BANACH SPACES WHICH ADMIT A NORM WITH THE UNIFORM KADEC-KLEE PROPERTY

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Studia Math. **112** *(1995) 267–277*

Abstract. Several results are established about Banach spaces $\mathfrak{X}$ which can be renormed to have the uniform Kadec-Klee property. It is proved that all such spaces have the complete continuity property. We show that the renorming property can be lifted from $\mathfrak{X}$ to the Lebesgue-Bochner space $L_2(\mathfrak{X})$ if and only if $\mathfrak{X}$ is super-reflexive. A basis characterization of the renorming property for dual Banach spaces is given.

1. INTRODUCTION

A sequence $\{x_n\}$ in a Banach space $\mathfrak{X}$ is *separated* (respectively, *ε-separated*) if $\inf\{\| x_n - x_m \| : n \neq m\} > 0$ (respectively, $\geq \varepsilon$). Recall that $\mathfrak{X}$ has the *Kadec-Klee* property if every separated weakly convergent sequence $\{x_n\}$ in the closed unit ball of $\mathfrak{X}$ converges to an element of norm strictly less than one. We say that $\mathfrak{X}$ has the *uniform Kadec-Klee* (UKK) property (or that $\mathfrak{X}$ has a UKK norm) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that every ε-separated weakly convergent sequence $\{x_n\}$ in the closed unit ball of $\mathfrak{X}$ converges to an element of norm less than $1-\delta$. This notion was introduced by Huff in [15]. Clearly, if $\mathfrak{X}$ has the Schur property (that is, if weak and norm sequential convergence are the same) or if $\mathfrak{X}$ is uniformly convex then $\mathfrak{X}$ has the UKK property. While uniformly convex spaces are necessarily reflexive, it turns out that many classical non-reflexive spaces, e.g. the Hardy spaces H_1 of analytic functions on the ball or on the polydisk [1], the Lorentz spaces $L_{p,1}(\mu)$ [5,9], and the trace class $\mathcal{C}_1$ [13,25], all have UKK norms.

The question of characterizing the Banach spaces which are isomorphic to uniformly convex spaces has been studied intensively. This paper takes up the related question raised in [15]: under what conditions does a Banach space possess an

1991 *Mathematics Subject Classification.* 46B22, 46B20, 46B28, 46G99.

S.J. Dilworth was supported in part by the NSF Workshop in Linear Analysis and Probability

Maria Girardi was supported in part by NSF DMS-9306460

Denka Kutzarova was supported in part by the Bulgarian Ministry of Education and Science under contract MM-213/92

equivalent norm with the UKK property? We call this property *UKK-ability*, and a Banach space having this property is said to be *UKK-able*.

Recall that a Banach space $\mathfrak{X}$ has the *complete continuity* property (CCP) if every bounded linear operator from L_1 into $\mathfrak{X}$ is completely continuous (i.e., maps weakly convergent sequences to norm convergent sequences). The CCP follows from the Radon-Nikodým property (RNP) because representable operators from L_1 into $\mathfrak{X}$ are completely continuous. While there are examples of spaces [8] with the Schur (and hence the UKK) property but not the RNP, it is proved in Section 2 below that UKK-ability *implies* the CCP. In Section 3 it is proved that the Lebesgue-Bochner spaces $L_p(\mathfrak{X})$ $(p > 1)$ are UKK-able if and only if $\mathfrak{X}$ is super-reflexive (this is the isomorphic version of a theorem of Partington [28]), from which the existence of a B-convex space which fails to be UKK-able is deduced. In the fourth section UKK-ability is discussed for Banach spaces with a basis (or a finite-dimensional decomposition). Using ideas of Prus [31] a basis characterization for the weak-star version of UKK-ability is given for *dual* Banach spaces. It is also proved that the space constructed by Gowers [10] which does not contain c_0, ℓ_1, or any infinite-dimensional reflexive subspace is UKK-able.

We wish to mention that UKK renormings were recently considered by Lancien, who earlier and independently proved Theorem 4 of this paper in his thesis [23].

Throughout the paper, $\mathfrak{X}$ denotes an arbitrary Banach space, $\mathfrak{X}^*$ the dual space, $S(\mathfrak{X})$ the unit sphere, and $Ba(\mathfrak{X})$ the closed unit ball of $\mathfrak{X}$. For any unexplained terminology the reader is referred to [8] or [26].

Finally, we wish to thank Haskell Rosenthal for several enlightening conversations about this subject matter during the conference on Banach space theory held in Ascona in September, 1993.

2. UKK implies CCP

First we establish some notation. Let L_1 denote the space of integrable functions on $[0,1]$ with respect to Lebesgue measure μ. Let

$$\left\{I_k^n = [\tfrac{k-1}{2^n}, \tfrac{k}{2^n})\colon n = 0,1,2,\dots \text{ and } k = 1,\dots,2^n\right\}$$

be the usual dyadic splitting of $[0,1]$. The Haar functions $\{h_j\}_{j\geq 1}$ are defined by

$$h_1 = 1_{I_1^0} \qquad \text{and} \qquad h_{2^n+k} = 2^n\left(1_{I_{2k-1}^{n+1}} - 1_{I_{2k}^{n+1}}\right)$$

for $n = 0,1,2,\dots$ and $k = 1,\dots,2^n$. The Rademacher functions $\{r_n\}_{n\geq 0}$ are defined by $r_0 = h_1$ and $r_n = 2^{1-n}\sum_{k=1}^{2^{n-1}} h_{2^{n-1}+k}$ for $n \geq 1$.

Theorem 1. *A UKK-able Banach space enjoys the CCP.*

Proof. Let $\mathfrak{X}$ be a Banach space failing the CCP. By the main result of [11] there is a norm one operator T from L_1 into $\mathfrak{X}$ along with an $\varepsilon > 0$ and a sequence $\{x_n^*\}_{n \geq 0}$ from $S(\mathfrak{X}^*)$ such that the following conditions hold:

(a) $x_n^*\,(Th_{2^n+k}) > \varepsilon$ for each $n \geq 0$ and $k = 1, \ldots 2^n$;

(b) the natural blocking $\mathfrak{X}_n \equiv \operatorname{span}(Th_j \colon 2^{n-1} < j \leq 2^n)$ for $n \geq 0$ of the images of the Haar functions forms a finite-dimensional decomposition with constant at most 2.

Fix $\delta > 0$. Since

$$\| T \| = \sup \left\{ \frac{T\,1_{I_k^n}}{\mu(I_k^n)} \colon n = 0, 1, 2, \ldots \text{ and } k = 1, \ldots , 2^n \right\} ,$$

there is a dyadic interval $A = I_{k_\delta}^{n_\delta}$ such that $1 - \delta < \left\| T\left(\frac{1_A}{\mu(A)}\right) \right\|$. We now restrict attention to this interval A.

Consider the sequence $\{x_n\}_{n > n_\delta}$ from $\mathfrak{X}$ given by

$$x_n = T\left(\frac{(1 + r_n)1_A}{\mu(A)}\right) .$$

Clearly, the sequence $\{x_n\}$ is contained in $Ba(\mathfrak{X})$ and converges weakly to $T\left(\frac{1_A}{\mu(A)}\right)$. Note that if $n > n_\delta$ then $T(r_n\,1_A)$ is in $\mathfrak{X}_n$. Thus condition (b) gives that for $m > n > n_\delta$,

$$\| x_n - x_m \| \left\| T\left(\frac{(r_n - r_m)1_A}{\mu(A)}\right) \right\| \frac{1}{2} \left\| T\left(\frac{r_n\,1_A}{\mu(A)}\right) \right\| .$$

For $n > n_\delta$, each $r_n 1_A$ has the form

$$r_n 1_A \;=\; 2^{1-n} \sum_{j \in J} h_{2^{n-1}+j}$$

for an appropriate set J of cardinality $2^{n-1}\mu(A)$. So condition (a) gives that

$$\| T(r_n 1_A) \| \, 2^{1-n} \sum_{j \in J} \varepsilon$$

Thus the sequence $\{x_n\}_{n > n_\delta}$ is $\frac{\varepsilon}{2}$-separated. Since $\delta > 0$ is arbitrary, it follows that $\mathfrak{X}$ does not have the UKK property. But since the CCP is an isomorphic property, in fact it follows that $\mathfrak{X}$ is not UKK-able. $\square$

Remark. Recall that a Banach space $\mathfrak{X}$ has the *point of continuity* property (PCP) if every bounded weakly-closed set $A \subset \mathfrak{X}$ has non-empty relatively weakly-open subsets of arbitrarily small diameter. Since the PCP is "separably determined" (see [2] or [32]), whenever $\mathfrak{X}$ fails the PCP there is a separable bounded weakly-closed set A and an $\epsilon > 0$ such that every $a \in A$ belongs to the weak closure of $A \setminus \{x : \|x - a\| < 2\epsilon\}$. If, in addition, $\mathfrak{X}$ does not contain ℓ_1, then by [27] and [3] the weak topology on A is sequentially determined, and so every $a \in A$ is the weak-limit of a sequence $\{x_n\} \subset A \setminus \{x : \|x - a\| < 2\epsilon\}$. Clearly, $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ which is ϵ-separated. Arguing now as in Theorem 1, it follows that $\mathfrak{X}$ is not UKK-able. Thus we have proved the following result.

Proposition 2. *If $\mathfrak{X}$ is UKK-able and does not contain ℓ_1 then $\mathfrak{X}$ has the PCP.*

We thank Haskell Rosenthal for showing us this result and its proof. The Bourgain-Rosenthal space [4] has the Schur property and fails the PCP (see [33]), and so the requirement in Proposition 2 that $\mathfrak{X}$ does not contain ℓ_1 cannot be eliminated.

As a further application of the result from [11], we give a characterization of the CCP for subspaces of Banach spaces with an unconditional basis: in this case the failure of the CCP is equivalent to isomorphic containment of c_0. Since $L_1(0,1)$ fails the CCP, this fact may be regarded as an extension of the theorem of Pełczyński [29] saying that $L_1(0,1)$ does not embed into a space with an unconditional basis. The result may also be obtained from a theorem of Wessel [35], although his methods are somewhat different. In fact, a stronger result is known: James [19, Theorem 4.5] has proved the same result for the PCP. He has also proved that there is no corresponding result for the RNP by constructing a Banach space $\mathfrak{X}$, which is a subspace of a space with an unconditional basis, such that $\mathfrak{X}$ fails the RNP and $\mathfrak{X}$ does not contain c_0 [20].

Theorem 3. *Suppose that $\mathfrak{X}$ embeds isomorphically into a Banach space with an unconditional basis. Then either $\mathfrak{X}$ contains c_0 or $\mathfrak{X}$ has the CCP.*

Proof. Suppose that $\mathfrak{X}$ fails the CCP and embeds into a space $\mathcal{Y}$ with an unconditional basis $\{y_n\}$ with basis constant K. As in the proof of Theorem 1, there is a norm one operator $T\colon L_1 \to \mathfrak{X}$ along with an $\varepsilon > 0$ and a sequence $\{x_n^*\}$ from $S(\mathfrak{X}^*)$ satisfying conditions (a) and (b) above. Fix a sequence $\{\gamma_n\}_{n\geq 1}$ such that $0 < \gamma_n < \varepsilon(4K\,2^n)^{-1}$.

First we construct, by induction on n, two increasing sequences $\{k_n\}_{n\geq 1}$ and $\{p_n\}_{n\geq 1}$ of positive integers such that for the sequence $\{f_n\}_{n\geq 1}$ in L_1 defined by

$$f_n \;=\; (f_0 \;+\; \cdots \;+\; f_{n-1})\, r_{k_n}$$

(where $f_0 = r_0$) and for the blocking $\{\mathcal{Y}_n\}_{n\geq 1}$ of $\mathcal{Y}$ defined by

$$\mathcal{Y}_n = \mathrm{sp}\{y_j \colon p_{n-1} \leq j < p_n\}$$

(where $p_0 = 1$) the following conditions hold:

(1) $\| Tf_n \| > \varepsilon$;
(2) $\| f_0 + \ldots + f_n \| < 2$;
(3) $d(Tf_n, \mathcal{Y}_n) < \gamma_n$.

Put $k_1 = 1$ so that $f_1 = r_1$. Note that condition (a) gives (1), while (2) is clear. Select p_1 such that $d(Tf_1, \mathcal{Y}_1) < \gamma_1$.

For $m \geq 1$, assume that $\{k_n\}_{n=1}^m$ and $\{p_n\}_{n=1}^m$ have been chosen so that $\{f_n\}_{n=1}^m$ satisfy (1), (2), and (3). As k tends to infinity, we have the following:

(i) $\| (f_0 + \ldots + f_m) + (f_0 + \ldots + f_m) r_k \|_{L_1} \to \| f_0 + \ldots + f_m \|_{L_1}$;
(ii) $(f_0 + \ldots + f_m) r_k \to 0$ weakly in L_1 .

So there exists $k_{m+1} > k_m$ such that for $f_{m+1} = (f_0 + \ldots + f_m) r_{k_{m+1}}$ the following hold:

(iii) $\| f_0 + \ldots + f_m + f_{m+1} \|_{L_1} < 2$;
(iv) Tf_{m+1} is within $\frac{1}{2}\gamma_{m+1}$ of some element from $\mathrm{sp}\{y_j \colon p_m \leq j < \infty\}$.

Now choose p_{m+1} such that $d(Tf_{m+1}, \mathcal{Y}_{m+1}) < \gamma_{m+1}$.

To verify condition (1), note that f_{m+1} has the form

$$f_{m+1} = f_0 \left[\prod_{j<m+1} (1 + r_{k_j}) \right] r_{m+1} \ .$$

The support of $\prod_{j<m+1}(1 + r_{k_j})$ has measure 2^{-m} and is the union of 2^{-m+k_m} dyadic intervals from $\{I_i^{k_m} : 1 \leq i \leq 2^{k_m}\}$ of the k_m^{th}-level. Thus, for some subset J of integers with cardinality $2^{-m-1+k_{m+1}}$, we have

$$f_{m+1} = 2^m \, 1_A \, r_{k_{m+1}} = 2^m \sum_{j\in J} 2^{1-k_{m+1}} \, h_{j+2^{-1+k_{m+1}}} \ .$$

Condition (a) gives that

$$\| Tf_{m+1} \| > 2^m \, 2^{-m-1+k_{m+1}} \, 2^{1-k_{m+1}} \, \varepsilon = \varepsilon \ .$$

This completes the induction. Let $x_n = Tf_n$. Condition (3) guarantees that $\{x_n\}$ is equivalent to some (unconditional) block basic sequence of $\{y_n\}$. Condition (2) guarantees that $\| x_1 + \ldots + x_n \| \leq 3$. Condition (1) just says that $\| x_n \| > \varepsilon$. Thus, $\{x_n\}$ is equivalent to the standard unit vector basis of c_0. $\square$

3. UKK-ability of Lebesgue-Bochner spaces

In this section $L_p(\mathfrak{X})$ denotes the Lebesgue-Bochner space of strongly-measurable $\mathfrak{X}$-valued functions defined on a separable *non-atomic* probability space, equipped with the L_p-norm. In [28] Partington proved that if $L_2(\mathfrak{X})$ is UKK (with its usual norm) then $\mathfrak{X}$ is uniformly convex. The isomorphic version of this result was recently proved by Lancien [23,24]. In this section we shall give a different proof of Lancien's result. In order to prove the theorem we require the following necessary condition for $\mathfrak{X}$ to be UKK-able, which is due to Huff [15].

Fact. For $\epsilon > 0$, let $B_\epsilon^{(0)} = Ba(\mathfrak{X})$, and for $n \geq 1$ define $B_\epsilon^{(n)}(\mathfrak{X})$ inductively thus:

$$B_\epsilon^{(n)}(\mathfrak{X}) = \{x \in \mathfrak{X} : x = w - \lim\{x_k\}, x_k \in B_\epsilon^{(n-1)}, \|x_j - x_k\| \geq \epsilon \quad (j \neq k)\}.$$

Suppose that $\mathfrak{X}$ admits an equivalent UKK norm. Then, for each $\epsilon > 0$, $B_\epsilon^{(n)}(\mathfrak{X}) = \emptyset$ for all sufficiently large n.

Theorem 4. *Let $1 < p < \infty$. Then $L_p(\mathfrak{X})$ admits an equivalent UKK norm if and only if $\mathfrak{X}$ is super-reflexive.*

Proof. Suppose that $\mathfrak{X}$ is not super-reflexive. Then, by a result of James [16], there exists $\epsilon > 0$ such that for every $N \geq 1$ there is a dyadic martingale difference sequence $\{d_k\}_{k=0}^N$, with $d_0 = 0$, adapted to the standard filtration of $\{0,1\}^N$, such that the corresponding martingale takes its values in $Ba(\mathfrak{X})$ and such that $\|d_k(\omega)\| \geq \epsilon$ for all $k \geq 1$ and for all $\omega \in \{0,1\}^N$. For a fixed integer $N \geq 1$, let , $_N = \cup_{k=1}^N \mathbb{N}^k$. We shall consider random variables defined on the Cantor group $\{0,1\}^{\Gamma_N}$ with its associated Haar probability measure μ. For each $1 \leq k \leq N$ and for each $(n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$, we define a random variable $D_{(n_1,\ldots,n_k)}$ thus:

$$D_{(n_1,\ldots,n_k)}(\omega) = d_k((\omega_{(n_1)}, \omega_{(n_1,n_2)}, \ldots, \omega_{(n_1,\ldots,n_k)}, 0, \ldots, 0))$$

for each $\omega \in \{0,1\}^{\Gamma_N}$. Observe that for each $N-$tuple $(n_1, n_2, \ldots, n_N)$, the random variables $D_{(n_1)}, D_{(n_1,n_2)}, \ldots D_{(n_1,\ldots,n_N)}$ have the same joint distribution as $d_1, d_2, \ldots, d_N$. Fix $n_1, \ldots, n_{N-1}$. For $k \geq 1$, let

$$X_k = (D_{(n_1)} + \cdots + D_{(n_1,\ldots,n_{N-1})}) + D_{(n_1,n_2,\ldots,n_{N-1},k)}.$$

It is easily seen that $\{D_{(n_1,\ldots,n_{N-1},k)}\}_{k=1}^\infty$ is weakly null in $L_p(\mathfrak{X})$, and so $\{X_k\}$ converges weakly to $D_{(n_1)} + \cdots + D_{(n_1,\ldots,n_{N-1})}$. Moreover, $\|X_k(\omega)\| \leq 1$ for all ω, and so $X_k \in Ba(L_p(\mathfrak{X}))$. For $j \neq k$, we have

$$\mu\{X_j = X_k\} = \mu\{\|X_j - X_k\| \geq 2\epsilon\} = \frac{1}{2},$$

and so $\|X_j - X_k\|_{L_p(\mathfrak{X})} > \epsilon$. Hence $B_\epsilon^{(1)}(L_p(\mathfrak{X}))$ contains every random variable of the form $D_{(n_1)} + \cdots + D_{(n_1,\ldots,n_{N-1})}$. Repeating this argument a total of N times we see that $B_\epsilon^{(N)}(L_p(\mathfrak{X}))$ contains the zero random variable. Since this holds for every $N \geq 1$, it follows from the Fact that $L_p(\mathfrak{X})$ does not admit a UKK norm. Conversely, suppose that $\mathfrak{X}$ is super-reflexive. Then $L_p(\mathfrak{X})$ is also super-reflexive, and so $L_p(\mathfrak{X})$ admits an equivalent uniformly convex (and hence UKK) norm. □

Corollary 5. *There exists a uniformly non-octahedral (in particular, a B-convex) space which does not admit an equivalent UKK norm.*

Proof. Let $\mathfrak{X}$ be the non-reflexive uniformly non-octahedral space constructed by James [17]. It is well-known that the property of being uniformly non-octahedral lifts from $\mathfrak{X}$ to $L_2(\mathfrak{X})$. On the other hand, since $\mathfrak{X}$ is non-reflexive, it follows from Theorem 4 that $L_2(\mathfrak{X})$ does not admit an equivalent UKK norm. □

Remarks. 1. Similarly, if $\mathfrak{X}$ is non-reflexive and of type two, [18], then $L_2(\mathfrak{X})$ is of type two but not UKK-able.

2. In [14] it is proved that if $L_2(\mathfrak{X})$ has the CPCP (which is the PCP for convex sets) then $\mathfrak{X}$ has the RNP. Corollary 5 can also be proved by combining this theorem with Proposition 2 above.

3. Since $(\sum_{n=1}^\infty \bigoplus \ell_\infty^n)_2$ has the UKK property [15] it follows that the UKK property does not imply any non-trivial super-property. On the other hand, every Banach space which is *super-UKK-able* is necessarily B-convex (since $L_1(0,1)$ is not UKK-able), and by Corollary 5 there exist Banach spaces which are B-convex but not super-UKK-able. Clearly, every super-reflexive space is super-UKK-able, and so one is led to pose the following question: is super-UKK-ability *equivalent* to super-reflexivity? A positive answer to this question would follow from a positive answer to the corresponding question for the super-CPCP, which was raised in [14].

4. UKK-ability for spaces with a basis

In [31] Prus characterizes reflexive UKK-able Banach spaces in terms of their basic sequences. While some results in [31] rely heavily on the weak compactness of the unit ball, using methods from [31] some partial information is obtained in this section concerning UKK-ability of non-reflexive Banach spaces with a basis. In particular, we prove that the space constructed by Gowers which is hereditarily non-reflexive and contains no copy of c_0 or ℓ_1 *does* admit an equivalent UKK norm.

First we recall some notation from [31]. Let $\{e_n\}$ be a basic sequence in a Banach space $\mathfrak{X}$ with coefficient functional sequence $\{e_n^*\}$ in $\mathfrak{X}^*$. An element $x \in [e_n] \equiv \text{span}\{e_n\}$ is said to be a *block* if $\text{supp}(x) = \{n : e_n^*(x) \neq 0\}$ is finite. A family $\{\mathfrak{X}_n\}$ of finite-dimensional subspaces of $[e_n]$ is a *blocking* of $\{e_n\}$ provided there exists an increasing sequence of integers $\{n_k\}$, $n_1 = 1$, such that $\mathfrak{X}_k = [e_i]_{i=n_k}^{n_{k+1}-1}$ for each k. We say that the blocks $y_1, y_2, \dots, y_n$ are disjoint (with respect to the blocking $\{\mathfrak{X}_k\}$) and write $y_1 < y_2 < \cdots < y_n$ if

$$\min\{m : y_i \in \sum_{j=1}^{m} \mathfrak{X}_j\} < \max\{m : y_{i+1} \in \sum_{j=m}^{\infty} \mathfrak{X}_j\}$$

for $i = 1, 2, \dots n-1$. Finally, we say that, for $1 \le p, q \le \infty$, the blocking $\{\mathfrak{X}_n\}$ satisfies a $(p,q)-$estimate provided there exist positive constants c and C such that

$$c\left(\sum_{i=1}^{n} \|y_i\|^p\right)^{1/p} \le \left\|\sum_{i=1}^{n} y_i\right\| \le C\left(\sum_{i=1}^{n} \|y_i\|^q\right)^{1/q}$$

for all disjoint blocks $y_1, y_2, \dots y_n$. The following result is essentially due to Prus. The reader is referred to [31, p.517] for the proof.

Theorem 6. *Let $\mathfrak{X}$ be a Banach space with a basis $\{e_n\}$. If there exists a blocking $\{\mathfrak{X}_n\}$ of the basis which satisfies a $(p,1)-$estimate for some $p < \infty$ then $\mathfrak{X}$ admits an equivalent UKK norm.*

In fact, the equivalent norm constructed on $\mathfrak{X}$ in Theorem 6 makes the blocking $\{\mathfrak{X}_n\}$ *monotone* and *boundedly complete*; and so, under this renorming $\mathfrak{X}$ is isometric to the dual space of $\mathcal{Y} = [e_n^*] \subset \mathfrak{X}^*$. Moreover, $\mathfrak{X}$ has the *weak-star* UKK property with respect to its predual $\mathcal{Y}$. (The weak-star UKK is in general stronger than the UKK property: it is defined in the obvious way by considering all ϵ-separated weak-star convergent sequences.) Thus, in looking for a *converse* to Theorem 6, it is appropriate to consider *dual* Banach spaces with the weak-star UKK property. To that end, say that a basis $\{e_n\}$ of a Banach space $\mathfrak{X}$ is *weakly nearly uniformly smooth* (WNUS) if there exists $c > 0$ such that for every normalized block basic sequence $\{x_n\}$ there exists $k > 1$ such that $\|x_1 + x_k\| < 2 - c$. We shall use the following result, which is again essentially due to Prus. The reader is referred to [31, pp.512-513] for the proof.

Theorem 7. *Let $\{e_n\}$ be a WNUS basis for a Banach space $\mathfrak{X}$. Then $\{e_n\}$ has a blocking $\{\mathfrak{X}_n\}$ which satisfies an $(\infty, q)-$estimate for some $q > 1$.*

We can now prove the converse of Theorem 6.

Theorem 8. *Suppose that $\mathfrak{X}^*$ has a basis and that $\mathfrak{X}^*$, with its usual dual norm, has the weak-star UKK property. Then $\mathfrak{X}^*$ has a basis which admits a blocking satisfying a $(p,1)-$estimate for some $p < \infty$.*

Proof. By [21] we may suppose that $\mathfrak{X}$ has a *shrinking* basis, so that $\{e_n^*\}$ is a basis for $\mathfrak{X}^*$. We shall show that $\{e_n\}$ has a blocking which satisfies an $(\infty, q)-$estimate for some $q > 1$, whence by a duality argument [30] the basis $\{e_n^*\}$ of $\mathfrak{X}^*$ has a blocking which satisfies a $(p, 1)-$ estimate, where $\frac{1}{p} + \frac{1}{q} = 1$. By Theorem 7, it suffices to prove that $\{e_n\}$ is WNUS. Choose $\delta > 0$ corresponding to $\epsilon = \frac{1}{4}$ in the definition of the weak-star UKK property; clearly, we may assume that $\delta < \frac{1}{4}$. Let $\{x_k\}$ be a normalized block basis with respect to $\{e_n\}$. Select $x_k^* \in Ba(\mathfrak{X}^*)$ such that $x_k^*(x_1 + x_k) = \|x_1 + x_k\|$. By passing to a subsequence and using the weak-star sequential compactness of $Ba(\mathfrak{X}^*)$ we may assume that $\{x_k^*\}$ converges weak-star to $x^* \in Ba(\mathfrak{X}^*)$. Hence there exists k_0 such that $|(x_k^* - x^*)(x_1)| < \delta/2$ for all $k > k_0$. The proof now divides into two cases.

Case 1. There exists $k > k_0$ such that $|x^*(x_k)| < \delta/2$ and $\|x_k^* - x^*\| < \frac{1}{2}$. Then

$$\begin{aligned}\|x_1 + x_k\| &= x_k^*(x_1 + x_k) \\ &= x^*(x_1 + x_k) + (x_k^* - x^*)(x_1 + x_k) \\ &\le |x^*(x_1)| + |x^*(x_k)| + |(x_k^* - x^*)(x_1)| + \|x_k^* - x^*\| \\ &< 1 + \frac{\delta}{2} + \frac{\delta}{2} + \frac{1}{2} \le \frac{7}{4}.\end{aligned}$$

Case 2. $\|x_k^* - x^*\| \ge \frac{1}{2}$ for all $k > k_0$. It follows from the weak-star lower semicontinuity of the norm and from the weak-star convergence of $\{x_k^*\}$ to x^* that $\{x_k^*\}$ has a subsequence which is $\frac{1}{4}$-separated. Hence, by the weak-star UKK property, $\|x^*\| < 1 - \delta$. So, provided $k > k_0$, we have

$$\begin{aligned}\|x_1 + x_k\| &= x_k^*(x_1 + x_k) \\ &\le |x^*(x_1)| + |(x_k^* - x^*)(x_1)| + |x_k^*(x_k)| \\ &\le (1 - \delta) + \frac{\delta}{2} + 1 \\ &\le 2 - \frac{\delta}{2}.\end{aligned}$$

So in both cases we have shown that $\mathfrak{X}$ is WNUS. $\square$

Remark. With only straightforward modifications to the proofs, Theorems 6, 7 and 8 remain valid if "basis" is replaced by "finite-dimensional decomposition" throughout.

Next we show that the space discovered by Gowers [10], which contains no copy of c_0, ℓ_1, or any infinite-dimensional reflexive Banach space, admits a UKK norm. To see this, we require the following criterion for a basis to admit a $(p, 1)-$estimate (cf. [22]).

Proposition 9. *Let $\mathfrak{X}$ be a Banach space with a basis $\{e_n\}$. Suppose that there exists $0 < c < 2$ such that $c\|x+y\| \geq \|x\| + \|y\|$ for all disjoint blocks $x, y \in \mathfrak{X}$. Then $\{e_n\}$ satisfies a $(p,1)-$estimate for some $p < \infty$.*

Proof. Consider the basic sequence $\{e_n^*\}$ in $\mathfrak{X}^*$. First we show that if x^*, y^* are disjoint blocks in $Ba([e_n^*])$, then $\|x^* + y^*\| \leq c$. Indeed, to derive a contradiction, suppose that there exist such blocks with $\|x^* + y^*\| > c$. We may suppose that $\mathrm{supp}(x^*) \leq n < \mathrm{supp}(y^*)$. Select $z \in Ba(\mathfrak{X})$ with $(x^* + y^*)(z) > c$. Write $z = x + y$, where $\mathrm{supp}(x) \leq n < \mathrm{supp}(y)$. Then

$$\begin{aligned} c &< (x^* + y^*)(z) \\ &= x^*(x) + y^*(y) \\ &\leq \|x\| + \|y\| \\ &\leq c\|x+y\| \\ &\leq c, \end{aligned}$$

which is a contradiction. By an argument of Gurarii and Gurarii [12] (see e.g. [7]) it now follows that $\{e_n^*\}$ satisfies an $(\infty, q)-$estimate for some $q > 1$, and hence by duality that $\{e_n\}$ satisfies a $(p,1)-$estimate for some $p < \infty$. $\square$

Corollary 10. *The space constructed by Gowers (without c_0, ℓ_1, or a reflexive subspace) admits a UKK norm.*

Proof. Let $f(x) = \sqrt{\log_2(x+1)}$. From the definition of the norm [10] it is clear that for disjoint blocks $x_1, x_2, \ldots, x_n$ with respect to the basis $\{e_n\}$ of the space one has

$$\left\| \sum_{i=1}^{n} x_i \right\| \geq \frac{1}{f(n)} \sum_{i=1}^{n} \|x_i\|.$$

In particular, if $x_1 < x_2$, then

$$c\|x_1 + x_2\| \geq \|x_1\| + \|x_2\|$$

for $c = \sqrt{\log_2 3} < 2$. So by Proposition 9 $\{e_n\}$ satisfies a $(p,1)-$estimate for some $p < \infty$, whence by Theorem 6 the space admits a UKK norm. $\square$

Remarks. 1. It can be shown (see e.g. [6]) that without passing to a blocking the basis of the Gowers space satisfies a $(p,1)-$estimate for *every* $p > 1$.

2. The original Tsirelson space [34] is reflexive and has the property that ℓ_∞^n is representable on blocks in every infinite-dimensional subspace. So, by the results of [30], no infinite-dimensional subspace is UKK-able.

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