each $E \in \Sigma$ there is an element $x_E \in \mathcal{X}$ satisfying

$$x^*_E(x_E) = \int_E x^* f \, d\mu$$

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for each $x^*$ in the dual space $\mathcal{X}^*$ of $\mathcal{X}$. The element $x_E$ is called the Pettis integral of $f$ over $E$ and we write
\[
P - \int_E f \, d\mu = x_E.
\]
It is clear that $L_1(\mathcal{X}) \subset \mathcal{P}_1(\mathcal{X})$, while the reverse inclusion holds if and only if $\mathcal{X}$ is finite dimensional (see e.g. [DG]).

If $f \in \mathcal{P}_1(\mathcal{X})$, then for each $x^* \in \mathcal{X}^*$ the function $x^* f \in L_1(\mathbb{R})$ and so there exists a set $A$ (which depends on $x^*$) of full measure such that
\[
\lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} x^* f(\omega) \, d\mu(\omega) = x^* f(t)
\]
for each $t \in A$. In his paper [P1] introducing the Pettis integral, Pettis phrased this by saying that the Pettis integral of a function in $\mathcal{P}_1(\mathcal{X})$ is pseudo-differentiable. He closed his paper by asking whether the Pettis integral of a function $f$ in $\mathcal{P}_1(\mathcal{X})$ enjoys the stronger property of being a.e. weakly differentiable; that is, does there necessarily exist a set $A$ (independent of $x^*$) of full measure such that
\[
\lim_{h \to 0} \frac{1}{h} \int_{t}^{t+h} x^* f(\omega) \, d\mu(\omega) = x^* f(t)
\]
for each $t \in A$ and $x^* \in \mathcal{X}^*$, or such that (which is the same thing of course)
\[
\text{weak} - \lim_{h \to 0} \frac{1}{h} \left[ P - \int_{t}^{t+h} f(\omega) \, d\mu(\omega) \right] = f(t)
\]
for each $t \in A$.

If $\mathcal{X}$ is finite dimensional, then the Pettis integral of a function in $\mathcal{P}_1(\mathcal{X})$ is a.e. weakly differentiable. R.S. Phillips [Ph] (for $\mathcal{X} = \ell_2$) and M.E. Munroe [M] (for $\mathcal{X} = C[0,1]$) each constructed an example of a function in $\mathcal{P}_1(\mathcal{X})$ whose Pettis integral is not a.e. weakly differentiable. G.E.F. Thomas [T, p. 131] conjectured that such a function in $\mathcal{P}_1(\mathcal{X})$ exists for every infinite-dimensional Banach space $\mathcal{X}$.

At the recent May 1993 Kent State University Functional Analysis Conference, Joe Diestel requested a further investigation into Pettis’s question. Independently, V. Kadets [K] recently constructed, for each infinite-dimensional Banach space $\mathcal{X}$, a function in $\mathcal{P}_1(\mathcal{X})$ whose Pettis integral fails to be a.e. weakly differentiable; specifically, it fails to be weakly differentiable on a set of positive, but not full, measure.

The main theorem of this paper constructs, for each infinite-dimensional Banach space $\mathcal{X}$, a function in $\mathcal{P}_1(\mathcal{X})$ whose Pettis integral is nowhere weakly differentiable. This theorem also addresses the degree of nondifferentiability of the Pettis integral. Our second theorem shows, for arbitrary Banach spaces, that the functions which
we construct are close to being optimal with respect to their degree of nondifferentiability. From these two theorems it follows (Corollaries 3 and 4) that the cotype of a space is closely tied to the degree of nondifferentiability of the Pettis integral.

Theorem 2 was shown to us by Nigel Kalton in answer to a question posed in a preliminary version of this paper. We are grateful to him for permission to include this result here.

To state our main result we introduce the collection \( \Psi \) of all increasing functions \( \psi : [0, \infty) \to [0, \infty) \) satisfying the growth condition

\[
\sum_{n=1}^{\infty} \psi(2^{-p_n-1}) \sqrt{2^{p_n}} < \infty ,
\]

(†)

for some increasing sequence \( \{p_n\}_{n=0}^{\infty} \) of integers. Examples of functions in \( \Psi \) are

\[
\psi(s) = s^{\frac{1}{p_n} + \epsilon},
\]

\[
\psi(s) = s^{\frac{1}{\log (1/s)}} \left( \frac{1}{\log (1/s)} \right)^{1+\epsilon}
\]

and

\[
\psi(s) = s^{\frac{1}{\log \log (1/s)}} \left( \frac{1}{\log \log (1/s)} \right)^{1+\epsilon}
\]

for \( p_n = n \) and any \( \epsilon > 0 \).

**Theorem 1.** Let \( X \) be an infinite-dimensional Banach space. For each \( \psi \in \Psi \), there exists \( f \in \mathcal{P}_1(X) \) such that

\[
\left\| \mathcal{P} - \int_{I} f \, d\mu \right\|_X \geq \psi(\mu(I))
\]

(‡)

for each interval \( I \) contained in \([0, 1]\).

**Remark.** Taking \( \psi(t) = t^{\frac{2}{3}} \) gives a Pettis integrable function \( f \) such that for each \( t \in \Omega \),

\[
\lim_{h \to 0} \left\| \frac{1}{h} \mathcal{P} - \int_{t}^{t+h} f(\omega) \, d\mu(\omega) \right\|_X = \infty .
\]

If the Pettis integral of this \( f \) were weakly differentiable at \( t \), then the above limit would be finite.

**Proof.** Let \( \{I_k^n : n = 0, 1, \ldots ; k = 1, \ldots, 2^n\} \) be the dyadic intervals on \([0, 1]\), i.e.

\[
I_k^n = \left( \frac{k-1}{2^n}, \frac{k}{2^n} \right).
\]

Define inductively a collection \( \{A_k^n : n = 0, 1, \ldots ; k = 1, \ldots, 2^n\} \) of disjoint sets of strictly positive measure such that \( A_k^n \subset I_k^n \) (e.g. appropriately chosen “fat Cantor” sets).
Fix $K > 1$. By a theorem of Mazur there is a basic sequence $\{x_n\}$ in $X$ with basis constant at most $K$. Take a blocking $\{F_n\}$ of the basis with each subspace $F_n$ of large enough dimension to find (using the finite-dimensional version of Dvoretzky’s Theorem [D]) a $2^n$-dimensional subspace $E_n$ of $F_n$ such that the Banach-Mazur distance between $E_n$ and $\ell_2^{2^n}$ is less than 2. Note that $\{E_n\}$ forms a finite-dimensional decomposition. Next find operators $T_n : \ell_2^{2^n} \to E_n$ such that $\|T_n\| \leq 2$ and $\|T_n^{-1}\| = 1$. Let $\{u_k^n : k = 1, \ldots, 2^n\}$ be the standard unit vectors of $\ell_2^{2^n}$ and let $e_k^n = T_n u_k^n$.

By the growth condition (†) on $\psi$, there is an increasing sequence $\{p_n\}_{n=0}^\infty$ of integers, with $p_0 = 0$, satisfying

$$\sum_{n=1}^\infty \psi(4 \cdot 2^{-p_n-1}) \sqrt{2^{p_n}} < \infty.$$ 

Define $f : [0,1] \to X$ by

$$f(\omega) = \sum_{n=1}^\infty \sum_{k=1}^{2^n} c_n \frac{1_{A_k^n}(\omega)}{\mu(A_k^n)} e_k^n ,$$

where

$$c_m = 2K \left[ \psi(4 \cdot 2^{-p_n-1}) \right] \cdot \delta_{m,p_n} ,$$

(here $\delta_{j,k}$ is the usual Kronecker delta symbol). Clearly, $f$ is strongly measurable.

The Pettis integral of $f$ is easily computable; namely,

$$\mathcal{P} - \int_E f \, d\mu = \sum_{n=1}^\infty \sum_{k=1}^{2^n} c_n \left( \int_E \frac{1_{A_k^n}}{\mu(A_k^n)} \, d\mu \right) e_k^n .$$

To see this, first note that the growth condition on $\psi$ guarantees that the above series does indeed converge to an element of $X$, since

$$\left\| \sum_{n=p}^q \sum_{k=1}^{2^n} c_n \left( \int_E \frac{1_{A_k^n}}{\mu(A_k^n)} \, d\mu \right) e_k^n \right\|_X = \left\| \sum_{n=p}^q \sum_{k=1}^{2^n} c_n \left( \int_E \frac{1_{A_k^n}}{\mu(A_k^n)} \, d\mu \right) T_n u_k^n \right\|_X$$

$$\leq 2 \sum_{n=p}^q c_n \left\| \sum_{k=1}^{2^n} \left( \int_E \frac{1_{A_k^n}}{\mu(A_k^n)} \, d\mu \right) u_k^n \right\|_{\ell_2^{2^n}}$$

$$= 2 \sum_{n=p}^q c_n \left[ \sum_{k=1}^{2^n} \left( \int_E \frac{1_{A_k^n}}{\mu(A_k^n)} \, d\mu \right)^2 \right]^{\frac{1}{2}}$$

$$\leq 2 \sum_{n=p}^q c_n \sqrt{2^n} ,$$
which approaches zero as \( p, q \to \infty \). Now fix \( E \in \Sigma \) and \( x^* \in \mathcal{X}^* \) and let \( e^n_k = \text{sign} (x^* e^n_k) \). Then

\[
\sum_{k=1}^{2^n} |x^* e^n_k| = \left| \sum_{k=1}^{2^n} e^n_k x^* T_n u^n_k \right| \leq \| T^* \| \| x^* \| \left\| \sum_{k=1}^{2^n} e^n_k u^n_k \right\| \leq 2 \| x^* \| (\sqrt{2^n}) ,
\]

and so

\[
\int_E \sum_{n=1}^\infty \sum_{k=1}^{2^n} c_n \frac{1}{A^n_k} \left( x^* (e^n_k) \right) d\mu = \sum_{n=1}^\infty \int_E \sum_{k=1}^{2^n} c_n \frac{1}{\mu(A^n_k)} x^* (e^n_k) d\mu
\]

\[
\leq \sum_{n=1}^\infty \sum_{k=1}^{2^n} c_n \left( \int_E \frac{1}{\mu(A^n_k)} d\mu \right) |x^* e^n_k| \leq \sum_{n=1}^\infty \sum_{k=1}^{2^n} c_n |x^* e^n_k| \leq 2 \| x^* \| \sum_{n=1}^\infty c_n (\sqrt{2^n}) < \infty .
\]

Thus we may interchange the integral and summation below to see that

\[
\int_E x^* f d\mu = \int_E \sum_{n=1}^\infty \sum_{k=1}^{2^n} c_n \frac{1}{\mu(A^n_k)} x^* (e^n_k) d\mu
\]

\[
= \sum_{n=1}^\infty \int_E \sum_{k=1}^{2^n} c_n \frac{1}{\mu(A^n_k)} x^* (e^n_k) d\mu = x^* \left( \sum_{n=1}^\infty \sum_{k=1}^{2^n} c_n \left( \int \frac{1}{\mu(A^n_k)} d\mu \right) e^n_k \right),
\]

as needed for (\(*\)).

Fix an interval \( I \in \Sigma \). Find a dyadic interval \( I^m_j \subset I \) such that \( 4 \mu(I^m_j) \geq \mu(I) \) and then find \( n \) such that \( p_{n-1} \leq m < p_n \). Let \( P \) be the natural projection from \( \sum \oplus E_j \) onto \( E_{p_n} \). Since \( \| P \| \leq 2K \),

\[
2K \left\| P - \int_I f d\mu \right\|_X \geq \left\| P \left( \mathcal{P} - \int_I f d\mu \right) \right\|_X
\]

\[
= c_{p_n} \left\| \sum_{k=1}^{2^{p_n}} \left( \int \frac{1}{\mu(A^n_k)} d\mu \right) e^n_k \right\|_X
\]

\[
\geq c_{p_n} \left\| \sum_{k=1}^{2^{p_n}} \left( \int \frac{1}{\mu(A^n_k)} d\mu \right) u^n_k \right\|_{L_2^{p_n}}
\]

\[
= c_{p_n} \left[ \sum_{k=1}^{2^{p_n}} \left( \int \frac{1}{\mu(A^n_k)} d\mu \right) \right]^{\frac{1}{2}} ,
\]
and so since $A_k^{p_n} \subset I_k^{p_n} \subset I_j^{m} \subset I$ for some $k$,

$$2K \left\| \mathcal{P} - \int f \, d\mu \right\|_{X} \geq c_{p_n} = 2K \psi \left( 4 \cdot 2^{-p_n-1} \right).$$

But $\psi$ is increasing and $4 \cdot 2^{-p_n-1} \geq 4 \cdot 2^{-m} \geq \mu(I)$ and so

$$\left\| \mathcal{P} - \int f \, d\mu \right\|_{X} \geq \psi(\mu(I)).$$

Thus $f$ satisfies the statement of the theorem.

The functions in $\Psi$ can be viewed as indicators of the degree of nondifferentiability (i.e. the poor “averaging behavior”) of the indefinite Pettis integral. For instance, taking

$$\psi(s) = s^{\frac{1}{t}} \left[ \frac{1}{\log \left( \frac{1}{s} \right)} \right]^{1+\epsilon},$$

we deduce from Theorem 1 that there exists $f \in \mathcal{P}_1(X)$ such that, not only do we have

$$\lim_{h \to 0} \left\| \frac{1}{h} \mathcal{P} - \int_{t}^{t+h} f(\omega) \, d\mu(\omega) \right\|_{X} = \infty,$$

but even worse,

$$\lim_{h \to 0} \frac{1}{h^{\frac{1}{t}}} \cdot \left[ \log \left( \frac{1}{h} \right) \right]^{1+\epsilon} \left\| \frac{1}{h} \mathcal{P} - \int_{t}^{t+h} f(\omega) \, d\mu(\omega) \right\|_{X} = \infty$$

for all $t \in \Omega$.

The next theorem shows that Theorem 1 comes close to describing the worst type of averaging behavior of the Pettis integral that can occur in an arbitrary infinite-dimensional Banach space. In particular, it shows that, for spaces on which the identity operator is $(2,1)$-summing (i.e., spaces with the Orlicz property), Theorem 1 fails to hold for the function $\psi(s) = s^{\frac{1}{t}}$. Thus, the growth condition (†) on $\psi \in \Psi$ can not be replaced by $\psi(s) = O(s^{\frac{1}{t}})$ as $s \to 0$. We do not know, however, whether it can be replaced by $\psi(s) = o(s^{\frac{1}{t}})$ as $s \to 0$.

**Theorem 2.** If the identity operator on an infinite-dimensional Banach space $X$ is $(q,1)$-summing for some $2 \leq q < \infty$, then, for every $f \in \mathcal{P}_1(X)$,

$$\left\| \mathcal{P} - \int_{t}^{t+h} f \, d\mu \right\|_{X} = o\left( h^{\frac{1}{t}} \right)$$

as $h \to 0^+$ for $\mu$-a.e. $t$.

The proof below, which uses a factorization theorem of Pisier [P], was pointed out to us by Nigel Kalton.
Proof. Fix \( f \in \mathcal{P}_1(\mathcal{X}) \) for an infinite-dimensional Banach space \( \mathcal{X} \). Consider the operator \( K : L_\infty \to \mathcal{X} \) given by
\[
K(g) = \mathcal{P} - \int_\Omega g(\omega) f(\omega) \, d\mu(\omega)
\]
We need to show that
\[
\| K \left( 1_{[0,t+h]} \right) - K \left( 1_{[0,t]} \right) \|_{\mathcal{X}} = o \left( h^\frac{1}{\epsilon} \right)
\]
as \( h \to 0^+ \) for \( \mu\)-a.e. \( t \). Fix \( \epsilon > 0 \).

Since \( K \) is compact and since the dual of \( L_\infty \) has the approximation property, there is \( \{\mu, DU, \text{Thm. VIII.3.6}\} \) a decomposition \( K = K_1 + K_2 \), with \( K_i \in \mathcal{L}(L_\infty, \mathcal{X}) \), such that \( K_1 \) has finite rank and \( K_2 \) has norm at most \( \epsilon^2 \). It is enough to show that there is some constant \( A \), which depends only on \( \mathcal{X} \) and \( q \), such that for each \( i \),
\[
\limsup_{h \to 0^+} h^{-\frac{1}{\epsilon}} \| K_i \left( 1_{[0,t+h]} \right) - K_i \left( 1_{[0,t]} \right) \|_{\mathcal{X}} \leq A \epsilon , \quad (\diamond)
\]
on a set of \( \mu \)-measure at least \( 1 - \epsilon^2 \).

Towards this, consider \( \{\mu, DU, \text{Thm. VIII.3.6}\} \) the natural surjective isometry \( \tau : L_\infty \to C(\Delta) \) for the appropriate extremally disconnected compact Hausdorff space \( \Delta \). Recall that \( \tau \) takes an indicator function of a Borel set in \([0,1]\) to an indicator function of a clopen set in \( \Delta \), say \( \tau \left( 1_A \right) = 1_{\hat{A}} \) in such a way that if \( A \subset B \subset \Omega \), then \( \hat{A} \subset \hat{B} \subset \Delta \) and \( \hat{B} \setminus \hat{A} = \hat{B} \setminus \hat{A} \). Let \( \hat{K}_i \) be the composite map:
\[
\hat{K}_i : C(\Delta) \xrightarrow{\tau^{-1}} L_\infty \xrightarrow{k_i} \mathcal{X} .
\]

First we deal with \( K_1 \). We assume, without loss of generality, that \( K_1 \) is of rank one. So the mapping \( \hat{K}_1 \) is of the form
\[
\hat{K}_1 (\varphi) = \left[ \int_\Delta \varphi \, d\lambda \right] x
\]
for some norm one element \( x \) in \( \mathcal{X} \) and a finite regular signed Borel measure \( \lambda \) on \( \Delta \). Thus
\[
\| K_1 \left( 1_{[0,t+h]} \right) - K_1 \left( 1_{[0,t]} \right) \|_{\mathcal{X}} = \| \hat{K}_1 \left( 1_{[0,t+h]} \right) - \hat{K}_1 \left( 1_{[0,t]} \right) \|_{\mathcal{X}}
\]
\[
= \left| \int_\Delta f \, d\lambda \right| = \left| \int_\Delta f \, d\lambda \right| = \left| \int_\Delta f \, d\lambda \right|
\]
where \( \alpha : [0,1] \to \mathbb{R} \) is given by \( \alpha(t) = \lambda \left( [0,t] \right) \). Since \( [0,t] \subset [0,t+h] \) for positive \( h \), the function \( \alpha \) is of bounded variation and so is differentiable \( \mu \)-almost everywhere. Thus,
\[
\| K_1 \left( 1_{[0,t+h]} \right) - K_1 \left( 1_{[0,t]} \right) \|_{\mathcal{X}} = O(h) \mu\text{-a.e. and so } (\diamond) \text{ holds for any } q > 1 .
\]
Now we deal with \( K_2 \). Fix \( 2 \leq q < \infty \). If the identity operator on \( \mathcal{X} \) is \((q,1)\)-summing, then [P, Cor. 2.7] there is a probability measure \( \nu \) on the Borel sets of \( \Delta \) such that the operator \( K_2^* \) admits a factorization of the form

\[
C(\Delta) \xrightarrow{\mathcal{K}_2^*} \mathcal{X}
\]

\[
J \mathcal{Y} \xrightarrow{\mathcal{I}_{\mathcal{X}}}
\]

\[
L_{q,1}(\nu)
\]

where \( J \) is the natural inclusion map and \( T \) is a bounded linear operator with operator norm at most \( C \| \mathcal{K}_2^* \| \leq C^2 \), where \( C \) depends only on \( \mathcal{X} \) and \( q \). Here, \( L_{q,1}(\nu) \) is the usual Lorentz space of all real-valued \( \nu \)-measurable functions \( f \) on \( \Delta \) for which the norm \( \| f \|_{q,1} \) is finite, where

\[
\| f \|_{q,1} = \int_0^\infty t^{\frac{1}{q}-1} f^*(t) \, dt
\]

and \( f^* \) is the non-increasing rearrangement of \( |f| \). As above

\[
\| K_2 \left( 1_{[0,t+h]} \right) - K_2 \left( 1_{[0,t]} \right) \|_{\mathcal{X}} = \| K_2 \left( 1_{(t,t+h]} \right) \|_{\mathcal{X}} \leq \| K_2^* \left( 1_{(t,t+h]} \right) \|_{\mathcal{X}} \leq C^2 \| J \left( 1_{(t,t+h]} \right) \|_{L_{q,1}(\nu)}.
\]

Since the non-increasing rearrangement of \( J \left( 1_{(t,t+h]} \right) \) is just the indicator function of the set \([0, \nu \left( t, t+h+1 \right)]\), we have

\[
\| J \left( 1_{(t,t+h]} \right) \|_{L_{q,1}(\nu)} = q \left[ \nu \left( t, t+h+1 \right) \right]^\frac{1}{q},
\]

and so

\[
h^{-\frac{1}{q}} \| K_2 \left( 1_{[0,t+h]} \right) - K_2 \left( 1_{[0,t]} \right) \|_{\mathcal{X}} \leq C q \varepsilon^2 \left[ \frac{|\beta(t+h) - \beta(t)|}{h} \right]^{\frac{1}{q}},
\]

where \( \beta: [0,1] \to \mathbb{R} \) is given by \( \beta(t) = \nu \left( 0, t \right) \). The function \( \beta \) is increasing and hence differentiable \( \mu \)-almost everywhere. Thus

\[
\limsup_{h \to 0^+} h^{-\frac{1}{q}} \| K_2 \left( 1_{[0,t+h]} \right) - K_2 \left( 1_{[0,t]} \right) \|_{\mathcal{X}} \leq C q \varepsilon^2 \left[ \beta'(t) \right]^{\frac{1}{q}}
\]

for \( \mu \)-a.e. \( t \). From \( \int_0^1 \beta'(t) \, dt \leq \beta(1) - \beta(0) \leq 1 \), it follows that \( \mu [\beta'(t) \geq \varepsilon^{-q}] \leq \varepsilon^q \).

Thus, on a set of measure at least \( 1 - \varepsilon^q \),

\[
\limsup_{h \to 0^+} h^{-\frac{1}{q}} \| K_2 \left( 1_{[0,t+h]} \right) - K_2 \left( 1_{[0,t]} \right) \|_{\mathcal{X}} \leq C q \varepsilon,
\]
which implies \((\emptyset)\) for \(K_2\).

Recall that the identity operator on a space with finite cotype \(q\) is \((q, 1)\)-summing. Indeed, cotype plays a major rôle in the unfolding drama. To see this, consider a space \(\mathcal{X}\) which contains a finite-dimensional decomposition \(\bigoplus E_n\) where the Banach-Mazur distance between \(E_n\) and \(\ell_p^n\) is less than \(M\) for each \(n\) for some fixed \(1 \leq p \leq \infty\) and \(M > 1\). By modifying Mazur's construction [see e.g. LT] of a basic sequence and using the fact (a simple compactness argument suffices) that finite representability of \(\ell_p\) is inherited by subspaces of finite codimension, it is possible to construct such a finite-dimensional decomposition in \(\mathcal{X}\) whenever \(\ell_p\) is finitely representable in \(\mathcal{X}\). By the Maurey-Pisier Theorem [MP], \(\ell_{q_0}\) is finitely representable in \(\mathcal{X}\) where \(2 \leq q_0 \leq \infty\) and

\[
q_0 = \inf \{ q : \mathcal{X} \text{ has cotype } q \} .
\]

In the same spirit as in the proof of Theorem 1 (and with similar notation), for \(1 \leq p \leq \infty\) let \(\Psi_p\) be the collection of all increasing functions \(\psi : [0, \infty) \to [0, \infty)\) satisfying the growth condition

\[
\sum_{n=1}^{\infty} \psi \left( 2^{p_n - 1} \right) \left[ 2^{p_n} \right]^{\frac{1}{p}} < \infty \quad \text{(\(\dagger_p\))}
\]

for some increasing sequence \(\{p_n\}_{n=0}^{\infty}\) of integers (following the convention that \(1/\infty\) is 0). For \(1 \leq p < \infty\), a typical function in \(\Psi_p\) is

\[
\psi(s) = s^{\frac{1}{p}} + \epsilon
\]

with \(p_n = n\) and for any \(\epsilon > 0\). For \(p = \infty\), \((\dagger_p)\) reduces to the condition

\[
\lim_{s \to 0^+} \psi(s) = 0 .
\]

Fix \(\psi \in \Psi_p\) and find an increasing sequence \(\{p_n\}_{n=0}^{\infty}\) of integers, with \(p_0 = 0\), satisfying

\[
\sum_{n=1}^{\infty} \psi \left( 4 \cdot 2^{-p_n - 1} \right) \left[ 2^{p_n} \right]^{\frac{1}{p}} < \infty
\]

(again, \(1/\infty\) is 0). Define \(f : [0, 1] \to \mathcal{X}\) by

\[
f(\omega) = \sum_{n=1}^{\infty} \sum_{k=1}^{2^n} c_n \frac{1_{A^n_k}(\omega)}{\mu(A^n_k)} e^n_k ,
\]

where

\[
c_m = 2K \left[ \psi \left( 4 \cdot 2^{-p_n - 1} \right) \right] \cdot \delta_{m, p_n} ,
\]
where $K$ is the finite-dimensional decomposition constant. Minor variations of the proof of Theorem 1 show that this function $f$ satisfies
\[
\left\| P - \int_I f \, d\mu \right\|_X \geqslant \psi(\mu(I))
\]
for each interval $I$ contained in $[0, 1]$.

Theorems 1 and 2, along with the above observations, give the following corollaries.

**Corollary 3.** Let $X$ be an infinite-dimensional Banach space with finite cotype and let $q_0 = \inf \{ q : \chi_0 \text{ has cotype } q \}$. Then the following hold.

1. If $p > q_0$, then for each $f \in \mathcal{P}_1(X)$, we have
\[
\left\| P - \int_t^{t+h} f \, d\mu \right\|_X = o(h^{1/p})
\]
as $h \to 0^+$ for $\mu$-a.e. $t$.

2. If $p < q_0$, then there is an $f \in \mathcal{P}_1(X)$ such that
\[
\left\| P - \int_t^{t+h} f \, d\mu \right\|_X \geqslant h^{1/p}
\]
for all $t \in [0, 1]$.

**Corollary 4.** For an infinite-dimensional Banach space $X$, the following are equivalent.

1. $X$ fails cotype.

2. For each $\psi \in \Psi_\infty$, there exists $f \in \mathcal{P}_1(X)$ such that
\[
\left\| P - \int_I f \, d\mu \right\|_X \geqslant \psi(\mu(I))
\]
for each interval $I$ contained in $[0, 1]$.

**Remark.** Note that Corollary 4 proves the existence of a reflexive Banach space for which the Pettis integral has essentially no kind of differentiability property whatsoever.

Theorem 1 can be reformulated by considering the indefinite Pettis integral
\[
g(t) = P - \int_0^t f(\omega) \, d\mu(\omega),
\]
and then expressing (†) as
\[
\| g(s) - g(t) \| \geqslant \psi(|s - t|). \tag{†'}
\]
Corollary 4 shows that if \( g \) is the indefinite integral of a Pettis-integrable function taking values in a space failing cotype, then there are (essentially) no restrictions on \( \psi \) in (1'). Since \( g(t) \) is always continuous [P1, Thm. 2.5], it is not unreasonable to inquire, in the case of an arbitrary infinite-dimensional Banach space, whether there are any restrictions on \( \psi \) which are attributable merely to the continuity of \( g \) as opposed to the additional fact that \( g \) is an indefinite Pettis integral. Our final result answers this question with a resounding no.

**Theorem 5.** Let \( \mathcal{X} \) be an infinite-dimensional Banach space and let \( \psi \in \Psi_\infty \). Then there exists a continuous function \( f: \Omega \to \mathcal{X} \) such that

\[
\|f(s) - f(t)\|_\mathcal{X} \geq \psi(|s - t|)
\]

for each \( s \) and \( t \) in \( \Omega \).

**Remark.** As Ralph Howard pointed out, Theorem 5 does not hold if \( \mathcal{X} \) is finite-dimensional. In fact, if \( f \) is a continuous function taking values in \( \mathbb{R}^n \) and satisfying the lower estimate given above, then an easy Hausdorff dimension argument (see e.g. [Kah]) shows that the function \( \psi \) must satisfy \( \liminf_{t \to 0} \psi(t)t^{-1/n} < \infty \) for every \( \epsilon > 0 \).

**Proof.** Find an increasing sequence \( \{p_n\}_{n=0}^\infty \) of integers with \( p_0 = 0 \) such that \( \sum_n \psi(2^{-p_n}) \) is finite and fix \( K > 1 \). Keeping with the notations and ideas of Theorem 1, find a finite-dimensional decomposition \( \{E_n\} \) in \( \mathcal{X} \) and, to avoid excessive superscripts, let \( J_k^n = I_k^n \) and likewise \( \bar{e}_k^n = e_k^n \) and \( \bar{u}_k^n = u_k^n \) for each admissible \( n \) and \( k \).

Consider the continuous piecewise-linear function

\[
f_n(\omega) = \sum_{k=1}^{2^{p_n}} 2^{p_n} \left[ \left( \frac{k}{2^{p_n}} - \omega \right) \bar{e}_k^n + \left( \omega - \frac{k-1}{2^{p_n}} \right) \bar{e}_{k+1}^n \right] 1_{J_k^n}(\omega).
\]

If \( \omega \in J_k^n \), then \( f_n(\omega) \) is of the form \( \alpha \bar{e}_k^n + (1 - \alpha) \bar{e}_{k+1}^n \) for some \( 0 \leq \alpha \leq 1 \). Thus the norm of \( f_n(\omega) \) is at most 2 for each \( \omega \in \Omega \). Define \( f: \Omega \to \mathcal{X} \) by

\[
f(\omega) = \sum_{n=2}^\infty c_n f_n(\omega),
\]

where

\[
c_{n+2} = 2K \psi(2^{-p_n}).
\]

Since each \( f_n \) is uniformly continuous and

\[
\left\| \sum_{n=p}^q c_n f_n(\omega) \right\| \leq 2 \sum_{n=p}^q c_n,
\]

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the choice of \( \{ p_n \} \) guarantees not only that \( f(\omega) \) is indeed in \( \mathcal{X} \) for each \( \omega \in \Omega \) but also that \( f \) is uniformly continuous.

Fix \( s, t \in \Omega \). Find \( p_n \) such that \( 2^{-p_n} < |s - t| \leq 2^{-p_{n-1}} \). Since \( s \) and \( t \) are in neither the same nor adjacent intervals of the partition \( \{ J_k^{n+1} \}_k \) of \( \Omega \), for appropriate distinct integers \( k - 1, k, j, j + 1 \),

\[
egin{align*}
  f_{n+1}(s) &= \alpha \tilde{\epsilon}_{k-1}^{n+1} + (1 - \alpha) \tilde{\epsilon}_k^{n+1} \\
  f_{n+1}(t) &= \beta \tilde{\epsilon}_{j+1}^{n+1} + (1 - \beta) \tilde{\epsilon}_j^{n+1}
\end{align*}
\]

for some \( 0 \leq \alpha, \beta \leq 1 \) and so

\[
\|f_{n+1}(s) - f_{n+1}(t)\|_{\mathcal{X}} \geq \|\alpha \tilde{u}_{k-1}^{n+1} + (1 - \alpha) \tilde{u}_k^{n+1} - \beta \tilde{u}_j^{n+1} - (1 - \beta) \tilde{u}_{j+1}^{n+1}\|_{\ell_2}
\]

\[
= \left[ (\alpha)^2 + (1 - \alpha)^2 + (\beta)^2 + (1 - \beta)^2 \right]^{\frac{1}{2}} 
\]

\[
\geq 1.
\]

Let \( P \) be the natural projection from \( \sum \oplus E_j \) onto \( E_{p_{n+1}} \). Since \( \psi \) is increasing, we see that

\[
2 K \|f(s) - f(t)\|_{\mathcal{X}} \geq \|P(f(s) - f(t))\|_{\mathcal{X}}
\]

\[
= c_{n+1} \| (f_{n+1}(s) - f_{n+1}(t)) \|_{\mathcal{X}}
\]

\[
\geq c_{n+1}
\]

\[
= 2 K \psi \left( 2^{-p_{n-1}} \right)
\]

\[
\geq 2 K \psi (|s - t|).
\]

Thus \( f \) satisfies the statement of the theorem. \( \blacksquare \)

Remark. Theorem 5 really only uses the existence of a basic sequence inside \( \mathcal{X} \), while Theorem 1 makes full use of Dvoretzky’s Theorem.

We close with a few observations. [DG, Ex. 3] constructs, for each fixed infinite-dimensional Banach space \( \mathcal{X} \), a strongly-measurable \( \mathcal{X} \)-valued function that is Pettis integrable but not Bochner-Lebesgue integrable; however, that function \textit{is} Bochner-Lebesgue integrable over \textit{any} interval not containing 0. Theorem 1 pushes this construction a bit further to give a Pettis integrable function that \textit{is not} Bochner-Lebesgue integrable over \textit{any} interval.

Consider the collection \( K(\mu, \mathcal{X}) \) of the \( \mu \)-continuous countably additive \( \mathcal{X} \)-valued vector measure with relatively compact range. If \( f \) is in \( \mathcal{P}_1(\mathcal{X}) \), then the corresponding measure \( \nu_f(E) = \mathcal{P} - \int_E f \, d\mu \) is in \( K(\mu, \mathcal{X}) \) [cf. DU, Thm. VIII.1.5]. The measure \( \nu_f(E) \) is of bounded semi-variation; furthermore, \( \nu_f(E) \) is of bounded variation if and only if \( f \) is in \( L_1(\mathcal{X}) \) [cf. DU, Thm. II.2A, Cor. 2.5]. Theorem 1 (consider the measure \( \nu_f \) corresponding to \( f \) as above) and [JK, Theorem 2] both
construct, for each fixed infinite-dimensional Banach space $X$, a vector measure in $K(\mu, X)$ that is of bounded semi-variation but of infinite variation on every interval. The measure in [JK, Theorem 2] cannot arise, however, as an indefinite Pettis integral, while the measure from Theorem 1 is (of course) precisely an indefinite Pettis integral.

References


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